High-energy heavy-ion scattering and the optical phase shift function

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We derive, within the framework of the Glauber approximation, an exact expression for the optical phase shift function $\bar{\chi}(b)$ which takes into account the center-of-mass correlation function. We calculate total and differential cross sections using the first-order term for $\bar{\chi}(b)$ and compare them with the usual (first-order) optical limit results and, for the lighter nuclei, with the exact Glauber results. The nucleus-nucleus elastic scattering amplitude does not exhibit the large-q divergence which characterizes the usual optical limit. Our results improve the calculated total and differential cross sections dramatically for light nuclei, and significantly for medium and heavy nuclei.

NUCLEAR REACTIONS Glauber approximation, high-energy scattering theory, center-of-mass effects ^{2}H, ^{4}He, ^{12}C, ^{16}O, ^{24}Mg, ^{40}Ca projectiles and targets, $E=2.1$ GeV/nucleon; calculated σ, ^{4}He (^{4}He,^{4}He), ^{12}C (^{12}C,^{12}C), $E=2.1$ GeV/nucleon; calculated $\sigma(\theta)$.

I. INTRODUCTION

Considerable interest in medium- and high-energy nucleus-nucleus collisions have been evoked in recent years by experiments planned or performed with the Berkeley bevalac.[1-3] Since the Glauber approximation[4,5] has had considerable success in describing high-energy hadron-nucleus scattering, it was natural to generalize it to nucleus-nucleus scattering.[6-9] There have now been a number of applications[6-18] of this generalization. These applications have been used to study such subjects as medium- and high-energy heavy-ion elastic scattering, high-energy total cross sections and their relation to factorization, and fragmentation cross sections.

There have been two principal ways in which the Glauber approximation has been applied to nucleus-nucleus collisions. Either the full Glauber multiple scattering series has been evaluated or an optical limit has been used. The former method is generally exceedingly tedious and prohibitively time consuming and has been used only in cases for which a projectile or target nucleus is very light ($A \lesssim 4$). More common is the optical limit phase shift function approach.

The usual optical limit phase shift function approach is obtained[8] by considering the Glauber multiple scattering series in the limit of large target and projectile mass numbers. A detailed treatment is given in Ref. 8. It is necessary to first construct the Glauber multiple scattering series. In practice, it is convenient to use wave functions which do not have the center-of-mass coordinates separated out, but which depend on all the A_1 coordinates of the target nucleus and all the A_2 coordinates of the incident nucleus. When the nuclear wave functions can be written as a product of an internal wave function and a center-of-mass wave function, the Glauber scattering amplitude can be expressed as the product of a matrix element with respect to nuclear wave functions involving all A_1 and A_2 coordinates and an extra correction factor[8] (the center-of-mass correlation function) which depends on the momentum transfer q. This function, $K(q)$, typically increases rapidly with increasing q and combines with the matrix element which is rapidly decreasing to yield elastic scattering angular distributions which exhibit maxima and minima which generally decrease with increasing q.

In constructing the optical phase shift function from the Glauber multiple scattering series, an approximate form for the Glauber matrix element is employed. Now when this approximate element is multiplied by the center-of-mass correlation function $K(q)$, the resulting angular distribution formally diverges (i.e., approaches infinity) as $q\rightarrow\infty$. Of course, for any fixed energy q is finite. The practical effect of this large-q "divergence" is that the angular distributions begin to increase very rapidly with q beyond some (physical) value of q. This unphysical result is a drawback of the usual optical limit approach.

In this work we derive an exact expression for the optical phase shift function $\bar{\chi}(b)$ which takes into account the center-of-mass correlation function. Using this result we then calculate the lowest order expression for $\bar{\chi}(b)$, corresponding to the usual optical limit. When either this new lowest order term for $\bar{\chi}(b)$ or the exact expression for $\bar{\chi}(b)$ is used, the resulting nucleus-nucleus elastic scattering amplitude does not exhibit the large-q divergence which characterizes the usual optical limit result.

