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# Correlation expansion of the optical potential

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The multiple scattering theory for the optical potential is examined. This series is arranged according to the number of target particles struck in forming the optical potential. The term which involves two target particles is summed as a three-body problem. Explicit formulas for calculating the optical potential in the fixed scatterer approximation are presented. Corrections to the fixed scatterer approximation, one a correction to closure, another a correction due to nonlocality in the two-body interaction, are presented. The relation between this work and other formal rearrangements of the multiple scattering series is presented.

[SCATTERING THEORY. Optical potential. Multiple scattering. Nucleon-nucleus scattering.]

## I. INTRODUCTION

Recently, the formal theory of elastic scattering of protons from finite nuclei has received[1-7] considerable attention. This attention is prompted by the existence of new accelerators and spectrometers which will allow the accurate measurement of elastic differential cross sections. In this work, we examine the multiple scattering series[8,9] for elastic scattering. We develop the series as an expansion for the optical potential: this is in distinction to approaches[10] which develop expansions for the elastic scattering $T$ matrix directly.

In developing a perturbative theory of the optical potential, two general attitudes seem to have emerged. One of these is that the leading term in the optical potential, which is of the form of a $T$ matrix folded with the target density, should contain as much of the physics as possible. This then generally results[1,2] in a $T$ matrix which describes the scattering of the incident particle with the target particle in the presence of the remaining nucleons. It is this $T$ matrix, itself a solution to a three-body problem, which must be folded with the target density to yield the lowest-order optical potential. Although much theoretical progress has been made along these lines, and unitarity relations indicate[3] that such an approach would yield a more accurate version of the impulse approximation, the numerical difficulties involved in using this approach are considerable and, to this time, have limited greatly any extensive applications[4] of the formal developments.

The second approach[5-8] may be characterized by the use of a simple two-body $T$ matrix, either the free $T$ matrix or a "closure" $T$ matrix, in the leading order optical potential, and the treatment of the corrections as higher-order terms in an expression for the optical potential. This is the approach adopted here. If including these corrections to the optical potential proves to be too difficult, then our approach will have the same practical disadvantage as Refs. 1 and 2.

In this work, the multiple scattering series for the pseudo-optical potential of Kerman, McManus, and Thaler (KMT) is arranged according to the number of distinct target particles which are involved in building the optical potential. We work in the closure approximation. The generalization to the use of other approximate propagators is presented in Appendix B. The lowest-order term for the pseudo-optical potential in which the incident particle interacts with one target particle is shown to yield the usual impulse approximation.

The second-order term, in which the incident particle interacts with two target nucleons within the optical potential, is shown to be the solution of a three-body problem. The physical significance for elastic scattering of truncating this series at any order is discussed. Explicit formulas for constructing the first- and second-order optical potentials in the fixed scatterer limit are presented. In particular, the second-order optical potential is shown to be related to the scattering from two fixed scatterers by a one (vector) dimensional integral equation.

Explicit formulas to calculate the leading corrections to the fixed scatterer approximation are also derived. There are two corrections of this type. The first is the correction due to the use of the closure approximation. This correction is seen to vanish in the forward direction and to be proportional to the derivative of the two-body $T$ matrix with respect to its energy parameter. A second correction to the fixed scatterer approximation is also derived. This correction arises only when the two-body interaction is nonlocal and is proportional to the ratio of the incident particle's mass to the sum of incident-plus-target particle masses. It is also proportional to the square of the range of the nonlocality divided by the wave length of the incident particle. Thus, this correction does not vanish at high energies. A rough estimate of the "convergence" of the correlation expansion is provided.

In Appendix A an alternate derivation of the central result of this work is presented. In Appendix C the techniques used in the text are employed to derive a correlation expansion of the optical potential itself. This result is a rearrangement of the perturbation series for the optical potential derived by Watson.[9]

In the text, it is noted how the results of Feshbach and Lambert[5] can readily be derived from our results. The work of Foldy and Walecka,[7] however, consists of a considerably different rearrangement of the multiple-scattering series. In Appendix D their treatment is examined and the relation to our approach is presented.

## II. CORRELATION EXPANSION OF THE OPTICAL POTENTIAL

In this section we develop an expansion for the pseudo-optical potential of Kerman, McManus, and Thaler[8] (KMT). The expansion of the optical potential is arranged such that all terms in the optical potential which involve the interaction of the incident particle with a single target nucleon will be summed as the leading term. The next order term is found by summing all of the terms *in the optical potential* in which the incident particle interacts with two distinct target particles. This term is shown in Sec. IV to be a solution to a three-body problem.

If the interaction of a projectile and a nucleon is governed by the two-body potential $v$, then the scattering of that projectile from a nucleus of $A$ identical nucleons (fermions) is given by a transition operator $T$ which satisfies the operator equation

$$T=\sum_{i=1}^{A} v_i+\sum_{i=1}^{A} v_i G^{(+)}(E)T\,, \tag{2.1}$$

where $v_i$ is the potential between the projectile and the $i$th nucleon. The many-body propagator $G^{(+)}(E)$ of Eq. (2.1) is

$$G^{(+)}(E)=[E-h_0-H_A+i\eta]^{-1}\,, \tag{2.2}$$

where $E$ is the parametric energy, $h_0$ is the projectile kinetic energy operator, $H_A$ is the $A$-body target nucleus Hamiltonian, and $i\eta$ is a stylized reminder of the outgoing wave prescription. We shall henceforth always assume outgoing waves without explicitly so noting.

We follow the procedure of Watson[9] and define $T$ to be $T=\sum T_i$, with $T_i$ given by

$$T_i=v_i+v_iG\sum_j T_j\,. \tag{2.3}$$

If we now define $t_i$ through the relation

$$t_i=v_i+v_i g t_i\,, \tag{2.4}$$

where the propagator $g$ is given by[11]

$$g=[E-h_0+i\eta]^{-1}\,, \tag{2.5}$$

then we may rewrite Eq. (2.3) as

$$T_i=t_i+t_iG\sum_{j\neq i} T_j+t_i(G-g)T_i\,. \tag{2.6}$$

In the closure approximation, we neglect the last term of Eq. (2.6), to obtain the approximate relation.

$$T_i^{(\text{clos})}=t_i+t_i g\sum_{j\neq i} T_j^{(\text{clos})}\,. \tag{2.7}$$

In Appendix B we generalize the development presented here in such a way that one is not limited to choosing $g$ as the closure propagator.

Iteration of the set of $A$ coupled equations given in Eq. (2.7) and summation over the particle index $i$, then yields, after some rearrangement

$$T^{(\mathrm{clos})}=\sum_i t_i+\sum_i t_i g\sum_{j\neq i} t_j+\sum_i t_i g\sum_{j\neq i} t_j g\sum_{k\neq j} t_k+\cdots$$
$$=\sum_i t_i+\sum_i\sum_{j\neq i}[t_i g t_j+t_i g t_j g t_i+t_i g t_j g t_i g t_j+\cdots]$$
$$+\sum_i\sum_{i\neq j}\sum_{\substack{k\neq i\\ k\neq j}}[t_i g t_j g t_k+t_i g t_j g t_i g t_k+t_i g t_j g t_k g t_i+t_i g t_j g t_k g t_j+\cdots]+\cdots \tag{2.8}$$

We now wish to find the pseudo-optical potential of KMT. An alternate approach would be to work with the optical potential itself. This approach follows very closely that of Watson[9] and is presented in Appendix C. The pseudo-optical potential of KMT is defined as the potential which, when inserted into a Lippmann-Schwinger equation, yields the pseudo-$T$ matrix, $T'=[(A-1)/A]T$. It is thus defined through the operator relation,

$$\left(\frac{A-1}{A}\right)T\equiv T'=U'+U'gPT', \tag{2.9}$$

where $P$ is the projector onto the nuclear ground state. From Eq. (2.9) we immediately obtain

$$U'=\left(\frac{A-1}{A}\right)T-\left(\frac{A-1}{A}\right)^2 TgPT+\left(\frac{A-1}{A}\right)^3 TgPTgPT+\cdots. \tag{2.10}$$

Insertion of the expression for $T$ of Eq. (2.8) into Eq. (2.10), then yields

$$U'=\left(\frac{A-1}{A}\right)\sum_i t_i+\left(\frac{A-1}{A}\right)\sum_i\sum_{j\neq i}\{t_i g t_j+t_i g t_j g t_i+\cdots\}+\left(\frac{A-1}{A}\right)\sum_i\sum_{j\neq i}\sum_{\substack{k\neq i\\ k\neq j}}\{t_i g t_j g t_k+\cdots\}$$
$$-\left(\frac{A-1}{A}\right)^2\sum_i t_i gP\sum_j t_j-\left(\frac{A-1}{A}\right)^2\sum_i t_i gP\sum_j t_j g\sum_{k\neq j} t_k-\left(\frac{A-1}{A}\right)^2\sum_i t_i g\sum_{j\neq i} t_j gP\sum_k t_k$$
$$-\left(\frac{A-1}{A}\right)^2\sum_i t_i gP\sum_j\sum_{k\neq j} t_j g t_k g t_j-\left(\frac{A-1}{A}\right)^2\sum_i t_i g\sum_{j\neq i} t_j g t_i gP\sum_k t_k$$
$$-\left(\frac{A-1}{A}\right)^2\sum_i t_i g\sum_{j\neq i} t_j gP\sum_k t_k g\sum_{l\neq k} t_l+\cdots. \tag{2.11}$$

At this point we recognize that we are dealing with $A$ target nucleons (fermions), so that the allowed physical states are completely antisymmetric in the exchange of the target particles. In order to make this explicit we introduce the projection operator $\Pi$, which projects onto the $A$-body Fock space of antisymmetrized states. The operator we seek is thus $T_a\equiv\Pi T\Pi$, which we may express in terms of the operator $U'_a\equiv\Pi U'\Pi$ as

$$\left(\frac{A-1}{A}\right)T_a=U'_a+U'_a gP\left(\frac{A-1}{A}\right)T_a. \tag{2.12}$$

From Eq. (2.11) we then observe that

$$U'_a=\Pi\{(A-1)t_1+(A-1)^2[t_1g(1-P)t_2+t_1g(1-P)t_2g(1-P)t_1+t_1g(1-P)t_2g(1-P)t_1g(1-P)t_2+\cdots]$$
$$+(A-1)^2(A-2)[t_1g(1-P)t_2g(1-P)t_3+t_1g(1-P)t_2g(1-P)t_1g(1-P)t_3+t_1g(1-P)t_2g(1-P)t_3g(1-P)t_1$$
$$+t_1g(1-P)t_2g(1-P)t_3g(1-P)t_2+\cdots]$$
$$+(A-1)^2(A-2)(A-3)[t_1g(1-P)t_2g(1-P)t_3g(1-P)t_4+\cdots]+\cdots\}\Pi. \tag{2.13}$$

The expansion of the optical potential given in Eq. (2.13) has now been rearranged according to the number of target particles which are struck by the incident particle. To make this more explicit, we keep in mind that we may only take matrix elements of $U'$ between antisymmetrized states, in order to rewrite $U'$ as

$$U'=\left(\frac{A-1}{A}\right)\{W^{(1)}+W^{(2)}+W^{(3)}+\cdots+W^{(A)}\}, \tag{2.14}$$

where

$$W^{(1)}=At_1, \tag{2.15}$$

$$W^{(2)}=A(A-1)[t_1g(1-P)t_2+t_1g(1-P)t_2g(1-P)t_1+t_1g(1-P)t_2g(1-P)t_1g(1-P)t_2+\cdots], \tag{2.16}$$

$$\begin{aligned}W^{(3)}=A(A-1)(A-2)[&t_1g(1-P)t_2g(1-P)t_3+t_1g(1-P)t_2g(1-P)t_1g(1-P)t_3\\&+t_1g(1-P)t_2g(1-P)t_3g(1-P)t_1+t_1g(1-P)t_2g(1-P)t_3g(1-P)t_2+\cdots].\end{aligned} \tag{2.17}$$

This expansion for $U'$, given in Eqs. (2.14)–(2.17), represents the correlation expansion we seek. It is a correlation expansion in the sense that each of the terms $W^{(i)}$ collects all terms in which the incident particle interacts with $i$ distinct particles within the optical potential.[12] This expansion is analogous to treatments of nuclear structure in terms of hole-line expansions. In that case the expansion is in terms of the number of occupied states participating in a given process, not in terms of powers of a reaction matrix. Here we expand in the number of struck nucleons and not in the powers of $t$. In the next section we explore the utility and meaning of this expansion. In Appendix A this result is rederived according to the method of KMT.

