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Faddeev-Yakubovsky equations for two distinct pairs of identical particles

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The problem of two distinct pairs of identical particles is considered in Faddeev-Yakubovsky formalism. A set of seven coupled integral equations is obtained which is simplified by assuming separable s-wave forms for the interaction between any two particles. An angular momentum analysis is carried out to obtain the equations satisfied by the partial wave components.

[NUCLEAR STRUCTURE Faddeev-Yakubovsky formalism, four-body equations, distinct pairs of identical particles, separable two-body potentials, partial wave analysis.]

I. INTRODUCTION

A mathematically rigorous approach to study the motion of a system of three particles was proposed by Faddeev[1] in 1960. Since then, this method has been used to investigate the properties of three-body systems consisting of nucleons and hyperons by a number of workers.[2] However, for the problem of motion of four particles, the variational method, in which one has to start with a trial wave function, was the only available one till the elegant works by Faddeev[3] and by Yakubovsky[4] were published. In their works, Faddeev and Yakubovsky clarified the difficulties associated with a correct formulation of the equations of motion of a four-body system. Similar nonvariational formalisms for dealing with four-particle dynamics have been proposed by Alt, Grassberger and Sandhas[5] and Sloan.[6] The Faddeev-Yakubovsky approach has been used by Kharchenko and Kuzmichev,[7] Narodetsky,[8] and Tjon[9] to obtain the equations satisfied by the components of four-body wave functions, which are required for investigating the properties of a system of four identical particles. Gibson and Lehman,[10] starting directly from the Schrödinger equation and utilizing the assumption of two-body separable potentials, have obtained the coupled two-variable integral equations that determine the four-nucleon bound state. Using an alternative field-theoretic approach, Fonseca and Shanley[11] have recently derived the integral equations for four-body scattering processes including breakup for the case of pairs of spinless identical particles n,n and a,a. They have introduced three spinless quasiparticles C,D, and α with s-wave couplings $D\to n+n$, $C\to a+a$, and $\alpha\to D+a$ only. They have avoided the particle exchange contribution to the three-body amplitude and arrived at the relevant equations for different four-body processes. This model has been used to study the four-nucleon problem[12] where the positions of the four-body bound states have been found out and a complete phase shift calculation has been performed. To our knowledge, no attempt has so far been made in Faddeev-Yakubovsky formalism to consider rigorously the problem where the particles are not all identical. In this work we derive the necessary dynamical equations within the formalism of Refs. 3, 4, and 7 for the case where the constituents of the four-body system are two different pairs of identical particles. It may be noted that in the Faddeev-Yakubovsky formalism all possible pair interactions are allowed. The symmetry properties of the system in our problem are quite different from that of four identical particles and we have a larger number of coupled integral equations with which to deal. Our equations will be useful to explore systems like ${}^{10}_{\Lambda\Lambda}\mathrm{Be}$ (in the $\alpha\alpha\Lambda\Lambda$ model).

In the next section, we give the four-particle integral equations and their modification for the case of two distinct pairs of identical particles. In Sec. III, we formulate the corresponding integral equations when the two-particle interaction is separable. The decomposition in terms of angular momenta is carried out in Sec. IV and the discussion appears in Sec. V.

II. FOUR-PARTICLE INTEGRAL EQUATIONS AND THE PROBLEM OF TWO PAIRS OF IDENTICAL PARTICLES

Following Faddeev[3] and Yakubovsky,[4] Kharchenko and Kuzmichev[7] have shown that the wave function