In Sec. II we present the expression for the optical phase shift function for hadron-nucleus collisions; in Sec. III we obtain it for nucleus-nucleus collisions. In Sec. IV we derive the first (lowest) order result for $\bar{\chi}(b)$. In Sec. V we apply our results to nucleus-nucleus total cross sections. In Sec. VI we investigate nucleus-nucleus elastic scattering angular distributions, and in the last section we make some concluding remarks.

II. HADRON-NUCLEUS SCATTERING

To set the stage for nucleus-nucleus collisions, let us very briefly review hadron-nucleus scattering. In practice it is convenient to be able to use a total wave function for the nuclear ground state, containing both internal and center-of-mass degrees of freedom. Let $\vec{s}_j$ be the projection of the position vector of nucleon j of the target nucleus onto the impact parameter plane. This wave function, $\psi(\{\vec{s}_j\})$ will be written as[8]

$$\psi(\{\vec{s}_j\}) = \mathcal{R}(\vec{R})\phi(\{\vec{s}_j'\}), \tag{1}$$

when $\vec{R}$ is the center-of-mass coordinate of the nucleus and $\vec{s}_j'$ is the coordinate of nucleon j relative to the nuclear center of mass. (In all our formulas we assume that the z integration—along the incident direction—has been performed.) This factorization does not describe a general type of wave function. It is applicable, however, to harmonic oscillator shell model wave functions and to product Gaussian wave functions. The wave function ϕ is the internal ground state wave function. The amplitude for elastic scattering of a hadron of incident momentum $\hbar k$ by a nucleus may be written as

$$F(q) = K(q)\,\frac{ik}{2\pi}\int d^2b\, e^{i\vec{q}\cdot\vec{b}}[1 - e^{i\chi(b)}], \tag{2}$$

where

$$K(q) = \left[\int d^2R\,|\mathcal{R}(\vec{R})|^2 e^{i\vec{q}\cdot\vec{R}}\right]^{-1}, \tag{3}$$

$$e^{i\chi(b)} = \langle \psi(\{\vec{s}_j\})|\prod_{j=1}^{A}[1 - \Gamma_j(\vec{b} - \vec{s}_j)]|\psi(\{\vec{s}_j\})\rangle, \tag{4}$$

$\hbar\vec{q}$ is the momentum transfer, and $\vec{b}$ is the impact parameter vector. The functions $\Gamma_j(\vec{b})$ are the hadron-nucleon profile functions related to the hadron-nucleon elastic scattering amplitudes $f_j(q)$ by

$$\Gamma_j(\vec{b}) = \frac{1}{2\pi i k_j}\int d^2q\, f_j(q)e^{-i\vec{q}\cdot\vec{b}}, \tag{5}$$

where $\hbar k_j$ is the incident momentum of the hadron. The function $\chi(b)$ is the optical limit phase shift function which is generally used and which we shall call the usual optical limit phase shift function.

In the case where the ground state wave functions are product wave functions and the hadron-nucleon interactions are equal ($\Gamma_j = \Gamma_N$) we obtain the result

$$i\chi(b) = A\ln[1 - C(b)], \tag{6}$$

where

$$C(b) = \frac{1}{2\pi i k_N}\int S(q)f_N(q)e^{-i\vec{q}\cdot\vec{b}}d^2q \tag{7}$$

and $S(q)$ is the nuclear form factor. Here A is the mass number of the target nucleus.

We point out that it has been common practice to approximate the optical phase shift function $\chi(b)$ given by Eq. (6) by

$$i\chi(b) \approx -AC(b). \tag{8}$$

This approximation leads to inaccurate results for the scattering amplitude $F(q)$ as well as a divergence in $F(q)$ for large q, and should not be made. In addition, we point out that there is no advantage (computational or otherwise) of Eq. (8) over Eq. (7). It is just as easy to numerically compute $F(q)$ via Eq. (7) as it is via Eq. (8). In fact, in certain cases $F(q)$ can be evaluated *analytically* with Eq. (7), whereas Eq. (8) generally leads to numerical integrations.