### III. INTERPRETATION OF THE CORRELATION EXPANSION OF THE OPTICAL OPERATOR IN THE MULTIPLE-SCATTERING SERIES

In order to interpret the result given in Eqs. (2.14)–(2.17), we will examine the correlation expansion term by term. We begin with the standard impulse approximation, i.e.,

$$U'_{(1)}=\left(\frac{A-1}{A}\right)W^{(1)}=(A-1)t_1. \tag{3.1}$$

Insertion of $U'_{(1)}$ into Eq. (2.9) yields the result that

$$\begin{aligned}\Pi T^{(\text{clos})}_{(1)}\Pi=\Pi\{&At_1+A(A-1)t_1gPt_1\\&+A(A-1)^2t_1gPt_1gPt_1\\&+A(A-1)^3t_1gPt_1gPt_1gPt_1+\cdots\}\Pi.\end{aligned} \tag{3.2}$$

Now we observe that Eq. (2.8) for $T^{(\text{clos})}$ is

$$T^{(\text{clos})}=\sum_i t_i+\sum_i t_ig\sum_{j\neq i}t_j+\sum_i t_ig\sum_{j\neq i}t_jg\sum_{k\neq j}t_k+\cdots, \tag{3.3}$$

so that we immediately are able to note that insertion of the projector onto the ground state $P$ for each intermediate scattering leads to

$$\begin{aligned}\Pi T_P^{(\text{clos})}\Pi&=\Pi\sum_i t_i\Pi+\Pi\sum_i t_igP\sum_{j\neq i}t_j\Pi\\&\quad+\Pi\sum_i t_igP\sum_{j\neq i}t_jgP\sum_{k\neq j}t_k\Pi+\cdots\\&=\Pi At_1\Pi+\Pi A(A-1)t_1gPt_1\Pi\\&\quad+\pi A(A-1)^2t_1gPt_1gPt_1\Pi+\cdots\\&=\Pi T^{(\text{clos})}_{(1)}\Pi.\end{aligned} \tag{3.4}$$

Thus we see that we obtain the interpretation of $T^{(\text{clos})}_{(1)}$ as the scattering that results from taking account of the multiple scattering of the projectile from any given target particle to all orders, with scattering from any one target particle to the next included, provided that the nucleus is in its ground state when the projectile scatters from the one target particle to the next. The achievement of the KMT pseudo-optical potential to this order is that just the scattering from one target particle goes into $U'$. If we insert this $U'$ into the one-body Lippmann-Schwinger equation, Eq. (2.9), and solve for $T$, we include all the scattering from target particle to target particle that proceeds through the ground state.

Let us now proceed to truncate $U'$ at the next order of the expansion of Eqs. (2.14)–(2.17), i.e., we take

$$\begin{aligned}U'_{(2)}&=\left(\frac{A-1}{A}\right)\{W^{(1)}+W^{(2)}\}\\&=\left(\frac{A-1}{A}\right)\{At_1+A(A-1)[t_1g(1-P)t_2+t_1g(1-P)t_2g(1-P)t_1+t_1g(1-P)t_2g(1-P)t_1g(1-P)t_2+\cdots]\}.\end{aligned} \tag{3.5}$$

Insertion of Eq. (3.5) into Eq. (2.9) yields the result that

$$\begin{aligned}\Pi T_{(2)}^{(\mathrm{clos})}\Pi &= \Pi\{At_1 + A(A-1)[t_1gt_2 + t_1gt_2gt_1 + \cdots]\\ &\quad + A(A-1)(A-2)[t_1gt_2gPt_1 + t_1gPt_1gt_2 - t_1gPt_1gPt_1 + \cdots] + \cdots\}\Pi\\ &= \Pi\{At_1 + A(A-1)[t_1gt_2 + t_1gt_2gt_1 + \cdots]\\ &\quad + A(A-1)(A-2)[t_1gt_2gt_1 - t_1g(1-P)t_2g(1-P)t_1 + \cdots] + \cdots\}\Pi\,. \end{aligned} \tag{3.6}$$

Now Eq. (3.6) may be compared with Eq. (2.8) for $T^{(\mathrm{clos})}$, which may be written as

$$\Pi T^{(\mathrm{clos})}\Pi = \Pi\{At_1 + A(A-1)[t_1gt_2 + t_1gt_2gt_1 + \cdots] + A(A-1)(A-2)[t_1gt_2gt_3 + \cdots] + \cdots\}\Pi\,. \tag{3.7}$$

If we wished to keep all terms in Eq. (3.7) in which the projectile scattered from any one or two target particles but did not scatter from a third particle, we would truncate Eq. (3.7) at the end of the first square bracket. However, if, as in the previous discussion, we wish to include scattering from a third particle which may occur when the nucleus is in its ground state, then other terms enter. The lowest-order term involving scattering from three particles is $\Pi t_1gt_2gt_3\Pi$. If we wish to exclude all scatterings from three different particles which do not proceed through the ground state after scattering from one or two different particles we would substract $t_1g(1-P)t_2g(1-P)t_3$ from $t_1gt_2gt_3$. Thus the appropriate modification of Eq. (3.7) would be

$$\begin{aligned}\Pi T_{PP}^{(\mathrm{clos})}\Pi &= \Pi\{At_1 + A(A-1)[t_1gt_2 + t_1gt_2gt_1 + \cdots]\\ &\quad + A(A-1)(A-2)[t_1gt_2gt_3 - t_1g(1-P)t_2g(1-P)t_3 + \cdots] + \cdots\}\Pi\\ &= \Pi\{At_1 + A(A-1)[t_1gt_2 + t_1gt_2gt_1 + \cdots]\\ &\quad + A(A-1)(A-2)[t_1gt_2gt_1 - t_1g(1-P)t_2g(1-P)t_1 + \cdots] + \cdots\}\Pi\\ &= \Pi T_{(2)}^{(\mathrm{clos})}\Pi\,. \end{aligned} \tag{3.8}$$

Thus we see that the $T$ matrix resulting from the second-order KMT optical potential includes all scatterings involving two particles. It also includes some of the scatterings involving more than two particles; all of these terms are included so long as they do not contain successive scattering from three different particles $(a,b,c)$ in which the scattering does not proceed through the ground state between particles $a$ and $b$ or $b$ and $c$. This means that terms such as $t_1gt_2Pt_3$, $t_1gt_2gPt_3gt_4$, and $t_1gt_2gPt_3gt_4gPt_5$ are included. Terms such as $t_1gt_2gt_3$, $t_1gt_2gt_3Pt_4$, and $t_1gt_2gPt_3gt_4gt_5$ are not included. This result, which we have explicitly demonstrated only to lowest relevant order, holds to all orders.

We may now interpret the truncation of $U'$ as $U'_{(n)}$. The scattering of the incident particle from $n$ target particles is included in $U'_{(n)}$. The scattering $T_{(n)}$ calculated from $U'_{(n)}$ as a pseudo-optical potential contains all terms which involve $n$ scatterings. Terms which involve more than $n$ scatterings are also included. These terms do not contain, however, successive scatterings from more than $n$ different target particles in which the scattering does not proceed through the ground state at some point between the $n$ scatterings.

This structure of the scattering amplitude which results from the use of $U'_{(n)}$ is also evident in the unitarity relations satisfied by $T_{(n)}$. In Ref. 13 it was shown that $T_{(n)}$ satisfies a unitarity relation to within terms which involve scattering from $n+1$ different particles. Thus the impulse approximation produces a $T$ matrix which satisfies unitarity to within terms which involve scattering from two different target nucleons, while the use of $U'_{(2)}$ will produce a $T$ matrix which will satisfy unitarity to within terms which involve scattering from three different target nucleons.

## IV. SECOND-ORDER POTENTIAL

In this section we shall discuss in some detail the second-order pseudo-optical potential, which in the present expansion is given by

$$U'_{(2)} = (A-1)t_1 + (A-1)^2[t_1g(1-P)t_2 + t_1g(1-P)t_2g(1-P)t_1 + t_1g(1-P)t_2g(1-P)t_1g(1-P)t_2 + \cdots]\,. \tag{4.1}$$

To understand this expression better let us discuss the scattering of a projectile from two fixed scatterers.

The $T$ matrix for an incident particle scattering from two infinitely massive noninteracting target particles may be written as

$$t_{12} = (v_1 + v_2) + (v_1 + v_2)gt_{12}\,. \tag{4.2}$$

This equation is poorly defined and difficult to study as its kernel is disconnected. We thus replace Eq. (4.2) by two coupled equations in the usual manner, i.e., we write $t_{12}$ as

$$t_{12} = \tau_1 + \tau_2 . \tag{4.3}$$

where $\tau_1$ and $\tau_2$ satisfy the equations

$$\tau_1 = v_1 + v_1 g \tau_1 + v_1 g \tau_2 , \tag{4.4}$$

and

$$\tau_2 = v_2 + v_2 g \tau_2 + v_2 g \tau_1 . \tag{4.5}$$

In terms of the two-body $T$ matrix defined in Eq. (2.4) these become the usual Watson-Faddeev equations given by

$$\tau_1 = t_1 + t_1 g \tau_2 , \tag{4.6}$$

and

$$\tau_2 = t_2 + t_2 g \tau_1 . \tag{4.7}$$

The substitution of Eq. (4.7) into Eq. (4.6) gives an uncoupled equation for $\tau_1$ with a connected kernel:

$$\tau_1 = t_1 + t_1 g t_2 + t_1 g t_2 g \tau_1 . \tag{4.8}$$

The iteration of this equation clearly produces a series which is quite similar to that contained in $U'_{(2)}$, Eq. (4.1), the only difference being the replacement of the propagator $g$ in Eq. (4.8) by the propagator $g(1-P)$ in Eq. (4.11).

It should be noted that, here, since the target particles are considered to be infinitely massive, they can absorb momentum, but not energy. Hence no energy can be transferred to the nucleus. However, the assumption that the constituent particles are infinitely massive also implies that all the nuclear excited states are degenerate in energy with the ground state and thus it takes no energy to excite these nuclear states, so that the propagator $g(1-P)$ is well defined.

We thus define the operators $\tau'_1$ and $\tau'_2$ by

$$\tau'_1 = t_1 + t_1 g(1-P)\tau'_2 , \tag{4.9}$$

and

$$\tau'_2 = t_2 + t_2 g(1-P)\tau'_1 , \tag{4.10}$$

and also define $t'_{12}$ to be

$$t'_{12} \equiv \tau'_1 + \tau'_2 . \tag{4.11}$$

The operator $t'_{12}$ thus represents the scattering from two fixed scattering centers where the incident particle propagates between the two particles via the propagator $g(1-P)$. For identical target particles, $U'_{(2)}$ may be written

$$U'_{(2)} = (A-1)t_1 + (A-1)^2\{\tfrac{1}{2} t'_{12} - t_1\} , \tag{4.12}$$

which follows immediately from the insertion of the iteration of Eqs. (4.9) and (4.10) into Eq. (4.11) to recover Eq. (4.1).