Ψ_A for a four-particle system defined by

$$\Psi_A = \lim_{\epsilon \to 0} i\epsilon \mathcal{G}(E_A + i\epsilon)\Phi_A\,, \tag{1}$$

where $\mathcal{G}$ represents the full Green's function for the system with energy E_A and Φ_A represents the wave function for the asymptotic state A, can be written as,

$$\Psi_A = \sum_{(ijk,l)} \Psi_A^{(ijk,l)} + \sum_{(ij,kl)} \Psi_A^{(ij,kl)} \tag{2}$$

with $\Psi_A^{(ijk,l)}$ and $\Psi_A^{(ij,kl)}$ satisfying the following coupled equations:

$$\Psi_A^{(ijk,l)} = \eta_A^{ijk} + \mathcal{G}_0(z)\{\mathfrak{M}_{ij,ij}(z)[\Psi_A^{(kjl,i)} + \Psi_A^{(ikl,j)} + \Psi_A^{(jk,il)} + \Psi_A^{(ki,jl)}] + \mathfrak{M}_{ij,jk}(z)[\Psi_A^{(ikl,j)} + \Psi_A^{(jil,k)} + \Psi_A^{(ki,jl)} + \Psi_A^{(ij,kl)}] + \mathfrak{M}_{ij,ki}(z)[\Psi_A^{(jil,k)} + \Psi_A^{(kjl,i)} + \Psi_A^{(ij,kl)} + \Psi_A^{(jk,il)}]\} \tag{3}$$

$$\Psi_A^{(ij,kl)} = \mathcal{G}_0(z)\{\mathfrak{N}_{ij,ij}(z)[\Psi_A^{(lkj,i)} + \Psi_A^{(kli,j)}] + \mathfrak{N}_{ij,kl}(z)[\Psi_A^{(jil,k)} + \Psi_A^{(ijk,l)}]\}\,, \tag{4}$$

where $z = E_A + i\epsilon$. The expression for the inhomogeneous part η_A^{ijk} when the particle 4 is scattered by the bound state of particles 1,2,3, can be found in Ref. 7. $\mathfrak{M}_{\alpha,\beta}$ and $\mathfrak{N}_{\alpha,\beta}$ satisfy Eqs. (9) and (10) of Ref. 7.

Kharchenko and Kuzmichev[7] have investigated a system consisting of four identical bosons. In this work, we consider the problem of two different pairs of identical particles with the pair interactions invariant under space inversion. Further, we will assume the space part of the wave function of any two particles to be of even parity. Using Eq. (1), the wave function of the four-particle state can be built from an asymptotic state of correct symmetry. To be specific, let the particles 1 and 2 form one pair of identical particles and the particles 3 and 4 form the other one.

First we start with the scattering of a particle on three particles in a bound state. The possible asymptotic wave functions are

$$\Phi_{123,4},\ \Phi_{231,4},\ \Phi_{312,4},\ \Phi_{214,3},\ \Phi_{142,3},\ \Phi_{421,3},$$
$$\Phi_{134,2},\ \Phi_{341,2},\ \Phi_{413,2},\ \Phi_{243,1},\ \Phi_{432,1},\ \Phi_{324,1}\,,$$

where the subscript (ijk,l) represents the case of the particle l being scattered by the bound state (ijk). For this type of asymptotic state any one of the following functions will have the proper symmetry required for the problem:

$$\Phi_{1a} = \Phi_{123,4} + \Phi_{214,3}\,, \tag{5a}$$

$$\Phi_{2a} = \Phi_{243,1} + \Phi_{324,1} + \Phi_{134,2} + \Phi_{413,2}\,, \tag{5b}$$

$$\Phi_{3a} = \Phi_{341,2} + \Phi_{432,1}\,, \tag{5c}$$

$$\Phi_{4a} = \Phi_{142,3} + \Phi_{421,3} + \Phi_{231,4} + \Phi_{312,4}\,. \tag{5d}$$

It may be noted that according to what has been stated above the asymptotic wave functions $\Phi_{ijk,l}$ and $\Phi_{jik,l}$ are identical.