III. NUCLEUS-NUCLEUS SCATTERING

In this section we generalize the results of Sec. II to nucleus-nucleus collisions. We write the amplitude for elastic scattering of two nuclei as

$$F(q) = \frac{ik}{2\pi}\int e^{i\vec{q}\cdot\vec{b}}[1 - e^{i\bar{\chi}(b)}]d^2b. \tag{9}$$

This defines the optical phase shift function $\bar{\chi}(b)$. Similarly, Eq. (1) becomes

$$\begin{aligned}\psi_{A_1}(\{\vec{s}_j\}) &= \mathcal{R}_{A_1}(\vec{R}_1)\phi_{A_1}(\{\vec{s}_j'\}),\\ \psi_{A_2}(\{\vec{s}_l\}) &= \mathcal{R}_{A_2}(\vec{R}_2)\phi_{A_2}(\{\vec{s}_l'\}),\end{aligned} \tag{10}$$

and Eq. (3) becomes

$$K(q) = \left[\int d^2R_1\,|\mathcal{R}_{A_1}(\vec{R}_1)|^2 e^{i\vec{q}\cdot\vec{R}_1} \times \int d^2R_2\,|\mathcal{R}_{A_2}(\vec{R}_2)|^2 e^{-i\vec{q}\cdot\vec{R}_2}\right]^{-1}. \tag{11}$$

One then obtains for $F(q)$ the result[8]

$$F(q)=K(q)\,\frac{ik}{2\pi}\int e^{i\vec{q}\cdot\vec{b}}[1-e^{i\chi(b)}]d^2b, \qquad (12)$$

where

$$e^{i\chi(b)}=\langle\psi_{A_1}(\{\vec{s}_j\})\psi_{A_2}(\{\vec{s}_l\})|\times\prod_{j=1}^{A_1}\prod_{l=1}^{A_2}[1-\Gamma_{jl}(\vec{b}-\vec{s}_j+\vec{s}_l)]\times|\psi_{A_1}(\{\vec{s}_j\})\psi_{A_2}(\{\vec{s}_l\})\rangle, \qquad (13)$$

in which

$$\Gamma_{jl}(\vec{b})=\frac{1}{2\pi i k_{jl}}\int e^{-i\vec{q}\cdot\vec{b}}f_{jl}(q)d^2q \qquad (14)$$

is the profile function for scattering of nucleon j by nucleon k. The function $\chi(b)$ is the usual optical phase shift function. From Eqs. (9) and (10) we obtain

$$e^{i\bar{\chi}(b)}=\frac{1}{(2\pi)^2}\int d^2q\,d^2b'e^{i\vec{q}\cdot(\vec{b}'-\vec{b})}\,K(q)e^{i\chi(b')}. \qquad (15)$$

This, together with Eqs. (13) and (14), is our general expression for the nucleus-nucleus phase shift function $\bar{\chi}(b)$ when the factorization given by Eq. (10) can be obtained. The factorization does not describe a general type of wave function. It is applicable, however, to harmonic oscillator shell model wave functions and product Gaussian wave functions.[19] We point out that Eqs. (12) and (13) are equivalent to Eqs. (9) and (15) as far as the scattering amplitudes are concerned, as both pairs of equations lead to the same scattering amplitude. However, $\chi(b)$ and $\bar{\chi}(b)$ are different.

In order to perform practical calculations with $\chi(b)$ or $\bar{\chi}(b)$, approximations to Eqs. (13) and (15) need to be made. It is usual to retain for $\chi(b)$ terms that are linear in Γ_{jk}. The usual (first-order) optical limit phase shift function is then given by

$$i\chi(b)\approx i\chi_1(b),$$

$$i\chi_1(b)=-\langle\psi_{A_1}\psi_{A_2}|\sum_{j=1}^{A_1}\sum_{l=1}^{A_2}\Gamma_{jl}(\vec{b}-\vec{s}_j+\vec{s}_l)|\psi_{A_1}\psi_{A_2}\rangle, \qquad (16)$$