We rewrite Eq. (4.12) as

$$U'_{(2)} = \left(\frac{A-1}{A}\right) \{\tfrac{1}{2} A(A-1)t'_{12} + [A - A(A-1)]t_1\} . \tag{4.13}$$

The interpretation of the quantity in the curly brackets is straightforward. The operator $t'_{12}$ represents the transition operator for the scattering of a projectile from a pair of target particles, with the propagator modification discussed above. The factor $\frac{1}{2}A(A-1)$ represents the number of independent pairs in the target. The second term represents the usual first-order term from which has been subtracted the extra number of $t_1$ (single particle) terms which have been included in the pair scattering part.

Solution of the scattering from two fixed scatters as given in Eqs. (4.6) and (4.7) is a soluble numerical problem, expecially if the two-body interaction is taken as separable. It is thus useful to rewrite $t'_{12}$ [which involves the solution of three-body equations with the propagator $g(1-P)$] in terms of the scattering amplitude $t_{12}$.

We may formally solve Eqs. (4.6) and (4.7) for $t_1$ and $t_2$ to give

$$t_1 = (1 + \tau_2 g)^{-1}\tau_1 \tag{4.14}$$

and

$$t_2 = (1 + \tau_1 g)^{-1}\tau_2 . \tag{4.15}$$

Substitution of these equations in Eqs. (4.9) and (4.10) gives

$$\tau'_1 = \tau_1 + \tau_1 g \tau'_2 - \tau_2 g \tau'_1 - \tau_1 g P \tau'_2 \tag{4.16}$$

and

$$\tau'_2 = \tau_2 + \tau_2 g \tau'_1 - \tau_1 g \tau'_2 - \tau_2 g P \tau'_1 . \tag{4.17}$$

If we now add Eqs. (4.15) and (4.16), we obtain

$$(\tau'_1 + \tau'_2) = (\tau_1 + \tau_2) - \tau_1 g P \tau'_2 - \tau_2 g P \tau'_1 . \tag{4.18}$$

Thus we have that

$$P t'_{12} P = P t_{12} P t_{12} P - \tfrac{1}{2} P t_{12} P g P t'_{12} P . \tag{4.19}$$

The matrix elements of this operator equation yield a simple one-body integral equation. Thus the solution of $t_{12}$ can in a simple way give the required matrix element of $t'_{12}$.

Instead of the three-body calculation necessary to calculate $t'_{12}$, one might undertake a Born series expansion of Eqs. (4.9) and (4.10). If this is done and one keeps only the lowest nonvanishing terms, one has

$$U'_{(2)} \approx (A-1)t_1 + (A-1)^2 t_1 g(1-P)t_2 . \tag{4.20}$$

This is exactly the formal result of Ref. 5.

If we keep all the terms in the expansion of $U'$ which involve three two-body $t$ matrices, we have from Eq. (2.13)

$$U' \approx (A-1)t_1 + (A-1)^2 t_1 g(1-P)t_2 + (A-1)^2 t_1 g(1-P)t_2 g(1-P)t_1 + (A-1)^2(A-2)t_1 g(1-P)t_2 g(1-P)t_3 . \quad (4.21)$$

This approximation has been calculated in Ref. 6. In our rearrangement of the multiple-scattering series, the term in Eq. (4.21) which involves scattering from particle 1 to 2 then back to 1 would be included in our second-order optical potential, while the term which scatters 1 to 2 to 3 would be the leading term in our third-order potential.

## V. EXPLICIT EVALUATION OF MATRIX ELEMENTS

Although we will show in Appendix B that the correlation expansion developed here does not require the use of the closure approximation, this approximation will greatly simplify the numerical evaluation of our formulas. At this point, we will present a specific way to evaluate the matrix elements involved in calculating the pseudo-optical potential in the fixed scatterer approximation. For a local potential the fixed scatterer and closure approximations are equivalent. For a nonlocal potential, they are not. Corrections to the fixed scatterer limit are presented in the next section. We emphasize that this is not a unique way of developing the numerics; one need not take the fixed scatterer limit, nor necessarily use the closure approximation. The treatment presented here, as we shall see, leads to a particularly convenient numerical approach. We begin by noting that the two-body interaction $v$ may be written in momentum space as

$$(\vec{k}'\vec{p}'\,|v\,|\,\vec{k}\vec{p}) = \delta(\vec{k}'+\vec{p}'-\vec{k}-\vec{p}) \times \left(\frac{M\vec{p}'-m\vec{k}'}{M+m}\,|v\,|\,\frac{M\vec{p}-m\vec{k}}{M+m}\right), \quad (5.1)$$

where we are using nonrelativistic kinematics. Momenta which are labeled $\vec{p}$ refer to the incident particle, those labeled $\vec{k}$ refer to target particle momenta, $M$ is the mass of a single target particle, and $m$ is the mass of the projectile.

It is important to note[14] that the use of the Galilean and translationally invariant form of the two-body potential, Eq. (5.1), together with the closure propagator in the two-body Lippmann-Schwinger equations, Eq. (2.4), does *not* in general yield the fixed scatterer approximation. For a local potential, we have

$$(\vec{k}'\vec{p}'\,|v_L\,|\vec{k}\vec{p}) = \delta(\vec{k}'+\vec{p}'-\vec{k}-\vec{p})v(\vec{p}'-\vec{p}) . \quad (5.2)$$

Substitution of this into Eq. (2.4) with the closure propagator then yields[7] the fixed scatterer approximation.

For a nonlocal potential, the fixed scatterer approximation requires an additional approximation: we must also take the limit of the nucleon mass $M$ going to infinity in Eq. (5.1). In the next section, we present the leading order correction terms to the optical potential which arise due to our use of the closure propagator and the fixed scatterer approximation.

If the mass of the target nucleon in Eq. (5.1) is taken to be infinite, we may write

$$(\vec{k}'\vec{p}'\,|v\,|\vec{k}\vec{p}) = \delta(\vec{k}'+\vec{p}'-\vec{k}-\vec{p})(\vec{p}'\,|v\,|\vec{p}) . \quad (5.3)$$

We then have, from Eq. (2.4), that the two-body $T$ matrix has a similar form:

$$(\vec{k}'\vec{p}'\,|t(E)\,|\vec{k}\vec{p}) = \delta(\vec{k}'+\vec{p}'-\vec{k}-\vec{p})(\vec{p}'\,|t(E)\,|\vec{p}) \quad (5.4)$$

and $(\vec{p}'\,|t(E)\,|\vec{p})$ satisfies

$$(\vec{p}'\,|t(E)\,|\vec{p}) = (\vec{p}'\,|v\,|\vec{p}) + \int d\vec{p}''(\vec{p}'|v\,|\vec{p}'')\frac{1}{E-E_{p''}+i\eta} \times (\vec{p}''|t(E)\,|\vec{p}) , \quad (5.5)$$

where, since we have used the closure propagator, $E_{p''}$ is given by

$$E_{p''} = \frac{\vec{p}''^2}{2m} . \quad (5.6)$$

The Fourier transform of Eq. (5.4) yields

$$(\vec{r}_0'\vec{r}_i'\,|t(E)\,|\vec{r}_0\vec{r}_i) = \delta(\vec{r}_i'-\vec{r}_i)(\vec{r}_0'-\vec{r}_i\,|t(E)\,|\vec{r}_0-\vec{r}_i) , \quad (5.7)$$

where $\vec{r}_0$ refers to the incident particle and $\vec{r}_i$ refers to the target particle. The matrix element $(\vec{r}_0'\,|t(E)\,|\vec{r}_0)$ is the Fourier transform of $(\vec{p}'\,|t(E)|\vec{p})$.

The lowest-order pseudo-optical potential, as given in Eq. (3.1), then becomes

$$(\vec{p}'\,|U_{(1)}|\vec{p}) = (A-1)(\vec{p}'\,|t(E)\,|\vec{p})\rho(\vec{p}'-\vec{p}) , \quad (5.8)$$

where $\rho(\vec{p}'-\vec{p})$ is the Fourier transform of the diagonal part of the single particle density $\rho(\vec{r})$ given by

$$\rho(\vec{r}) = \int d\vec{r}_2 \cdots d\vec{r}_A \phi^2(\vec{r},\vec{r}_2,\ldots\vec{r}_A) . \quad (5.9)$$

The factorization in Eq. (5.8) is a result of our use of the closure propagator[7] *and* (for a nonlocal potential) the infinite mass limit used in going from Eq. (5.1) to Eq. (5.3).

In order to calculate the second-order term in the optical potential, we have suggested that one first calculate the scattering from two fixed scattering centers, Eqs. (4.6) and (4.7). These equations written explicitly in coordinate space are

$$(\vec{r}_0', \vec{r}_1\vec{r}_2 | \tau_1 | \vec{r}_0, \vec{r}_1\vec{r}_2) = (\vec{r}_0' - \vec{r}_1 | t_1 | \vec{r}_0 - \vec{r}_1) + \int d\vec{r} d\vec{r}' (\vec{r}_0' - \vec{r}_1 | t_1 | \vec{r} - \vec{r}_1) \int \frac{d\vec{k}}{(2\pi)^3} \frac{e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}}{E - E_k + i\eta} (\vec{r}', \vec{r}_1\vec{r}_2 | \tau_2 | \vec{r}_0, \vec{r}_1\vec{r}_2) \tag{5.10}$$

and a similar equation for $\tau_2$ in terms of $\tau_1$. If we introduce the center-of-mass and relative coordinates for $\vec{r}_1$ and $\vec{r}_2$ given by

$$\vec{R} = \tfrac{1}{2}(\vec{r}_1 + \vec{r}_2) \tag{5.11}$$

and

$$\vec{r}_{rel} = \vec{r}_1 - \vec{r}_2 \tag{5.12}$$

then Eq. (5.7) may be solved by introducing the related quantities $\bar{\tau}_1$ and $\bar{\tau}_2$. To this end we must first solve

$$(\vec{r}_0', \vec{r}_{rel} | \bar{\tau}_1 | \vec{r}_0, \vec{r}_{rel}) = (\vec{r}_0' - \tfrac{1}{2}\vec{r}_{rel} | t_1 | \vec{r}_0 - \tfrac{1}{2}\vec{r}_{rel}) + \int d\vec{r} d\vec{r}' (\vec{r}_0' - \tfrac{1}{2}\vec{r}_{rel} | t_1 | \vec{r} - \tfrac{1}{2}\vec{r}_{rel}) \int \frac{d\vec{k}}{(2\pi)^3} \frac{e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}}{E - E_k + i\eta} (\vec{r}', \vec{r}_{rel} | \bar{\tau}_2 | \vec{r}_0, \vec{r}_{rel}), \tag{5.13}$$

and the accompanying analogous equation for $\bar{\tau}_2$. These are the standard Faddeev equations for the scattering of a particle of mass $m$ from two infinitely massive particles located a distance $\vec{r}_{rel}$ apart. The usual techniques for solving these equations may be used. In particular, for separable $t_1$ and $t_2$, the angular momentum decomposed form of Eq. (5.13) leads to a one dimensional integral equation.