For the case of scattering between two pairs of bound particles, the possible asymptotic wave functions are

$$\Phi_{12,34},\ \Phi_{34,12},\ \Phi_{23,14},\ \Phi_{14,23},\ \Phi_{31,24},\ \Phi_{24,31}\,,$$

where the subscript (ij,kl) stands for the scattering of the combination of particles (ij) by the combination (kl). Corresponding to this type of asymptotic states, the functions which have the proper symmetry are

$$\Phi_{1b} = \Phi_{12,34}\,, \tag{6a}$$

$$\Phi_{2b} = \Phi_{34,12}\,, \tag{6b}$$

$$\Phi_{3b} = \Phi_{23,14} + \Phi_{14,23} + \Phi_{31,24} + \Phi_{24,31}\,. \tag{6c}$$

For the asymptotic functions of the type $\Phi_{ij,kl}$ we have, of course,

$$\Phi_{ij,kl} = \Phi_{ji,kl} = \Phi_{ij,lk} = \Phi_{ji,lk}\,.$$

We can choose any one of the Φ's given in Eqs. (5) and (6) to be the state Φ_A for determining Ψ_A of Eq. (1). Choosing Φ_{1a} of Eq. (5a) to be the required asymptotic state and writing Ψ_{1a} as a sum of components in the form given in Eq. (2), $\Psi_{1a}^{(ijk,l)}$ and $\Psi_{1a}^{(ij,kl)}$ will satisfy equations similar to Eqs. (3) and (4), respectively. These 18 equations are not all independent, and utilizing the symmetry of the pairs of particles can be reduced to a set of seven coupled integral equations given below:

$$\Psi_{1a}^{1a} = \Phi_{1a}^{(12)} + \mathcal{G}_0(z)\{[\mathfrak{M}_{12,12}(z) + \mathfrak{M}_{12,23}(z)](\Psi_{1a}^{2a} + \chi_{1a}^{3b}) + 2\mathfrak{M}_{12,31}(z)(\Psi_{1a}^{1a} + 2\chi_{1a}^{1b})\}\,, \tag{7a}$$

$$\Psi_{1a}^{2a} = 2\mathcal{G}_0(z)\{2\mathfrak{M}_{23,23}(z)(\Psi_{1a}^{3a} + 2\chi_{1a}^{2b}) + [\mathfrak{M}_{23,34}(z) + \mathfrak{M}_{23,42}(z)](\Psi_{1a}^{4a} + \chi_{1a}^{3b})\}\,, \tag{7b}$$

$$\Psi_{1a}^{3a} = \mathcal{G}_0(z)\{[\mathfrak{M}_{34,34}(z) + \mathfrak{M}_{34,41}(z)](\Psi_{1a}^{4a} + \chi_{1a}^{3b}) + 2\mathfrak{M}_{34,13}(z)(\Psi_{1a}^{3a} + 2\chi_{1a}^{2b})\}\,, \tag{7c}$$

$$\Psi_{1a}^{4a}=2\Phi_{1a}^{(23)}+2\mathcal{G}_0(\mathfrak{z})\{2\mathfrak{M}_{41,41}(\mathfrak{z})(\Psi_{1a}^{1a}+2\chi_{1a}^{1b})+[\mathfrak{M}_{41,12}(\mathfrak{z})+\mathfrak{M}_{41,24}(\mathfrak{z})](\Psi_{1a}^{2a}+\chi_{1a}^{3b})\}, \tag{7d}$$

$$\chi_{1a}^{1b}=\mathcal{G}_0(\mathfrak{z})[\mathfrak{N}_{12,12}(\mathfrak{z})\Psi_{1a}^{3a}+\mathfrak{N}_{12,34}(\mathfrak{z})\Psi_{1a}^{1a}], \tag{7e}$$

$$\chi_{1a}^{2b}=\mathcal{G}_0(\mathfrak{z})[\mathfrak{N}_{34,34}(\mathfrak{z})\Psi_{1a}^{1a}+\mathfrak{N}_{34,12}(\mathfrak{z})\Psi_{1a}^{3a}], \tag{7f}$$

$$\chi_{1a}^{3b}=2\mathcal{G}_0(\mathfrak{z})\mathfrak{N}_{23,23}(\mathfrak{z})(\Psi_{1a}^{2a}+\Psi_{1a}^{4a}), \tag{7g}$$

where

$$\Phi_{1a}^{(ij)}=\Phi_{123,4}^{(ij)}+\Phi_{214,3}^{(ij)}$$

with

$$\Psi_{ijk,l}^{(ij)}\equiv\mathcal{G}_0(\mathfrak{z})V_{ij}\Phi_{ijk,l}.$$

The superscripts on the Ψ's and the χ's stand for the same type of linear combination which has been used to define the subscripts in Eqs. (5) and (6).