which, by means of Eq. (5), may be written as

$$i\chi_1(b)=-\sum_{j=1}^{A_1}\sum_{l=1}^{A_2}\frac{1}{2\pi i k_{jl}}\int e^{-i\vec{q}\cdot\vec{b}}S_{A_1}(\vec{q})\times S_{A_2}(-\vec{q})f_{jl}(q)d^2q, \qquad (17)$$

where S_{A_1} and S_{A_2} are the nuclear form factors. Unfortunately, for Gaussian form factors and nucleon-nucleon scattering amplitudes this result leads to a divergence in $F(q)$ for large q.[8] It is likely that for most reasonable form factors and NN scattering amplitudes it will also lead to a divergence in $F(q)$ for large q. Since total cross sections depend only on the forward elastic scattering amplitude $F(0)$, Eq. (17) has often been used in their evaluation. In addition, angular distributions at values of q for which $F(q)$ given by Eqs. (12) and (17) is still decreasing have often been calculated using these equations.

If we perform the corresponding approximation for $\bar{\chi}(b)$, we obtain the result

$$i\bar{\chi}(b)\approx i\bar{\chi}_1(b),$$

$$i\bar{\chi}_1(b)=-\frac{1}{(2\pi)^2}\int e^{i\vec{q}\cdot(\vec{b}'-\vec{b})}K(q)\times\langle\psi_{A_1}\psi_{A_2}|\sum_{j=1}^{A_1}\sum_{l=1}^{A_2}\Gamma_{jl}(\vec{b}'-\vec{s}_j+\vec{s}_l)|\psi_{A_1}\psi_{A_2}\rangle\times d^2b'd^2q, \qquad (18)$$

which, by means of Eqs. (16) and (17), may be written as

$$i\bar{\chi}_1(b)=\frac{1}{(2\pi)^2}\int e^{i\vec{q}\cdot(\vec{b}'-\vec{b})}i\chi_1(b)K(q)d^2b'd^2q \qquad (19)$$

$$=-\frac{1}{2\pi i}\sum_{j=1}^{A_1}\sum_{l=1}^{A_2}\int e^{-i\vec{q}\cdot\vec{b}}K(q)S_{A_1}(\vec{q})S_{A_2}(-\vec{q})\times k_{jl}^{-1}f_{jl}(q)d^2q. \qquad (20)$$

This result does not lead to a divergence in $F(q)$ for large q^2. This is due to the translational invariance of the Glauber approximation scattering amplitude obtained with $\bar{\chi}\approx\bar{\chi}_1$. This translational invariance is destroyed by the approximation $\chi\approx\chi_1$.[20]

In the simple case where the nuclear ground state wave functions are given by

$$\psi_{A_1}=\prod_{j=1}^{A_1}\varphi_{A_1}(\vec{r}_j), \qquad (21)$$

$$\psi_{A_2}=\prod_{l=1}^{A_2}\varphi_{A_2}(\vec{r}_l), \qquad (22)$$

and the nucleon-nucleon interactions are equal ($\Gamma_{jk}=\Gamma_{NN}$), we obtain

$$i\chi_1(b)=-A_1A_2\,\langle\varphi_{A_1}(\vec{r}_j)\varphi_{A_2}(\vec{r}_l)|\Gamma_{NN}(\vec{b}-\vec{s}_j+\vec{s}_l)|\varphi_{A_1}(\vec{r}_j)\varphi_{A_2}(\vec{r}_l)\rangle \qquad (23)$$

$$=-\frac{A_1A_2}{2\pi i k_N}\int e^{-i\vec{q}\cdot\vec{b}}S_{A_1}(\vec{q})S_{A_2}(-\vec{q})f_{NN}(q)d^2q \qquad (24)$$

and

$$i\bar{\chi}_1(b) = -\frac{A_1 A_2}{(2\pi)^2}\int e^{i\vec{q}\cdot(\vec{b}'-\vec{b})} d^2b' K(q) \langle \varphi_{A_1}(\vec{r}_j)\varphi_{A_2}(\vec{r}_l) | \Gamma_{NN}(\vec{b}' - \vec{s}_j + \vec{s}_l) | \varphi_{A_1}(\vec{r}_j)\varphi_{A_2}(\vec{r}_l)\rangle d^2q \quad (25)$$

$$= -\frac{A_1 A_2}{2\pi i k_N}\int e^{-i\vec{q}\cdot\vec{b}} K(q) S_{A_1}(\vec{q}) S_{A_2}(-\vec{q}) f_{NN}(q) d^2q\,. \quad (26)$$