The general operator $\tau_1$ may be found from the solution to Eq. (5.13) via the relation

$$(\vec{r}_0', \vec{r}_1\vec{r}_2 | \tau_1 | \vec{r}_0, \vec{r}_1\vec{r}_2) = (\vec{r}_0' - \vec{R}(\vec{r}_1, \vec{r}_2), \vec{r}_{rel}(\vec{r}_1, \vec{r}_2) | \bar{\tau}_1 | \vec{r}_0 - \vec{R}(\vec{r}_1, \vec{r}_2), \vec{r}_{rel}(\vec{r}_1, \vec{r}_2)). \tag{5.14}$$

The ground state matrix element of $Pt_{12}'P$ [which we will denote as $(\vec{r}_0' | \{t_{12}'\} | \vec{r}_0)$] is then given by the one (vector) dimensional equation, Eq. (4.18), which is

$$(\vec{r}_0' | \{t_{12}'\} | \vec{r}_0) = (\vec{r}_0' | \{t_{12}\} | \vec{r}_0) - \frac{1}{2} \int d\vec{r} d\vec{r}' (\vec{r}_0' | \{t_{12}\} | \vec{r}) \int \frac{d\vec{k}}{(2\pi)^3} \frac{e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}}{E - E_k + i\eta} (\vec{r}' | \{t_{12}'\} | \vec{r}_0), \tag{5.15}$$

where $(\vec{r}_0' | \{t_{12}\} | \vec{r}_0)$ is the ground state matrix element of $Pt_{12}P$ given by

$$(\vec{r}_0' | \{t_{12}\} | \vec{r}_0) = \int d\vec{r}_1 d\vec{r}_2 (\vec{r}_0', \vec{r}_1\vec{r}_2 | t_{12} | \vec{r}_0, \vec{r}_1\vec{r}_2) \rho_2(\vec{r}_1, \vec{r}_2). \tag{5.16}$$

The diagonal two-body density $\rho_2(\vec{r}_1, \vec{r}_2)$ is given by

$$\rho_2(\vec{r}_1, \vec{r}_2) = \int d\vec{r}_3 \cdots d\vec{r}_A \, \phi^2(\vec{r}_1, \vec{r}_2, \vec{r}_3 \cdots \vec{r}_A). \tag{5.17}$$

This may be combined with Eq. (5.8) and substituted into Eq. (4.13) to yield the optical potential through second order.

The approximation to the optical potential which arises from keeping only the second-order term in the Born expansion of $t_{12}'$ produces a particularly simple result. This approximation, given in Eq. (4.20), may be written explicitly as

$$(\vec{r}_0' | U_{(2)}' | \vec{r}_0) \simeq (A-1) \int d\vec{r}_1 (\vec{r}_0' - \vec{r}_1 | t_i | \vec{r}_0 - \vec{r}_1) \rho(\vec{r}_1) + (A-1)^2 \int d\vec{r}_1 d\vec{r}_2 (\vec{r}_0' - \vec{r}_1 | f | \vec{r}_0 - \vec{r}_2) C_2(\vec{r}_1, \vec{r}_2), \tag{5.18}$$

where $C_2(\vec{r}_1, \vec{r}_2)$ is the two-body correlation function as defined and studied in Ref. 2 and is given by

$$C_2(\vec{r}_1, \vec{r}_2) = \rho_2(\vec{r}_1, \vec{r}_2) - \rho(\vec{r}_1)\rho(\vec{r}_2). \tag{5.19}$$

The function $(\vec{r}_0' - \vec{r}_1 | f | \vec{r}_0 - \vec{r}_2)$ in Eq. (5.18) is given by

$$(\vec{r}_0' - \vec{r}_1 | f | \vec{r}_0 - \vec{r}_2) = \int d\vec{r} d\vec{r}' (\vec{r}_0' - \vec{r}_1 | t_1 | \vec{r} - \vec{r}_1) \int \frac{d\vec{k}}{(2\pi)^3} \frac{e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}}{(E - E_k + i\eta)} (\vec{r}' - \vec{r}_2 | t_2 | \vec{r}_0 - \vec{r}_2). \tag{5.20}$$

**Again we emphasize that only the diagonal part of the density, the two-body density, and the correlation function enter because we have used the closure approximation and, for a nonlocal potential,** the infinite target mass approximation. The approximation given in Eq. (5.18) is identical to the approximation to the optical potential derived in Ref. 2. In that work additional approximations to the integrations in Eq. (5.20) are made, and the resulting optical potential is calculated and studied in detail.

From Eqs. (5.18) and (5.20) it is straightforward to estimate the convergence of the correlation expansion developed here. This estimate is presented in Sec. VII.

## VI. CORRECTIONS TO THE FIXED SCATTERER APPROXIMATION

We have, in the previous sections, presented the formal theory of higher-order corrections to the impulse approximation where at each state we have used the closure approximation. There is no reason, *a priori*, to assume that the higher-order corrections in the multiple-scattering series (for all targets, energies, and projectiles) will be more significant than the corrections which arise from using the closure propagator. In this section, we, therefore, present a perturbative technique for improving the closure approximation. In addition, we have noted that, for a nonlocal potential, the fixed scatterer approximation requires an additional approximation. Such corrections are also discussed in this section.

We remind the reader once more that we have developed our expansion in terms of a particular propagator. It is also possible to use other propagators, as outlined in Appendix B, for example. No matter what the choice, however, one must cancel a many-body propagator with a two-body propagator. Thus, corrections of the type which, here, we shall present for the particular choice of the closure propagator, will always be required.

These corrections can readily be derived, if we start with the formula for the optical potential from Ref. 15, which is also derived in Appendix A, viz.

$$U'=(A-1)t_1+(A-1)t_1(\pi-P)GU'+t_1(\pi G-g)U' , \tag{6.1}$$

where $G$, $t$, and $g$ are defined in Eqs. (2.2), (2.4), and (2.5), respectively. This equation is an exact integral equation for the pseudo-optical potential of KMT in terms of the two-body scattering amplitude $t$. The closure approximation arises from the replacement of the many-body operator $G$ by the "closure" propagator $g$. We may derive the correction to this approximation by using

$$G=g+gH_AG . \tag{6.2}$$

We then have for Eq. (6.1)

$$\begin{aligned} U'=&(A-1)t_1+(A-1)t_1(\Pi-P)gU'\\ &+t_1(\Pi-1)gU'+(A-1)t_1(\Pi-P)gH_AGU'\\ &+t_1\Pi gH_AGU' . \end{aligned}\tag{6.3}$$

The first three terms represent the closure approximation to $U'$, and the last two terms provide the corrections to closure $\Delta U'$:

$$\Delta U'=(A-1)t_1(\Pi-P)gH_AGU'+t_1\Pi gH_AGU' . \tag{6.4}$$

If we now approximate $U'$ by

$$U'\approx(A-1)t_1 \tag{6.5}$$

and

$$G\approx g , \tag{6.6}$$

Eq. (6.4) becomes

$$\Delta U'\approx(A-1)At_1g\Pi H_Agt_1 , \tag{6.7}$$

where we have used $PH_A=0$.

We now take the ground state matrix elements of this expression to obtain the following expression for the change in $U'$:

$$(\vec{p}'|\Delta U'|\vec{p})=\left(\frac{A-1}{A}\right)\int d\vec{p}''(\vec{p}'|t|\vec{p}'')g(\vec{p}'')g(\vec{p}'')(\vec{p}''|t|\vec{p})\,(\phi|\sum_i e^{-i(\vec{p}'-\vec{p}'')\cdot\vec{r}_i}H_A\sum_j e^{-i(\vec{p}''-\vec{p})\cdot\vec{r}_j}|\phi) , \tag{6.8}$$

where $\phi$ represents the target ground state wave function and the implied matrix element is to be taken by integrating over $\vec{r}_1\cdots\vec{r}_A$, and where $g(\vec{p}'')$ is the closure propagator in momentum space, viz.,

$$g(\vec{p}'')=\left(E-\frac{\vec{p}''^2}{2m}+i\eta\right)^{-1} . \tag{6.9}$$

This matrix element may be written as

$$(\phi|\sum_i e^{-i(\vec{p}'-\vec{p}'')\cdot\vec{r}_i}H_A\sum_j e^{-i(\vec{p}''-\vec{p})\cdot\vec{r}_j}|\phi)=\tfrac{1}{2}(\phi|\sum_i e^{-i(\vec{p}'-\vec{p}'')\cdot\vec{r}_i},\left[H_A,\sum_j e^{-i(\vec{p}''-\vec{p})\cdot\vec{r}_j}\right]|\phi) . \tag{6.10}$$

For a local potential, this double commutator involves only the kinetic energy part of $H_A$ and is given by

$$\left[\sum_i e^{-i(\vec{p}'-\vec{p}'')\cdot\vec{r}_i},\left[H_A,\sum_j e^{-i(\vec{p}''-\vec{p})\cdot\vec{r}_j}\right]\right]=\delta_{ij}\sum_j\frac{1}{M}(\vec{p}''-\vec{p}')\cdot(\vec{p}''-\vec{p})e^{-i(\vec{p}'-\vec{p})\cdot\vec{r}_j} . \tag{6.11}$$

The change in $U'$ is then given by[16]

$$(\vec{p}'|\Delta U'|\vec{p}) = \frac{(A-1)}{2M}\int d\vec{p}''(\vec{p}''-\vec{p})\cdot(\vec{p}''-\vec{p}')(\vec{p}'|t|\vec{p}'')g(\vec{p}'')g(\vec{p}'')(\vec{p}''|t|\vec{p})\rho(\vec{p}'-\vec{p}) . \tag{6.12}$$

This is a general formula for $\Delta U'$ which, given the two-body $t$ matrix in the closure approximation and the target density, may be used to calculate the correction to the optical potential.

We may now roughly estimate the size of this correction. We note that in the integrand in Eq. (6.12) there are two factors which are both forward peaked. The product $(\vec{p}'|t|\vec{p}'')(\vec{p}''|t|\vec{p})$ will peak near the value of $\vec{p}''$ given by

$$\vec{p}'' \approx \tfrac{1}{2}(\vec{p}+\vec{p}') . \tag{6.13}$$

We thus replace $\vec{p}''$ in the expression $(\vec{p}''-\vec{p})\cdot(\vec{p}''-\vec{p}')$ by its approximate value $\frac{1}{4}(\vec{p}'-\vec{p})^2$ determined by Eq. (6.13). The change in $U'$ then becomes

$$(\vec{p}'|\Delta U'|\vec{p}) \approx \frac{(A-1)}{8M}(\vec{p}'-\vec{p})^2\rho(\vec{p}'-\vec{p}) \times \int d\vec{p}''(\vec{p}''|t|\vec{p}'')g(\vec{p}'')g(\vec{p}'') \times (\vec{p}''|t|\vec{p}) . \tag{6.14}$$

We now use the identity for the off-shell $t$ matrix[17]

$$t(E)g_B g_B t(E) = -\frac{\partial}{\partial E}t(E) \tag{6.15}$$

to rewrite Eq. (6.14) as

$$(\vec{p}'|\Delta U'|\vec{p}) \approx \frac{(A-1)}{8M}(\vec{p}'-\vec{p})^2\rho(\vec{p}'-\vec{p}) \times \left\{\frac{\partial}{\partial E}(\vec{p}'|t(E)|\vec{p})\right\} . \tag{6.16}$$

From this formula we see that this correction vanishes in the forward direction where $\vec{p}'=\vec{p}$. Away from the forward direction, the change in $U'$ is of order

$$\frac{\Delta U}{U} \sim \frac{(\vec{p}'-\vec{p})^2}{8M\Delta E} , \tag{6.17}$$

where $\Delta E$ is an energy parameter characteristic of the rate of change of $t(E)$. For nucleons incident on a nucleus, this correction is very small for two reasons. First, one is not generally interested in scattering through momentum transfer as great as $8M$. Secondly, the nucleon-nucleon interaction is a very smooth function of energy and thus $\Delta E$ will be a large number. For pions incident on a nucleus, for example, Eq. (6.17) is maximized at the resonance where $\Delta E \sim \frac{1}{2}\Gamma \sim 60$ MeV. On the resonance, for a large momentum transfer of ~200 MeV/$c$, this correction is then about 10%.[18]

Before proceeding it is necessary to note that the use of the first order approximation to $U'$, Eq. (6.5), may result in an underestimate of the effect. If one uses the second-order optical potential one allows for the possibility of having a highly excited intermediate state so that the value of $H_A$ could be large. This would enhance the correction. However, such a term involves at least a double-scattering event followed by an action of $H_A$ followed by another scattering from one of the struck two nucleons. Because $H_A$ rearranges one of the struck nucleons, such a term may be thought of as a part of the third-order optical potential, and such terms are conventionally believed to be small.