If instead of choosing Φ_{1a} as the asymptotic state to be used in Eq. (1), we choose any of the other six states defined in Eqs. (5) and (6) and proceed in the manner explained above, we will arrive at six other sets of seven coupled integral equations. The integral equations belonging to these six sets will differ from those belonging to the set of Eqs. (7) in their inhomogeneous parts. However, the kernels of corresponding equations in the different sets will be identical.

For bound state problems, solutions are to be found for the integral equations after dropping out the inhomogeneous parts. It will be sufficient, therefore, to consider the set of Eqs. (7) and henceforth we will limit our discussions to this set only. The subscript $1a$ attached to the wave functions in Eqs. (7) will also be omitted in the rest of the paper.

The Eqs. (7) are given in operator form. To get corresponding equations in momentum space one has to introduce a set of relative momentum variables. Taking the momentum representation of the component states $\Psi^{(ijk,l)}$ and $\Psi^{(ij,kl)}$ in terms of their characteristic set of variables[6] and combining these representations in the same way in which we have obtained Eqs. (7) from Eqs. (3) and (4), we get the integral equations for the wave functions in momentum space. These equations without their inhomogeneous parts are given below:

$$\begin{aligned}\Psi^{1a}(\vec{k},\vec{p},\vec{q})=\left(z_1-\frac{k^2}{2\mu_{12}}-\frac{p^2}{2\mu_{12,3}}\right)^{-1}\int\Big\{&2M_{12,31}(\vec{k}\vec{p};\vec{K}_1'\vec{P}_1';z_1)\Psi^{1a}\left(\vec{k}',\vec{q}+\frac{m_b}{2m_a+m_b}\vec{q}',\vec{q}'\right)\\&+[M_{12,12}(\vec{k}\vec{p};\vec{K}_2'\vec{P}_2';z_1)+M_{12,23}(\vec{k}\vec{p};\vec{K}_3'\vec{P}_3';z_1)]\Psi^{2a}\left(\vec{k}',\vec{q}+\frac{m_b}{2m_b+m_a}\vec{q}',\vec{q}'\right)\\&+4M_{12,31}(\vec{k}\vec{p};\vec{K}_4'\vec{P}_4';z_1)\chi^{1b}(\vec{k}',\vec{q}-\tfrac{1}{2}\vec{q}',\vec{q}')\\&+[M_{12,12}(\vec{k}\vec{p};\vec{K}_5'\vec{P}_5';z_1)+M_{12,23}(\vec{k}\vec{p};\vec{K}_6'\vec{P}_6';z_1)]\\&\times\chi^{3b}\left(\vec{k}',\vec{q}-\frac{m_b}{m_a+m_b}\vec{q}',\vec{q}'\right)\Big\}\frac{d\vec{k}'}{(2\pi)^3}\cdot\frac{d\vec{q}'}{(2\pi)^3},\end{aligned} \tag{8a}$$

$$z_1=\mathfrak{z}-\frac{q^2}{2\mu_{123,4}};$$

$$\begin{aligned}\Psi^{2a}(\vec{k},\vec{p},\vec{q})=2\left(z_2-\frac{k^2}{2\mu_{23}}-\frac{p^2}{2\mu_{23,4}}\right)^{-1}\int\Big\{&2M_{23,23}(\vec{k}\vec{p};\vec{K}_7'\vec{P}_7';z_2)\Psi^{3a}\left(\vec{k}',\vec{q}+\frac{m_a}{2m_b+m_a}\vec{q}',\vec{q}'\right)\\&+[M_{23,34}(\vec{k}\vec{p};\vec{K}_8'\vec{P}_8';z_2)+M_{23,42}(\vec{k}\vec{p};\vec{K}_9'\vec{P}_9';z_2)]\Psi^{4a}\left(\vec{k}',\vec{q}+\frac{m_a}{2m_a+m_b}\vec{q}',\vec{q}'\right)\\&+4M_{23,23}(\vec{k}\vec{p};\vec{K}_{10}'\vec{P}_{10}';z_2)\chi^{2b}(\vec{k}',\vec{q}-\tfrac{1}{2}\vec{q}',\vec{q}')\\&+[M_{23,34}(\vec{k}\vec{p};\vec{K}_{11}'\vec{P}_{11}';z_2)+M_{23,42}(\vec{k}\vec{p};\vec{K}_{12}'\vec{P}_{12}';z_2)]\\&\times\chi^{3b}\left(\vec{k},\vec{q}-\frac{m_a}{m_a+m_b}\vec{q}',\vec{q}'\right)\Big\}\frac{d\vec{k}'}{(2\pi)^3}\cdot\frac{d\vec{q}'}{(2\pi)^3},\end{aligned} \tag{8b}$$