If we assume nuclear ground state form factors given by

$$S_{A_i}(q) = e^{-q^2 R_i^2/4}, \quad i = 1,2\,, \quad (27)$$

and nucleon-nucleon elastic scattering amplitudes

$$f_{NN}(q) = \frac{k\sigma_N}{4\pi}(i+\rho)e^{-(1/2)aq^2} \quad (28)$$

typical at high energies, we obtain

$$i\chi_1(b) = -A_1 A_2 y e^{-b^2/R^2}\,, \quad (29)$$

where

$$y = \sigma_N(1 - i\rho)/2\pi R^2\,, \quad (30)$$

$$R^2 = R_1{}^2 + R_2{}^2 + 2a\,. \quad (31)$$

This is the usual first-order optical limit result often used in analyses of nucleus-nucleus scattering. The new first-order optical limit result is then

$$i\bar{\chi}_1(b) = -\frac{A_1 A_2 y}{(2\pi)^2}\int e^{i\vec{q}\cdot(\vec{b}'-\vec{b})} K(q) e^{-b'^2/R^2} d^2b' d^2q \quad (32)$$

$$= -\tfrac{1}{2}A_1 A_2 y R^2 \int_0^\infty J_0(qb) e^{-q^2R^2/4} K(q) q\, dq\,, \quad (33)$$

which is to be compared with the usual first-order optical limit result $i\chi_1(b)$ given by Eq. (29).

Let us next consider nuclear ground state form factors given by

$$S_{A_i}(q) = (1 + \delta_i q^2) e^{-q^2 R_i^2/4}, \quad i = 1,2\,. \quad (34)$$

This is the form obtained from harmonic oscillator wave functions for s-shell and p-shell nuclei. For harmonic oscillator wave functions, the function $K(q)$ is given by

$$K(q) = \exp\left[\frac{q^2}{4}\left(\frac{R_1{}^2}{A_1} + \frac{R_2{}^2}{A_2}\right)\right]. \quad (35)$$

The usual first-order optical phase shift function $\chi_1(b)$ is then evaluated to be

$$i\chi_1(b) = -A_1 A_2 \frac{\sigma_N(1-i\rho)}{2\pi R^2} e^{-b^2/R^2} \times \left\{ 1 + \frac{4(\delta_1+\delta_2)}{R^2} + \frac{32\delta_1\delta_2}{R^4} - \frac{b^2}{R^2}\left[\frac{4(\delta_1+\delta_2)}{R^2} + \frac{64\delta_1\delta_2}{R^4}\right] + \frac{16\delta_1\delta_2 b^4}{R^8}\right\} \quad (36)$$

$$\equiv g(b,R)\,, \quad (37)$$

where Eqs. (36) and (37) define $g(b,R)$. The result for Gaussian form factors is obtained by setting $\delta_1 = \delta_2 = 0$. The new first-order optical phase shift function $\bar{\chi}_1(b)$ is given simply by

$$i\bar{\chi}_1(b) = g(b,r)\,, \quad (38)$$

where

$$r^2 = R_1{}^2(1 - 1/A_1) + R_2{}^2(1 - 1/A_2) + 2a\,. \quad (39)$$

Thus we see that the effect of properly including the center-of-mass correlation function in the calculation of $\bar{\chi}_1(b)$ is, in the case of harmonic oscillator wave functions, simply to replace R_i^2 by $R_i^2(1 - A_i^{-1})$ in the usual expression for the first-order optical phase shift function. However, since the approximation $\bar{\chi} \approx \bar{\chi}_1$ does not destroy the translational invariance of the Glauber approximation scattering amplitude, the resulting scattering amplitude does not diverge for large q, as it does when $i\chi_1(b) = g(b,R)$ is used in Eq. (12).