As we noted at the beginning of Sec. V, for a nonlocal potential the fixed scatterer approximation requires, in addition to the closure approximation, the limit of an infinite target nucleon mass in Eq. (5.1). We will now examine the leading correction to this approximation.

We begin by "expanding" the $\delta$ function in Eq. (5.1):

$$(\vec{r}_0'\vec{r}_i'|v|\vec{r}_0\vec{r}_i) = \delta\left(\frac{m\vec{r}_0'+M\vec{r}_i'-m\vec{r}_0-M\vec{r}_i}{m+M}\right)(\vec{r}_0'-\vec{r}_i'|v|\vec{r}_0-\vec{r})$$
$$\approx \delta(\vec{r}_i'-\vec{r}_i)(\vec{r}_0'-\vec{r}_i'|v|\vec{r}_0-\vec{r}_i) + \left[\frac{m}{m+M}(\vec{r}_0'-\vec{r}_i'-\vec{r}_0+\vec{r}_i)\cdot\vec{\nabla}_{\vec{r}_i}\delta(\vec{r}_i'-\vec{r}_i)\right](\vec{r}_0'-\vec{r}_i'|v|\vec{r}_0-\vec{r}_i) . \tag{6.18}$$

If we define the last term in this equation as $\Delta v$, we may use the off-shell two potential formula to estimate the change in the two-body $T$ matrix due to $\Delta v$. We have

$$t(E^+) = t_0(E^+) + \Omega_0{}^\dagger(E^-)\Delta v\Omega(E^+) , \tag{6.19}$$

where $t_0(E^+)$ is given explicitly by Eqs. (5.5) and (4.8), and $\Omega(E^+)$ and $\Omega_0(E^+)$ are the usual wave operators given by

$$\Omega(E^+) = 1 + \frac{1}{E-h_0+i\eta}t(E^+) , \tag{6.20}$$

$$\Omega_0(E^+) = 1 + \frac{1}{E-h_0+i\eta}t_0(E^+) .$$

The change in $t(E^+)$ is thus given by

$$\Delta t(E^+) = \Omega_0{}^\dagger(E^-)\Delta v \Omega(E^+) \approx \Omega_0{}^\dagger(E^-)\Delta v \Omega_0(E^+) . \tag{6.21}$$

Because we are using the fixed scatterer approximation, $\Omega_0(E^+)$ is diagonal in the target nucleon coordinates, i.e.,

$$(\vec{r}_0'\vec{r}_i' | \Omega_0(E^+) | \vec{r}_0\vec{r}_i) = \delta(\vec{r}_i' - \vec{r}_i)(\vec{r}_0' | \Omega_0(\vec{r}_i, E^+) | \vec{r}_0) . \tag{6.22}$$

The lowest-order correction to the pseudo-optical potential is then given by

$$\begin{aligned}(\vec{r}_0' | \Delta U' | \vec{r}_0) = \frac{m}{m+M}(A-1) \int d\vec{r}_i' d\vec{r}_i (\vec{r}_i' | \rho | \vec{r}_i) \\ \times \int d\vec{r}_0'' d\vec{r}_0''' (\vec{r}_0' | \Omega_0{}^\dagger(\vec{r}_i', E^-) | \vec{r}_0''')(\vec{r}_0''' - \vec{r}_i' | v | \vec{r}_0'' - \vec{r}_i) \\ \times [(\vec{r}_0''' - \vec{r}_i' - \vec{r}_0'' + \vec{r}_i) \cdot \vec{\nabla}_{\vec{r}_i'} \delta(\vec{r}_i' - \vec{r}_i)](\vec{r}_0'' | \Omega_0(\vec{r}_i, E^+) | \vec{r}_0) .\end{aligned} \tag{6.23}$$

We now define the correction to $U'$ without the distortion factors, $\Omega(E^+)$ and $\Omega^\dagger(E^-)$, as $\Delta U_B'$

$$(\vec{r}_0' | \Delta U_B' | \vec{r}_0) \equiv \frac{m}{m+M}(A-1) \int d\vec{r}_i' d\vec{r}_i (\vec{r}_i' | \rho | \vec{r}_i)(\vec{r}_0' - \vec{r}_i' | v | \vec{r}_0 - \vec{r}_i)[(\vec{r}_0' - \vec{r}_i' - \vec{r}_0 + \vec{r}_i) \cdot \vec{\nabla}_{\vec{r}_i'} \delta(\vec{r}_i' - \vec{r}_i)] . \tag{6.24}$$

At this point it is convenient to work in the variables

$$\vec{\mathcal{R}} = \vec{r}_i' - \vec{r}_i , \quad \vec{\mathcal{R}}_0 = \vec{r}_0' - \vec{r}_0 , \quad \vec{R} = \tfrac{1}{2}(\vec{r}_i' + \vec{r}_i) , \quad \vec{R}_0 = \tfrac{1}{2}(\vec{r}_0' + \vec{r}_0) . \tag{6.25}$$

In terms of these variables, $\Delta U_B$ becomes

$$\Delta U_B'(\vec{R}_0, \vec{\mathcal{R}}_0) = \frac{m}{m+M}(A-1) \int d\vec{R}\, d\vec{\mathcal{R}}\, v(\vec{R}_0 - \vec{R}, \vec{\mathcal{R}}_0 - \vec{\mathcal{R}}) \rho(\vec{R}, \vec{\mathcal{R}})(\vec{\mathcal{R}}_0 - \vec{\mathcal{R}}) \cdot \vec{\nabla}_{\vec{\mathcal{R}}} \delta(\vec{\mathcal{R}}) , \tag{6.26}$$

where

$$\begin{aligned}(\vec{r}' | v | \vec{r}) &= v[\tfrac{1}{2}(\vec{r}' + \vec{r}), \vec{r}' - \vec{r}] , \\ (\vec{r}' | \rho | \vec{r}) &= \rho[\tfrac{1}{2}(\vec{r}' + \vec{r}), \vec{r}' - \vec{r}] .\end{aligned} \tag{6.27}$$

If we now introduce the Fourier transform of $v$ and $\rho$, we have

$$\begin{aligned}\Delta U_B'(\vec{R}_0, \vec{\mathcal{R}}_0) = \frac{m}{m+M}(A-1)(2\pi)^{-6} \int d\vec{R}\, d\vec{\mathcal{R}}\, d\vec{K}\, d\vec{k}\, d\vec{P}\, d\vec{p}\, v(\vec{k}, \vec{K}) e^{+i\vec{k}\cdot(\vec{R}_0 - \vec{R})} \rho(\vec{p}, \vec{P}) e^{+i\vec{P}\cdot\vec{\mathcal{R}}} e^{+i\vec{p}\cdot\vec{R}} \\ \times \{-i\vec{\nabla}_{\vec{K}} e^{+i\vec{K}\cdot(\vec{\mathcal{R}}_0 - \vec{\mathcal{R}})}\} \cdot \{\nabla_{\vec{\mathcal{R}}} \delta(\vec{\mathcal{R}})\} ,\end{aligned} \tag{6.28}$$

where

$$(\vec{k}' | v | \vec{k}) = v[\vec{k}' - \vec{k}, \tfrac{1}{2}(\vec{k}' + \vec{k})] = (2\pi)^{-3} \int d\vec{R}\, d\vec{\mathcal{R}}\, e^{-i(\vec{k}' - \vec{k})\cdot\vec{R}} e^{(-i/2)(\vec{k}' + \vec{k})\cdot\vec{\mathcal{R}}} v(\vec{R}, \vec{\mathcal{R}}) , \tag{6.29}$$

with a similar expression for $\rho(P, p)$. Straightforward manipulations then give $\Delta U_B'$ in momentum space as

$$(\vec{p}' | \Delta U_B' | \vec{p}) = \frac{m}{m+M}(A-1) \int d\vec{P}\, \rho(\vec{p}' - \vec{p}, \vec{P})\{\vec{P} - \tfrac{1}{2}(\vec{P}' + \vec{P})\} \cdot \vec{\nabla}_{(\vec{p}' + \vec{p})/2} v[\vec{p}' - \vec{p}; \tfrac{1}{2}(\vec{P}' + \vec{P})] . \tag{6.30}$$

If the density is constructed from orbitals such that the time reversed orbitals are filled pairwise, then we have[19]

$$\int d\vec{P}\, \vec{P} \rho(\vec{p}' - \vec{p}; \vec{P}) = 0 . \tag{6.31}$$

In this case, we have

$$(\vec{p}' | \Delta U_B' | \vec{p}) = -\frac{m}{m+M}(A-1)\{\tfrac{1}{2}(\vec{p}' + \vec{p}) \cdot \vec{\nabla}_{(\vec{p}' + \vec{p})/2} v[\vec{p} - \vec{p}, \tfrac{1}{2}(\vec{p}' + \vec{p})]\} \rho(\vec{p}' - \vec{p}) , \tag{6.32}$$

where $\rho(\vec{p}' - \vec{p})$ is the Fourier transform of the diagonal coordinate space density Eq. (5.10). If we insert Eq. (6.32) into Eq. (6.23), we find that the correction to $U'$ is given by

$$(\vec{p}'|\Delta U'|\vec{p}) = -\frac{m}{m+M}(A-1)\int d\vec{p}''d\vec{p}'''\psi_{\vec{p}'}^{(-)*}(\vec{p}''')\{\tfrac{1}{2}(\vec{p}'''+\vec{p}'')\cdot\vec{\nabla}_{(\vec{p}'''+\vec{p}'')/2}v[\vec{p}'''-\vec{p}'';\tfrac{1}{2}(\vec{p}'''+\vec{p}'')]\}$$
$$\times\rho(\vec{p}'''-\vec{p}'')\psi_{\vec{p}}^{(+)}(\vec{p}''), \qquad (6.33)$$

where $\psi_{\vec{p}}^{(+)}(\vec{p}')$ are scattering states in momentum space with outgoing (+) or incoming (−) boundary conditions, which describe scattering of the incident projectile from a single fixed target particle.

First, one should notice that for a local potential, the two-body potential in momentum space is a function of momentum transfer $\vec{p}'-\vec{p}$, alone. In this case, $\Delta U'$ clearly vanishes, as we knew in advance it must. Secondly, the correction is proportional to $m/(m+M)$; for nucleon-nucleus scattering this is $\frac{1}{2}$, while for pion-nucleus scattering this is about $\frac{1}{8}$.

The momentum dependence of the correction is complicated by the distorting wave functions $\psi_{\vec{p}}^{(\pm)}$. A qualitative estimate of the behavior of this term can probably be obtained by approximating $\Delta U$ by $\Delta U_B$, Eq. (6.32). In this approximation we have that this correction is proportional to $\vec{p}'+\vec{p}$, and thus relative to the leading term in the optical potential; its importance grows as the energy of the incident projectile increases. This correction is thus unique in the sense that its importance does not manifestly vanish at high energies.