$$z_2=\mathfrak{z}-\frac{q^2}{2\mu_{234,1}};$$

$$\Psi^{3a}(\vec{k},\vec{p},\vec{q})=\frac{1}{2}\left(z_3-\frac{k^2}{2\mu_{34}}-\frac{p^2}{2\mu_{34,1}}\right)^{-1}\left(z_2-\frac{k^2}{\mu_{23}}-\frac{p^2}{2\mu_{23,4}}\right)\xi^{2a}(\vec{k},\vec{p},\vec{q}),\quad z_3=\mathfrak{z}-\frac{q^2}{2\mu_{341,2}}; \tag{8c}$$

$$\Psi^{4a}(\vec{k},\vec{p},\vec{q})=2\left(z_4-\frac{k^2}{2\mu_{41}}-\frac{p^2}{2\mu_{41,2}}\right)^{-1}\left(z_1-\frac{k^2}{2\mu_{12}}-\frac{p^2}{2\mu_{12,3}}\right)\xi^{1a}(\vec{k},\vec{p},\vec{q}),\quad z_4=\mathfrak{z}-\frac{q^2}{2\mu_{412,3}}; \tag{8d}$$

$$\chi^{1b}(\vec{k},\vec{\kappa},\vec{s})=\left(z_5-\frac{k^2}{2\mu_{12}}-\frac{\kappa^2}{2\mu_{34}}\right)^{-1}\int\left[N_{12,12}(\vec{k}\vec{\kappa};\tfrac{1}{2}\vec{s}+\vec{q}',\vec{k}';z_5)\Psi^{3a}\left(\vec{k}',-\vec{s}-\frac{2m_b}{2m_b+m_a}\vec{q}',\vec{q}'\right)\right.$$
$$\left.+N_{12,34}(\vec{k}\vec{\kappa};\tfrac{1}{2}\vec{s}+\vec{q}',\vec{k}';z_5)\Psi^{1a}\left(\vec{k}',-\vec{s}-\frac{2m_a}{2m_a+m_b}\vec{q}',\vec{q}'\right)\right]\frac{d\vec{k}'}{(2\pi)^3}\cdot\frac{d\vec{q}'}{(2\pi)^3},$$
$$z_5=\mathfrak{z}-\frac{s^2}{2\mu_{12,34}}; \tag{8e}$$

$$\chi^{2b}(\vec{k},\vec{\kappa},\vec{s})=\left(z_6-\frac{k^2}{2\mu_{34}}-\frac{\kappa^2}{2\mu_{12}}\right)^{-1}\left(z_5-\frac{k^2}{2\mu_{12}}-\frac{\kappa^2}{2\mu_{34}}\right)\xi^{1b}(\vec{k},\vec{\kappa},\vec{s}),\quad z_6=\mathfrak{z}-\frac{s^2}{2\mu_{34,12}}; \tag{8f}$$