IV. TOTAL CROSS SECTIONS

In this section we show how the correct inclusion of the center-of-mass correlations in the first-order optical phase shift function affects the calculated total cross sections. The total cross section is obtained from the forward elastic scattering amplitude $F(0)$ by means of the optical theorem

$$\sigma_{\text{tot}} = (4\pi/k)\,\text{Im}F(0)\,. \quad (40)$$

Since we have seen in Sec. III that for harmonic oscillator wave functions the main effect of the center-of-mass correlations is to multiply R_i^2 by $(1 - A_i^{-1})$, and since total cross sections depend roughly on R_i^n with $2 < n < 3$, we would expect the total cross sections calculated with $\bar{\chi}_1(b)$ to be somewhat smaller than the cross sections calculated with the usual optical phase shift function $\chi_1(b)$. (We point out that various multiple scattering contributions are included via the first-order optical phase shift functions χ_1 and $\bar{\chi}_1$.)

In Table I we show the calculated nucleus-nucleus total cross sections for an incident energy of 2.1 GeV/nucleon for a variety of incident and target nuclei, together with the available measurements.[2] Since the main purpose of our calculations is to compare various theoretical results, the precise values of the input parameters used

TABLE I. Nucleus-nucleus total cross sections at 2.1 GeV/nucleon. Columns 2 and 3 are calculated with the usual phase shift function $\chi_1(b)$. Columns 4 and 5 are calculated with the new phase shift function $\overline{\chi}_1(b)$. Column 6 is calculated via the "exact" Glauber multiple scattering series. Columns 3 and 5 are calculated with harmonic oscillator wave functions for α, ^{12}C, and ^{16}O.

Nuclei	$\sigma_{tot}(\chi_1)$ (mb)		$\sigma_{tot}(\overline{\chi}_1)$ (mb)		σ_{tot}("exact") (mb)	σ_{tot}(expt.) (mb)
d-d	162		154		160	158 ± 0.8
d-α	294		267		263	272 ± 1.5
						262 ± 1.8
d-^{12}C	729		643		621	644 ± 3.5
						617 ± 3.0
d-^{16}O	910		801		776	
d-^{24}Mg	1208		1057		1021	
d-^{40}Ca	1724		1523		1481	
α-α	429	429	386	386	386	408 ± 2.5
α-^{12}C	902	895	834	834		835 ± 5
						826 ± 5.9
α-^{16}O	1097	1087	1023	1020		
α-^{24}Mg	1387		1307			
α-^{40}Ca	1939		1851			
^{12}C-^{12}C	1605	1564	1518	1502		1347 ± 39
^{12}C-^{16}O	1880	1825	1789	1761		
^{12}C-^{24}Mg	2272		2180			
^{12}C-^{40}Ca	3010		2914			
^{16}O-^{16}O	2180	2109	2087	2043		
^{16}O-^{24}Mg	2607		2512			
^{16}O-^{40}Ca	3402		3305			
^{24}Mg-^{24}Mg	3077		2983			
^{24}Mg-^{40}Ca	3949		3854			
^{40}Ca-^{40}Ca	4940		4845			

are not critical. Nevertheless, we have tried to use realistic values. The nucleon-nucleon input parameters are obtained from nucleon-nucleon scattering measurements[21,22] and are $\sigma = 42.7$ mb, $\rho = -0.28$, and $a = 6.2$ $(\mathrm{GeV}/c)^{-2}$.