In order to elucidate this further, let us examine a particular model for the two-body interaction. If we take

$$(\vec{p}'|v|\vec{p}) = V_L(\vec{p}'-\vec{p}) + (\vec{p}'|V_{NL}|\vec{p})$$
$$= V_L(\vec{p}'-\vec{p}) + V_0(\vec{p}'-\vec{p})a^3e^{-a^2|\vec{p}'+\vec{p}/2|^2} \qquad (6.34)$$

with $a$ the range of the nonlocality, then Eq. (6.28) gives

$$(\vec{p}'|\Delta U_B'|\vec{p}) = \left(\frac{m}{m+M}\right)(A-1)a^2\tfrac{1}{2}(\vec{p}'+\vec{p})^2$$
$$\times(\vec{p}'|v_{NL}|\vec{p})\rho(\vec{p}'-\vec{p}). \qquad (6.35)$$

The expansion parameter that relates the size of this term to the leading term is thus

$$\lambda = \left(\frac{m}{m+M}\right)p^2a^2\left(\frac{V_{NL}}{V}\right), \qquad (6.36)$$

where $V_{NL}/V$ is the ratio of the strength of the nonlocal part of the two-body interaction to the total strength of the interaction. For pion-nucleus scattering, the pion-nucleon interaction is approximately separable so that $V_{NL}\sim V$, and the range $a$ is about (500 MeV)$^{-1}$; thus $\lambda$ is 0.1 for an energy $E_\pi \approx 500$ MeV. For nucleon-nucleus scattering, we might estimate the range of the nonlocality to be 0.2 fm (~1 GeV$^{-1}$) in which case $\lambda$ is 0.1 for the kinetic energy of $T_N \approx 100\ \text{MeV}\times V/V_{NL}$, where the ratio of $V/V_{NL}$ is not reliably known.

The physical interpretation of Eq. (6.36) is that one cannot "fix" a particle which interacts nonlocally to within the range of the nonlocality. If the incident wavelength of particle zero is much larger than the nonlocal range of the interaction, then "fixing" the target particle within this range is sufficient. When the incident wavelength becomes comparable to the nonlocal range, then the inability to "fix" the target particle will cause significant corrections. The increase in the correction term, Eq. (6.36), with $k^2$ is clearly a result of our expanding this correction. If this estimate produces a large correction, one, of course, must include higher-order terms in the series to achieve a quantitative estimate of the correction.

## VII. ESTIMATE OF CONVERGENCE OF CORRELATION EXPANSION

In this section we provide a rough estimate of the convergence of the correlation expansion for the optical potential. We shall consider the truncation of our expansion for the optical potential given in Eq. (5.18). This equation written in momentum space is

$$\langle\vec{p}'|U'_{(2)}|\vec{p}\rangle = \langle\vec{p}'|V^{(1)}|\vec{p}\rangle + \langle\vec{p}'|V^{(2)}|\vec{p}\rangle \qquad (7.1)$$

with

$$\langle\vec{p}'|V^{(1)}|\vec{p}\rangle = (A-1)\langle\vec{p}'|t|\vec{p}\rangle\rho(\vec{p}'-\vec{p}) \qquad (7.2)$$

and

$$\langle\vec{p}'|V^{(2)}|\vec{p}\rangle = (A-1)^2\int d\vec{p}''\langle\vec{p}'|t|\vec{p}''\rangle g_E(\vec{p}'')$$
$$\times\langle\vec{p}''|t|\vec{p}\rangle C(\vec{p}'-\vec{p}'',\vec{p}-\vec{p}''), \qquad (7.3)$$

where $C(\vec{k}',\vec{k})$ is the Fourier transform of the correlation function defined in Eq. (5.19). An estimate of the convergence of the expansion will then be provided by the magnitude of $R$ defined by

$$R \equiv \frac{|V^{(2)}|}{|V^{(1)}|}. \qquad (7.4)$$

In order to obtain an estimate of the magnitude of $V^{(1)}$ and $V^{(2)}$ we will make a forward scattering approximation to $T$ matrices, viz.

$$\langle\vec{p}'|t|\vec{p}\rangle \approx t_E(0). \qquad (7.5)$$

This is a standard approximation often made in Eq. (7.2), where in coordinate space the two-body $T$ matrix is of shorter range than the nuclear radius. We shall also make this approximation in

Eq. (7.3). Here the approximation can at best be qualitative as we are interested in changes in the correlation function which (in coordinate space) occur over a range of ~0.5 fm which is comparable to the range of the $T$ matrix. Making this approximation yields

$$\langle \vec{r}_0' | V^{(1)} | \vec{r}_0 \rangle = (A-1) t_E(0)(2\pi)^3 \rho(\vec{r}_0)\delta(\vec{r}_0' - \vec{r}_0) \quad (7.6)$$

and

$$\langle \vec{r}_0' | V^{(2)} | \vec{r}_0 \rangle = (A-1)^2 t_E{}^2(0)(2\pi)^6 g_E(\vec{r}_0' - \vec{r}_0) C(\vec{r}_0', \vec{r}_0) . \quad (7.7)$$

In order to estimate the magnitude of $V^{(2)}$, we shall consider an approximately equivalent local potential. This potential is given by[9]

$$\int \langle \vec{r}_0' | V^{(2)} | \vec{r}_0 \rangle \psi(\vec{r}_0) d\vec{r}_0 \cong \left( \int \langle r_0' | V^{(2)} | \vec{r}_0 - \vec{s} \rangle e^{i\vec{k}\cdot\vec{s}} d\vec{s} \right) \psi(\vec{r}_0') . \quad (7.8)$$

Corrections to this approximation are reasonably small for $\pi$-nucleus scattering, and should be even smaller for nucleon-nucleus scattering, where $V_L/E$ is even smaller. Upon substitution of Eq. (7.7), we have

$$V_L^{(2)}(\vec{r}_0) = (A-1)^2 t_E{}^2(0)(2\pi)^6 \times \int d\vec{s}\, C(\vec{r}_0, \vec{r}_0 - \vec{s}) g_E(\vec{s}) e^{i\vec{k}\cdot\vec{s}} . \quad (7.9)$$

We now assume a Jastrow form[20] for the correlation function

$$C(\vec{r}_0, \vec{r}_0 - \vec{s}) = \rho^2(\vec{r}_0) H(s) . \quad (7.10)$$

This gives in Eq. (7.9)

$$V_L^{(2)}(\vec{r}_0) = (A-1)^2 t_e{}^2(0)(2\pi)^6 \rho^2(\vec{r}_0) \times \int H(s) g_E(\vec{s}) e^{i\vec{k}\cdot\vec{s}} d\vec{s} . \quad (7.11)$$

If we now use an explicit form for $g_E(\vec{s})$ and perform the angular integration, this becomes

$$V_L^{(2)}(\vec{r}_0) = -(A-1)^2 t_E{}^2(0)(2\pi)^6 \rho^2(\vec{r}_0) E_k \times \frac{1}{ik} \int_0^\infty H(s)(1 - e^{2iks}) ds . \quad (7.12)$$

The exponential term in the integral will tend to oscillate, and for reasonably large values of $k$ the integral will be dominated by the 1, giving

$$V_L^{(2)}(\vec{r}_0) = (A-1)^2 t_E{}^2(0)(2\pi)^6 \rho^2(\vec{r}_0) E_k \frac{l_c}{ik} , \quad (7.13)$$

where $l_c$ is a correlation length defined by

$$l_c \equiv \int_0^\infty H(s) ds . \quad (7.14)$$

We now approximate $A\rho(\vec{r}_0) \approx (A-1)\rho(\vec{r}_0) \approx 0.17$ fm$^{-3}$, the density of nuclear matter, and we take $t_E(0)$ to correspond to a total nucleon-nucleon cross section of 40 mb. This gives

$$R = 0.66 \text{ fm}^{-2} \left( \frac{l_c}{k} \right) . \quad (7.15)$$

For correlation length of 0.5 fm, one finds that $R \approx 10\%$ for a 400 MeV proton, and that the correlation term should be expected to decrease with increasing incident momentum.

## VIII. CONCLUSIONS

We have seen how the multiple scattering series for the KMT pseudo-optical potential may be systematically arranged according to the number of target particles which are actively involved in building the optical potential. In the closure approximation, the first term was seen to yield the usual impulse approximation. The second term, in which the incident particle interacts with two target particles in the expansion for the optical potential, was seen to be the solution of a three-body problem. For nucleon-nucleus elastic scattering at intermediate energies, the second Born approximation to this three-body problem has been calculated in Ref. 5. The conclusion reached there is that although the effects of correlations are not negligible, they are masked by other unknowns in the problem, the largest unknown being the spin dependence of the nucleon-nucleon interaction. Whether the summation of the second-order term as a three-body problem will alter this conclusion cannot be decided without further numerical investigation.

We have also derived expressions which correct for the closure approximation and the fixed scatterer approximation. Reliable calculations of these corrections may allow one to investigate correlative effects at lower energies or larger momentum transfer than has been possible, heretofore. We have not, however, included the antisymmetrization[21] of the incident nucleon with the target nucleons. A completely systematic theory of nucleon-nucleus elastic scattering requires the ability to calculate all such exchange effects quantitatively.

The necessity of the inclusion of correction terms and exchange effects in order to examine correlations in elastic scattering might be taken as an indication that elastic scattering is not the best place to try to learn about correlations. One might try to find a reaction in which the correlation effects were much larger. One must recall, however, that the correlation function is a ground state to ground state matrix element, and it is only in elastic scattering that one measures ground state to ground state matrix elements directly. Thus, in any inelastic scattering the extraction of a target correlation function requires further detailed knowledge of the relation of the final state to the target ground state. At the moment, this fact seems to limit one to elastic scattering or to double charge exchange to the double analog state as possible probes of nuclear correlations.

## APPENDIX A

In the text, the expansion for the pseudo-optical potential was derived using algebraic techniques similar to those originally used by Watson and coworkers.[4] Here we present an alternate derivation of our results which follows more closely the approach of KMT.

According to KMT, which deals only with a target of $A$ identical fermions, Eq. (2.1) is written as

$$\Pi T \Pi = \Pi \sum_i v_i \Pi + \Pi \sum_i v_i G T \Pi , \quad \text{(A1)}$$

where $\Pi$ is a projector onto antisymmetrized $A$-body states. From Eq. (A1) we obtain

$$\Pi T \Pi = A \Pi v \Pi + A \Pi v \Pi G T \Pi , \quad \text{(A2)}$$

where $v$ represents any one of the $A$ identical $v_i$. Bearing in mind that only matrix elements of $T$ between antisymmetric states will be taken, we rewrite Eq. (A2) as

$$T = Av + Av \Pi G T . \quad \text{(A3)}$$

We now define the two-body scattering $T$ matrix by

$$t = v + vgt , \quad \text{(A4)}$$

where, to be specific, we shall consider $g$ to be the "closure" propagator, Eq. (2.5). We use Eq. (A4) to eliminate $v$ from Eq. (A3) to obtain

$$T = At + (A-1)t \Pi G T + t(\Pi G - g)T . \quad \text{(A5)}$$

The definition of the pseudo-$T$ matrix $T'$ by

$$T' = \left(\frac{A-1}{A}\right) T \quad \text{(A6)}$$

yields

$$T' = (A-1)t + (A-1)t \Pi G T' + t(\Pi G - g)T' . \quad \text{(A7)}$$

The pseudo-optical potential operator of KMT is defined by

$$T' = U' + U' G P T' . \quad \text{(A8)}$$

Substitution of Eq. (A8) into Eq. (A7) so as to eliminate $T'$ yields the exact operator for $U'$:

$$U' = (A-1)t + (A-1)t(\Pi - P)GU' + t(\Pi G - g)U' . \quad \text{(A9)}$$

In the closure approximation, the full many-body propagator $G$ is approximated by the closure propagator $g$; this yields for Eq. (A9)

$$U' = (A-1)t + (A-1)t\bar{g}U' , \quad \text{(A10)}$$

where $\bar{g}$ is defined by

$$\bar{g} \equiv g(\Pi - P) + \frac{1}{A-1} g(\Pi - 1) . \quad \text{(A11)}$$

Iteration of Eq. (A10) gives

$$U' = (A-1)t + (A-1)^2 t\bar{g}t + (A-1)^3 t\bar{g}t\bar{g}t + \cdots . \quad \text{(A12)}$$

At this point we shall examine the terms in Eq. (A12) one at a time. The first term is

$$\begin{aligned}(A-1)\Pi t \Pi &= (A-1)\Pi \frac{1}{A} \sum_i t_i \Pi \\ &= (A-1)\Pi t_1 \Pi . \end{aligned} \quad \text{(A13)}$$