$$\chi^{3b}(\vec{k},\vec{\kappa},\vec{s})=2\left(z_7-\frac{k^2}{2\mu_{23}}-\frac{\kappa^2}{2\mu_{41}}\right)^{-1}\int\left[N_{23,23}\left(\vec{k}\vec{\kappa};\frac{m_a}{m_a+m_b}\vec{s}+\vec{q}',\vec{k}';z_7\right)\Psi^{2a}\left(\vec{k}',-\vec{s}-\frac{m_a+m_b}{2m_b+m_a}\vec{q}',\vec{q}'\right)\right.$$
$$\left.+N_{23,23}\left(\vec{k}\vec{\kappa};\frac{m_b}{m_a+m_b}\vec{s}+\vec{q}',\vec{k}';z_7\right)\Psi^{4a}\left(\vec{k}',-\vec{s}-\frac{m_a+m_b}{2m_a+m_b}\vec{q}',\vec{q}'\right)\right]$$
$$\times\frac{d\vec{k}'}{(2\pi)^3}\cdot\frac{d\vec{q}'}{(2\pi)^3},\quad z_7=\mathfrak{z}-\frac{s^2}{2\mu_{23,41}}; \tag{8g}$$

where

$$\vec{K}_1'=\frac{m_b}{m_a+m_b}\vec{k}'+\frac{m_b}{2(m_a+m_b)}\vec{q}+\frac{2m_a+m_b}{2(m_a+m_b)}\vec{q}'=\vec{K}_4'+\frac{2m_a+m_b}{2(m_a+m_b)}\vec{q},$$

$$\vec{P}_1'=\vec{k}'-\frac{m_b}{2(2m_a+m_b)}\vec{q}-\tfrac{1}{2}\vec{q}'=\vec{P}_4'-\tfrac{1}{2}\vec{q},$$

$$\vec{K}_2'=\tfrac{1}{2}\vec{k}'+\frac{m_a}{2(m_a+m_b)}\vec{q}+\frac{2m_a+m_b}{2(m_a+m_b)}\vec{q}'=\vec{K}_5'+\frac{2m_a+m_b}{2(m_a+m_b)}\vec{q},$$

$$\vec{P}_2'=\vec{k}'-\frac{m_am_b}{(m_a+m_b)(2m_a+m_b)}\vec{q}-\frac{m_b}{m_a+m_b}\vec{q}'=\vec{P}_5'-\frac{m_b}{m_a+m_b}\vec{q},$$

$$\vec{K}_3'=\frac{m_a}{m_a+m_b}\vec{k}'+\frac{m_am_b}{(m_a+m_b)^2}\vec{q}+\frac{m_b(2m_a+m_b)}{(m_a+m_b)^2}\vec{q}'=\vec{K}_6'+\frac{m_b(2m_a+m_b)}{(m_a+m_b)^2}\vec{q},$$

$$\vec{P}_3'=\vec{k}'-\frac{m_a^2}{(m_a+m_b)(2m_a+m_b)}\vec{q}-\frac{m_a}{m_a+m_b}\vec{q}'=\vec{P}_6'-\frac{m_a}{m_a+m_b}\vec{q}.$$

The expressions for $(\vec{K}_7'\vec{P}_7')$, $(\vec{K}_8'\vec{P}_8'),\ldots,(\vec{K}_{12}'\vec{P}_{12}')$ are similar to those for $(\vec{K}_1'\vec{P}_1')$, $(\vec{K}_2'\vec{P}_2'),\ldots,(\vec{K}_6'\vec{P}_6')$, respectively, but with the replacement $m_a\leftrightarrow m_b$. In Eq. (8c), the function ξ^{2a} has a form similar to the right-hand side of Eq. (8b) but with $M_{23,23}$, $M_{23,34}$, and $M_{23,42}$ replaced by $M_{34,13}$, $M_{34,34}$, and $M_{34,41}$, respectively. Also, z_2 is replaced by z_3. The functions ξ^{1a} and ξ^{1b} occurring in Eqs. (8d) and (8f) have the form of the right-hand sides of Eqs. (8a) and (8e), respectively, with the replacements $M_{12,12}\to M_{41,12}$, $M_{12,23}\to M_{41,24}$, $M_{12,31}\to M_{41,41}$, and $z_1\to z_4$ in the first case and $N_{12,12}\to N_{34,12}$, $N_{12,34}\to N_{34,34}$, $z_5\to z_6$ in the second one.