In column 1 we indicate the nuclei involved in the collisions. In columns 2 and 3 we present the total cross sections calculated using the usual first-order optical phase shift function $\chi_1(b)$. In columns 4 and 5 the cross sections are calculated using the new first-order optical phase shift function $\overline{\chi}_1(b)$. Although the exact Glauber result for deuteron-nucleus and α-α cross sections can be calculated with relative ease, we show the optical limit for these cross sections for the sake of completeness. In columns 2 and 4 we use Gaussian form factors, and in columns 3 and 5 we use harmonic oscillator wave functions for ^{4}He, ^{12}C, and ^{16}O. The values of R_i were obtained from electron scattering measurements[23-25] with center-of-mass and finite proton size corrections taken into account.[26] (The rms values used are shown in Table II.) The quantities presented in column 6 correspond to those in columns 2 and 4, except that here the cross sections are calculated using the exact Glauber multiple scattering series. The computer time required to calculate these "exact" cross sections was too prohibitive to obtain the results for other than deuteron-nucleus and α-α collisions. In column 7 we show the available data.

TABLE II. Root-mean-squared radii.

A	$\langle r_A^2 \rangle^{1/2}$ (fm)
2	2.17
4	1.71
12	2.453
16	2.71
24	2.98
40	3.50

We observe from columns 2 and 4 or 3 and 5 that the new first-order phase shift function $\overline{\chi}_1(b)$ produces a reduction in σ_{tot} of the order of ~10% for deuteron-nucleus and α-α collisions. The effect decreases with the size of the systems involved in the collision, being only ~2% for ^{40}Ca-^{40}Ca collisions. We also note that for deuteron-nucleus and α-α collisions, where the exact (Glauber) results have been calculated, the cross sections obtained using $\overline{\chi}_1(b)$ (column 4) differ from these exact results (column 6) by between 0 and 4%. On the other hand, when these cross sections are calculated using $\chi_1(b)$, the results (column 2) differ from these exact results by between 1 and 18%. Thus, by means of the very simple modification of the usual optical phase shift function we obtain significantly improved results.

If we compare the cross sections presented in columns 5 and 6 with the data shown in column 7, we note that the results are in good qualitative agreement, but that there is room for improvement. We also note that the large discrepancy between theory and experiment for the ^{12}C-^{12}C cross section[2,14] is reduced considerably by use of a harmonic oscillator wave function and by use of the new first-order optical phase shift function $\bar{\chi}_1(b)$.

V. ELASTIC SCATTERING ANGULAR DISTRIBUTIONS

In Fig. 1 we show the differential cross section $d\sigma/d|t|$ as a function of t, the squared four-momentum transfer, for α-α elastic scattering at an incident energy of 2.1 GeV/nucleon. The solid curve is the exact Glauber result. The dashed curve is obtained using the new first-order optical phase shift function $\bar{\chi}_1(b)$ in Eq. (9). The dotted curve is obtained with the usual first-order optical phase shift function $\chi_1(b)$ in Eq. (12). We note that

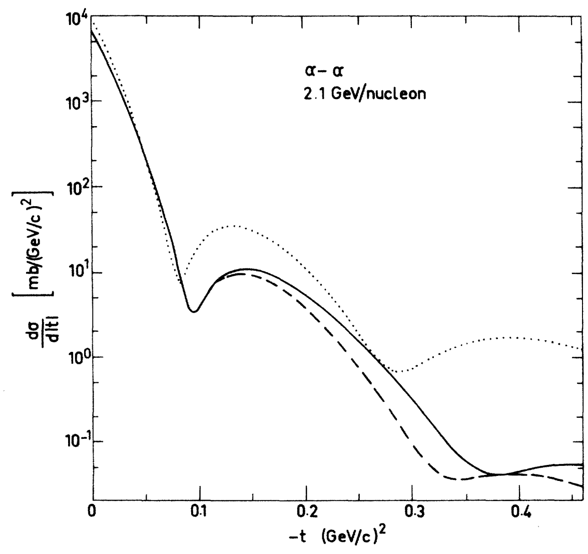

FIG. 1. Differential cross sections for ^4He-^4He elastic scattering at an incident energy of 2.1 GeV/nucleon as a function of t, the squared four-momentum transfer. The solid curve is the exact Glauber multiple scattering result. The dashed curve is obtained using the new first order optical phase shift function $\bar{\chi}_1(b)$. The dotted curve is obtained using the usual first order optical phase shift function $\chi_1(b)$.