The second term is

$$\begin{aligned}
(A-1)^2 t\bar{g}t &= (A-1)^2 \Pi t g(\Pi - P) t \Pi - (A-1)\Pi t g(1-\Pi) t \Pi \\
&= \frac{(A-1)^2}{A^2} \Pi \sum_i t_i g(\Pi - P) \sum_j t_j \Pi - \frac{(A-1)}{A} \Pi \sum_i t_i g t_i \Pi + \frac{(A-1)}{A^2} \Pi \sum_i t_i g \Pi \sum_j t_j \Pi \\
&= \frac{(A-1)}{A} \Pi \sum_i t_i g(\Pi - P) \sum_j t_j \Pi - \frac{(A-1)}{A} \Pi \sum_i t_i g t_i \Pi + \frac{(A-1)}{A^2} \Pi \sum_i t_i g P \sum_j t_j \Pi \\
&= \Pi \left[ \frac{(A-1)}{A} \sum_i t_i g(1-P) \sum_j t_j - \frac{(A-1)}{A} \sum_i t_i g t_i + \frac{A-1}{A^2} \sum_i t_i g P \sum_j t_j \right] \Pi \\
&= \Pi[(A-1)^2 t_1 g(1-P) t_2 + (A-1) t_1 g(1-P) t_1 - (A-1) t_1 g(1-P) t_1] \Pi \\
&= (A-1)^2 \Pi t_1 g(1-P) t_2 \Pi .
\end{aligned} \quad \text{(A14)}$$

Straightforward continuation of the process yields

$$U' = \frac{(A-1)}{A}\{At_1 + A(A-1)[t_1g(1-P)t_2 + t_1g(1-P)t_2g(1-P)t_1 + \cdots]$$
$$+A(A-1)(A-2)[t_1g(1-P)t_2g(1-P)t_3 + \cdots] + \cdots$$
$$+A(A-1)(A-2)\cdots 2\cdot 1[t_1g(1-P)t_2g(1-P)\cdots t_{(A-1)}g(1-P)t_A]\}$$
$$= \frac{(A-1)}{A}(W^{(1)} + W^{(2)} + \cdots W^{(A)}). \tag{A15}$$

This is the central result derived in the text.

## APPENDIX B

We have, until now, discussed the "correlation" expansion in terms of the closure approximation. It is not difficult to reformulate this discussion without incorporating this approximation. We find it instructive to begin by working with the transition operator, and then to apply the same attitudes to the optical operator. Thus we recall Eq. (2.1)

$$T = \sum_i v_i + \sum_i v_i G T. \tag{B1}$$

We then define the operator $\bar{t}_i$ by means of the relation[22]

$$\bar{t}_i = v_i + v_i G \bar{t}_i , \tag{B2}$$

where $G$ is again the many-body propagator of Eq. (2.2). Elimination of $v_i$ in favor of $\bar{t}_i$ in Eq. (B1) by means of Eq. (B2), then leads to the familiar Watson series:

$$T = \sum_i \bar{t}_i + \sum_i \bar{t}_i G \sum_{j\neq i} \bar{t}_j + \sum_i \bar{t}_i G \sum_{j\neq i} \bar{t}_j G \sum_{k\neq j} \bar{t}_k + \cdots . \tag{B3}$$

We may then easily reorder the series of Eq. (B3) into the form

$$T = T^{(1)} + T^{(2)} + T^{(3)} + \cdots , \tag{B4}$$

with

$$T^{(1)} = \sum_i \bar{t}_i , \tag{B5}$$

$$T^{(2)} = \sum_i \sum_{j\neq i} (\bar{t}_i G \bar{t}_j + \bar{t}_i G \bar{t}_j G \bar{t}_i + \bar{t}_i G \bar{t}_j G \bar{t}_i G \bar{t}_j + \cdots), \tag{B6}$$

$$T^{(3)} = \sum_i \sum_{j\neq i} \sum_{\substack{k\neq i \\ k\neq j}} (\bar{t}_i G \bar{t}_j G \bar{t}_k + \bar{t}_i G \bar{t}_j G \bar{t}_k G \bar{t}_i + \bar{t}_i G \bar{t}_j G \bar{t}_k G \bar{t}_j + \bar{t}_i G \bar{t}_j G \bar{t}_i G \bar{t}_k + \cdots). \tag{B7}$$

The series for $T^{(2)}$ is clearly in the form

$$T^{(2)} \equiv \sum_i \sum_{j\neq i} T_{ij}^{(2)} = \sum_{i>j} (T_{ij}^{(2)} + T_{ji}^{(2)}) \equiv \sum_{i>j} \tilde{T}_{ij}^{(2)} , \tag{B8}$$

where $\tilde{T}_{ij}^{(2)}$ is given by

$$\tilde{T}_{ij}^{(2)} = \bar{t}_i G \bar{t}_j + \bar{t}_i G \bar{t}_j G \bar{t}_i + \bar{t}_i G \bar{t}_j G \bar{t}_i G \bar{t}_j + \cdots + \bar{t}_j G \bar{t}_i + \bar{t}_j G \bar{t}_i G \bar{t}_j + \bar{t}_j G \bar{t}_i G \bar{t}_j G \bar{t}_i + \cdots . \tag{B9}$$

We then note that $\tilde{T}_{ij}^{(2)}$ is related to the solution of the problem of the scattering of the projectile from the two target particles ($i$ and $j$), still considered, however, to be in the nucleus. Let us then formulate this scattering problem in order to compare that result with Eq. (B9).

The scattering of the projectile from two target particles ($i$ and $j$), is given by

$$\bar{t}_{ij} = (v_i + v_j) + (v_i + v_j) G \bar{t}_{ij} , \tag{B10}$$

where $\bar{t}_{ij}$ is written with a bar to emphasize that the propagator in Eq. (B10) is the many-body propagator of Eq. (2.2). If we then use Eq. (B2) in the standard way to eliminate $v_i$ and $v_j$ in Eq. (B10) in favor of $\bar{t}_i$ and $\bar{t}_j$, we find that

$$\bar{t}_{ij} = (\bar{t}_i + \bar{t}_j) + (\bar{t}_i G \bar{t}_j + \bar{t}_j G \bar{t}_i) + (\bar{t}_i G \bar{t}_j G \bar{t}_i + \bar{t}_j G \bar{t}_i G \bar{t}_j) + \cdots , \tag{B11}$$

as expected from Eq. (B3). Comparison of Eq. (B11) with Eq. (B9) leads to the identification of $\tilde{T}_{ij}^{(2)}$ as

$$\tilde{T}_{ij}^{(2)} = \bar{t}_{ij} - \bar{t}_i - \bar{t}_j \tag{B12}$$

or

$$T^{(2)} = \sum_{i>j} (\bar{t}_{ij} - \bar{t}_i - \bar{t}_j). \tag{B13}$$

For a target of $A$ identical fermions we see that

$$T = A\bar{t}_1 + \tfrac{1}{2}A(A-1)(\bar{t}_{12} - 2\bar{t}_1) + \cdots . \tag{B14}$$

The first term represents the $A$ single scattering terms, the second term represents the $\frac{1}{2}A(A-1)$ scatterings from pairs (from which the single particle scatterings have been removed, since they have already been included in the first terms).

It is of interest to note that to make the closure approximation at this point, we need only replace $G \equiv (E - h_0 - H_A + i\eta)^{-1}$ by $g = (E - h_0 + i\eta)^{-1}$, wherever it appears. In that case $\bar{t}_i$ and $\bar{t}_{ij}$ become $t_i$ and $t_{ij}$, respectively, where $t_i$ satisfies the Lippmann-Schwinger equation given by Eq. (2.4) and $t_{ij}$ is given by Eq. (4.2). In this approximation $t_i$ represents the scattering of the projectile from a fixed target particle ($i$), and $t_{ij}$ represents the scattering of the projectile from a pair of fixed target particles ($i$ and $j$).

In an exactly corresponding way, we may begin with the Lippmann-Schwinger equation for the optical operator

$$U=\sum_i v_i+\sum_i v_i G(1-P)U. \tag{B15}$$

In this case we follow Watson and write

$$\hat{t}_i=v_i+v_i G(1-P)\hat{t}_i\ , \tag{B16}$$

to obtain the Watson series for $U$, viz.

$$U=\sum_i \hat{t}_i+\sum_i \hat{t}_i G(1-P)\sum_{j\neq i}\hat{t}_j + \sum_i \hat{t}_i G(1-P)\sum_{j\neq i}\hat{t}_j G(1-P)\sum_{k\neq j}\hat{t}_k+\cdots . \tag{B17}$$

This series may be resummed, as before, to give

$$U=U^{(1)}+U^{(2)}+U^{(3)}+\cdots , \tag{B18}$$

where

$$U^{(1)}=\sum_i \hat{t}_i\ ,$$

$$U^{(2)}=\sum_i\sum_{j\neq i}[\hat{t}_i G(1-P)\hat{t}_j+\hat{t}_i G(1-P)\times\hat{t}_j G(1-P)\hat{t}_i+t_i G(1-P)\hat{t}_j G(1-P)\times\hat{t}_i G(1-P)\hat{t}_j+\cdots]\ , \tag{B19}$$

and so on. We then observe that

$$U^{(2)}=\sum_{i>j}(\hat{t}_{ij}-\hat{t}_i-\hat{t}_j)\ , \tag{B20}$$

where $\hat{t}_{ij}$ satisfies the relation

$$\hat{t}_{ij}=(v_i+v_j)+(v_i+v_j)G(1-P)\hat{t}_{ij}\ . \tag{B21}$$

Thus, the only difference between the treatment of the multiple scattering of the transition operator $T$ and the optical operator $U$ is that the propagator $G$ is replaced by $G(1-P)$ wherever it appears, again just as anticipated.

In the KMT formulation the operator $U'$ appears. This operator is defined in Eq. (2.9). Combining the definition Eq. (2.9) with Eq. (B1), we obtain

$$T'=\left(\frac{A-1}{A}\right)\sum_i v_i+\sum_i v_i GT'\ . \tag{B22}$$

If $T'$ is then eliminated between Eq. (2.9) and Eq. (B22), the relation

$$U'=\left(\frac{A-1}{A}\right)\sum_i v_i + \sum_i v_i GU'-\left(\frac{A-1}{A}\right)\sum_i v_i GPU'$$

results. The definition

$$U''=\left(\frac{A}{A-1}\right)U' \tag{B23}$$

gives

$$U''=\sum_i v_i+\sum_i v_i GU''-\left(\frac{A-1}{A}\right)\sum_i v_i gPU'' \tag{B24}$$

or

$$U_i''=v_i+v_i G\sum_j U_j''-\left(\frac{A-1}{A}\right)v_i GP\sum_j U_j''\ , \tag{B25}$$

where

$$U''\equiv\sum_i U_i''\ . \tag{B26}$$

We may eliminate $v_i$ in favor of $\bar{t}_i$ in Eq. (B25) to obtain

$$U_i''=\bar{t}_i+\bar{t}_i G(1-P)\sum_{j\neq i}U_j'' + \frac{1}{A}\ \bar{t}_i GP\left[\sum_j U_j''-AU_i''\right]\ . \tag{B27}$$

At this point we then particularize our discussion to the case where all the target particles are identical fermions. In that case $U, U', U'', T, T'$, etc. are all taken to be operators which operate only on completely antisymmetric target states. The operator $\Pi$ has been defined as the projector onto the space of antisymmetric target states. We then note that

$$U_i''\Pi=\left(\bar{t}_i+\bar{t}_i G(1-P)\sum_{j\neq i}U_j''\right)\Pi\ , \tag{B28}$$

since

$$P\left(\sum_j U_j''-AU_i\right)\Pi=0\ . \tag{B29}$$

Thus in the case that the target states are completely antisymmetric, we may substitute the simpler relation

$$U_i''=\bar{t}_i+\bar{t}_i G(1-P)\sum_{j\neq i}U_j'' \tag{B30}$$

for Eq. (B27). This is the approach taken by KMT, although the algebra here is slightly different.