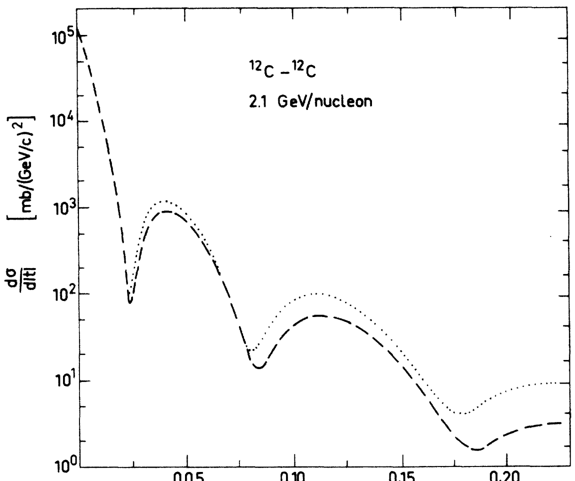

FIG. 2. Differential cross sections for ^{12}C-^{12}C elastic scattering at an incident energy of 2.1 GeV/nucleon as a function of t. The dashed (dotted) curve is obtained using the new (usual) first order optical phase shift function.

up through the second maximum [$-t\sim 0.14$ (Gev/$c)^2$] the results obtained with the new optical phase shift function $\bar{\chi}_1(b)$ are very close to the exact Glauber results, whereas the results obtained with the usual optical phase shift function $\chi_1(b)$ differ from the exact Glauber results by as much as a factor of 5. At large momentum transfers (not shown) the cross section obtained with $\chi_1(b)$ *increases*, whereas the exact cross section and that obtained with $\bar{\chi}_1(b)$ continue to decrease. The cross section obtained with $\chi_1(b)$ attains an *absolute* minimum of ~0.1 mb/(GeV/$c)^2$ at $-t\approx 1.2$ (GeV/$c)^2$. Beyond this value of t the cross section rises and, after a very shallow relative minimum of ~0.4 mb/(GeV/$c)^2$ at $-t\approx 1.8$ (GeV/$c)^2$, it rises *monotonically*.

In Fig. 2 we show the differential cross section for ^{12}C-^{12}C elastic scattering at 2.1 GeV/nucleon. The dashed curve is obtained using the new first-order optical phase shift function $\bar{\chi}_1(b)$ in Eq. (9). The dotted curve is obtained with the usual first-order optical phase shift function $\chi_1(b)$ in Eq. (12). Harmonic oscillator wave functions were used. As expected, the center-of-mass effects are smaller for this heavier system than they were for α-α scattering. Nevertheless, one still observes differences of a factor of ~2 near the first minimum [$-t\approx 0.026$ (GeV/$c)^2$], and beyond the second minimum [$-t\gtrsim 0.085$ (GeV/$c)^2$], and a factor of ~3 beyond the third minimum [$-t\gtrsim 0.18$ (GeV/$c)^2$]. The difference increases markedly at larger momentum transfers (not shown). At very large momentum transfers the cross section obtained with the usual optical phase shift function χ_1 increases.

VI. CONCLUSIONS

We have investigated collisions between nuclei within the framework of the Glauber approximation. The usual optical limit result for the phase shift function leads to a nucleus-nucleus elastic scattering amplitude whose modulus formally increases without bound as $q \to \infty$. This tendency to increase exhibits itself in the physical region of momentum transfers by the appearance of scattering intensities and cross sections which are too large. This drawback of the usual first-order optical limit result is avoided by introducing the center-of-mass correlations in each order of the optical phase shift function. The use of the first-order term of this modified phase shift function leads to significant improvement in the results for total cross sections and elastic scattering differential cross sections.

Using the methods of this paper, the effects of higher-order terms in the optical phase shift function may be calculated. These terms will contain certain classes of multiple scattering not contained in the first-order optical phase shift function $\bar{\chi}_1$. Furthermore, for second- and higher-order optical phase shift functions, correlation effects may be significant. Investigations of these aspects of nucleus-nucleus collisions are in progress.

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