The definition Eq. (B26) together with Eq. (B30) implies, in the usual way, that

$$U''=\sum_i\bar{t}_i+\sum_i\bar{t}_i G(1-P)\sum_{j\neq i}\bar{t}_j +\sum_i\bar{t}_i G(1-P)\sum_{j\neq i}\bar{t}_j G(1-P)\sum_{k\neq j}\bar{t}_k+\cdots \tag{B31}$$

or

$$U' = \left(\frac{A-1}{A}\right)\Big[\sum_i \bar{t}_i + \sum_i \bar{t}_i G(1-P) \sum_{j\neq i} \bar{t}_j + \sum_i \bar{t}_i G(1-P) \times \sum_{j\neq i} \bar{t}_j G(1-P) \sum_{k\neq j} \bar{t}_k + \cdots \Big] . \quad \text{(B32)}$$

Resummation of the series of Eq. (B32) as

$$U' = U^{(1)\prime} + U^{(2)\prime} + U^{(3)\prime} \quad \text{(B33)}$$

then yields

$$U^{(1)\prime} = \frac{A-1}{A}\sum_i \bar{t}_i , \quad \text{(B34)}$$

$$U^{(2)\prime} = \frac{A-1}{A} \sum_{i>j} (\tilde{t}_{ij} - \bar{t}_i - \bar{t}_j) , \quad \text{(B35)}$$

and so on. The operator $\tilde{t}_{ij}$ which appears in Eq. (B35) is given by the series

$$\tilde{t}_{ij} = \bar{t}_i + \bar{t}_i G(1-P)\bar{t}_j + \bar{t}_i G(1-P)\bar{t}_j G(1-P) \times \bar{t}_i + \cdots + \bar{t}_j + \bar{t}_j G(1-P)\bar{t}_i - \bar{t}_j G(1-P)\bar{t}_i G(1-P)\bar{t}_j + \cdots , \quad \text{(B36)}$$

or equivalently $\tilde{t}_{ij} = \Lambda_{ij} + \Lambda_{ji}$, where $\Lambda_{ij}$ and $\Lambda_{ji}$ satisfy the coupled equations

$$\Lambda_{ij} = \bar{t}_i + \bar{t}_i G(1-P)\Lambda_{ji} \quad \text{(B37)}$$

and

$$\Lambda_{ji} = \bar{t}_j + \bar{t}_j G(1-P)\Lambda_{ij} . \quad \text{(B38)}$$

Again we recognize that the two coupled equations above are equivalent to the two uncoupled equations

$$\Lambda_{ij} = \bar{t}_i + \bar{t}_i G(1-P)\bar{t}_j + \bar{t}_i G(1-P)\bar{t}_j G(1-P)\Lambda_{ij} , \quad \text{(B39)}$$

$$\Lambda_{ji} = \bar{t}_j + \bar{t}_j G(1-P)\bar{t}_i + \bar{t}_j G(1-P)\bar{t}_i G(1-P)\Lambda_{ji} \quad \text{(B40)}$$

and that iteration of these equations yields the series Eq. (B36). The pair of equations (B37)–(B38) are, of course, the Watson-Faddeev equations for the scattering of the projectile from the pair of particles $ij$ with a modified propagator.

## APPENDIX C

In the paper we developed an expansion for the pseudo-optical potential operator of KMT. The techniques used there can just as readily be used to obtain an expansion for the optical operator of Watson, defined by

$$T = U + UGPT . \quad \text{(C1)}$$

As has been noted in Ref. 15 the operators $U'$ and $U$ are related by the integral equation

$$U' = \left(\frac{A-1}{A}\right) U + \frac{1}{A} UPGU' , \quad \text{(C2)}$$

which permits one to determine the approximation for $U$ which is exactly equivalent to a given approximation for $U'$. The relation, Eq. (C2), can be used to generate an expansion for $U$ from our expansion for $U'$. It is simpler, however, to derive the expansion directly from Eq. (C1). Using Eq. (1.1) and Eq. (C1) we may derive an integral equation for $U$ which becomes, in the closure approximation,

$$U = Av + Av(1-P)\Pi g U . \quad \text{(C3)}$$

In this paper, we have demonstrated that the equation for the many-body $T$ matrix given by

$$T = Av + Avg\,\Pi T , \quad \text{(C4)}$$

may be rearranged according to a hierarchy of the number of target particles participating in the scattering process. This rearrangement is

$$T = At_1 + A(A-1)(\tfrac{1}{2}t_{12} - t_1) + \cdots , \quad \text{(C5)}$$

where $t_1$ was given by

$$t_1 = v_1 + v_1 g t_1 \quad \text{(C6)}$$

and

$$\tfrac{1}{2}t_{12} = t_1 + t_1 g t_2 + t_1 g t_2 g t_1 + \cdots . \quad \text{(C7)}$$

The similarity between Eqs. (C3) and (C4) allows us to immediately rewrite Eq. (C3) as

$$U = A\tilde{t}_1 + A(A-1)(\tfrac{1}{2}\tilde{t}_{12} - \tilde{t}_1) , \quad \text{(C8)}$$

where

$$\tilde{t}_1 = v_1 + v_1 g(1-P)\tilde{t}_1 \quad \text{(C9)}$$

and

$$\tfrac{1}{2}\tilde{t}_{12} = \tilde{t}_1 + \tilde{t}_1 g(1-P)\tilde{t}_2 + \tilde{t}_1 g(1-P)\tilde{t}_2 g(1-P)\tilde{t}_1 + \cdots . \quad \text{(C10)}$$

These equations, Eqs. (C8)–(C10), provide a correlation expansion of the optical operator $U$. However, we find the expansion of $U'$ preferable because it is an expansion in $t$, while the expansion of $U$ is an expansion in $\tilde{t}$, defined in Eq. (C9).

## APPENDIX D

In this paper we have arranged the multiple scattering series for the pseudo-optical potential of KMT according to the number of distinct target particles which are involved in constructing the optical potential. We have seen that a Born approximation to a three-body problem will yield the result of Refs. 5 and 6. In Appendix B we have shown how a similar arrangement may be performed for the optical potential itself; this yields a resummation of higher-order terms in the Watson[9] expansion of the optical potential. Foldy and

Walecka[7] have proposed an alternate rearrangement of the multiple scattering series which is not closely related either to this work or to that of Refs. 5, 6, 8, and 9. The techniques used here, however, can be readily adapted to derive their results. In this Appendix, we rederive their results and compare their rearrangement of the multiple scattering series with the one derived here.

Foldy and Walecka begin with the multiple scattering series in the closure approximation, Eq. (2.8), rewritten slightly:

$$T=\sum_i t_i+\sum_i t_i g \sum_{j\neq i} t_j+\sum_i t_i g\sum_{j\neq i} t_j g \sum_{k\neq i,j} t_k + \sum_i t_i g \sum_{j\neq i} t_j g t_i + \cdots . \qquad \text{(D1)}$$

They then keep only single particle density matrix contributions to those terms in which a given particle participates no more than once. This is exactly equivalent to inserting the projection operator $P$ into Eq. (D1) in the following manner:

$$T_{\text{FW}}=At_1+A(A-1)t_1 g\, Pt_1 + A(A-1)(A-2)t_1 gPt_1 gPt_1 + \cdots + A(A-1)t_1 g t_2 g t_1+\cdots , \qquad \text{(D2)}$$

where we have made use of the fact that we are only going to take matrix elements with fully antisymmetrized $A$-body states.

At this point, they make the approximation that the number of target particles struck is much smaller than the total number of particles in the target. This mathematically consists of approximating $A(A-1)$ by $A^2$, $A(A-1)(A-2)$ by $A^3$, etc. With this approximation, Eq. (D2) becomes

$$T_{\text{FW}}\approx At_1+A^2t_1gPt_1+A^3t_1gPt_1gPt_1+\cdots + A(A-1)t_1gt_2gt_1+\cdots . \qquad \text{(D3)}$$

This equation then is broken into two parts defined by

$$T_{\text{FW}}^{(\text{opt})}=At_1+A^2t_1gPt_1+A^3t_1g\, Pt_1gPt_1 ,$$
$$T_{\text{FW}}^{(\text{res})}=A(A-1)t_1gt_2t_1+\cdots . \qquad \text{(D4)}$$

The ground state matrix elements of the optical potential part $T_{\text{FW}}^{(\text{opt})}$ are clearly a solution of a Lippmann-Schwinger equation with an optical potential defined by

$$\langle \vec{p}\,' \,|\, U_{\text{FW}}^{(\text{opt})} \,|\, \vec{p}\rangle = A\langle \vec{p}\,' | t | \vec{p}\rangle\, \rho(\vec{p}\,'-\vec{p}) , \qquad \text{(D5)}$$

where we have already noted that the factorization in Eq. (D4) results from the use of the closure propagator, and, in the case of a nonlocal potential, the assumption of an infinite target particle mass. The second part of the $T$ matrix $T_{\text{FW}}^{(\text{res})}$ is termed the "local field" correction or the "rescattering" term.

Recently[15] the accuracy of Eq. (D5) as an approximation to the optical potential has been examined numerically. It was found that for proton scattering from $^4$He, $^{12}$C, and $^{40}$Ca at energies from 90 MeV to 1.05 GeV, Eq. (D5) is not an accurate approximation to the optical potential. The use of this optical potential in a Lippmann-Schwinger equation clearly generates the approximation for elastic scattering

$$T\approx T_{\text{FW}}^{(\text{opt})} \qquad \text{(D6)}$$

with $T_{\text{FW}}^{(\text{opt})}$ defined in Eq. (D3). A comparison with Eq. (D2) shows that the correction terms to this approximation are

$$\Delta T=At_1g\, Pt_1+A(3A-2)t_1g\, Pt_1gPt_1 + \cdots + T_{\text{FW}}^{(\text{res})} . \qquad \text{(D7)}$$

The first set of correction terms might be referred to as "counting" terms.

It is precisely these correction terms which are eliminated by the use of the KMT pseudo-optical potential or by the Watson two-body $T$ matrix defined in Eq. (C9). This is readily seen by returning to Eq. (D2) and including that part of the Foldy-Walecka "rescattering" term which proceeds from one particle to another through the ground state in the first series. This gives

$$T=At_1+A(A-1)t_1g\, Pt_1+A(A-1)^2t_1g\, Pt_1g\, Pt_1 + A(A-1)[t_1g\, Pt_2g(1-P)t_1+t_1g\,(1-P)t_2gPt_1 + t_1g\,(1-P)t_2g(1-P)t_1] , \qquad \text{(D8)}$$

from which it is clear how the KMT pseudo-optical potential exactly sums the first series in this expression. The second term in this expression has here been shown to be a part of a systematic correlation expansion.

Foldy and Walecka then propose two corrections to Eq. (D5). One is the rescattering correction, the leading term of which is given in Eq. (D4). The second correction is the correlation correction to the approximation used in going from Eq. (D1) to Eq. (D2).

We feel that the arrangement of the multiple scattering series presented in this work or in Refs. 5 and 6 yields a more systematic approach to calculating elastic scattering. These works may be viewed as breaking the Foldy-Walecka rescattering correction into two pieces. The first is used to eliminate the counting correction of Eq. (D7), while the remaining piece quite naturally combines with the Foldy-Walecka correlation correction to yield the correlation expansion developed here. Thus in these works, there is in the closure approximation only a single correction term to the leading order optical potential and this is a correlation correction. In the Foldy-Walecka arrangement, there are three corrections to their leading order optical potential: the rescattering, correlation, and counting corrections. All three must be included to exactly reproduce the term in Eq. (D1) which is second order in the two-body $T$ matrix.

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†Present address.

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