The three-particle scattering amplitude $M_{\alpha,\beta}(\vec{k}\vec{p};\vec{k}'\vec{p}';z)$ and the scattering amplitude of two noninteracting pairs $N_{\alpha,\beta}(\vec{k}\vec{\kappa};\vec{k}'\vec{\kappa}';z)$ satisfy the integral equations

$$M_{\alpha,\beta}(\vec{k}\vec{p};\vec{k}'\vec{p}';z)=(2\pi)^3\langle\vec{k}\,|t_\alpha\left(z-\frac{\mathfrak{m}}{2\mathfrak{m}_\alpha m_\alpha}p^2\right)|\vec{k}'\rangle\,\delta(\vec{p}-\vec{p}')\delta_{\alpha\beta}$$
$$+\sum_{\gamma\neq\alpha}\int\langle\vec{k}\,|t_\alpha\left(z-\frac{\mathfrak{m}}{2\mathfrak{m}_\alpha m_\alpha}p^2\right)\left|\frac{m_\gamma}{\mathfrak{m}_\alpha}\vec{p}+\vec{p}''\right\rangle\left(z-\frac{p^2}{2\mu_\gamma}-\frac{p''^2}{2\mu_\alpha}-\frac{m_\alpha}{\mu_\gamma\mathfrak{m}_\gamma}\vec{p}\cdot\vec{p}''\right)^{-1}$$
$$\times\left\langle\vec{p}+\frac{m_\alpha}{\mathfrak{m}_\gamma}\vec{p}'',\vec{p}''\,\right|M_{\gamma,\beta}(z)\,|\vec{k}',\vec{p}'\rangle\,\frac{d\vec{p}''}{(2\pi)^3},\quad\alpha(\beta\text{ or }\gamma)=ij,jk,ki; \tag{9a}$$

$$N_{\alpha,\alpha}(\vec{k}\vec{\kappa};\vec{k}'\vec{\kappa}';z)=(2\pi)^3\langle\vec{k}\,|t_\alpha\left(z-\frac{\kappa^2}{2\mu_\beta}\right)|\vec{k}'\rangle\,\delta(\vec{\kappa}-\vec{\kappa}')+\int\langle\vec{k}|t_\alpha\left(z-\frac{\kappa^2}{2\mu_\beta}\right)|\vec{k}''\rangle\left(z-\frac{\kappa^2}{2\mu_\beta}-\frac{k''^2}{2\mu_\alpha}\right)^{-1}\langle\vec{\kappa},\vec{k}''\,|N_{\beta,\alpha}(z)|\vec{k}',\vec{\kappa}'\rangle\frac{d\vec{k}''}{(2\pi)^3}, \tag{9b}$$

where $\beta\neq\alpha$;

$$N_{\alpha,\beta}(\vec{k}\vec{\kappa};\vec{k}'\vec{\kappa}';z)=\int\langle\vec{k}\,|t_\alpha\left(z-\frac{\kappa^2}{2\mu_\beta}\right)|\vec{k}''\rangle\left(z-\frac{\kappa^2}{2\mu_\beta}-\frac{k''^2}{2\mu_\alpha}\right)^{-1}\langle\vec{\kappa},\vec{k}''\,|N_{\beta,\beta}(z)|\vec{k}',\vec{\kappa}'\rangle\frac{d\vec{k}''}{(2\pi)^3}, \tag{9c}$$

for $\alpha\neq\beta$, $\alpha(\beta)=ij,kl$;

In Eqs. (8) and (9) $m_a=m_1=m_2$ and $m_b=m_3=m_4$, where m_i is the mass of the ith particle and the reduced masses μ are defined as in Ref. 7. The $\mathfrak{m}_\alpha$'s and m_α's occurring in Eqs. (9) are defined as $\mathfrak{m}_\alpha=\mathfrak{m}-m_\alpha$ and $m_\alpha=m_{ij}=\mathfrak{m}-m_i-m_j$ where $\mathfrak{m}$ is the sum of the masses of three particles i, j, and k.

III. REDUCTION OF FOUR-BODY EQUATIONS IN CASE OF SEPARABLE POTENTIALS

The integral equations (8) get simplified if the two-particle interactions have separable form. Separable interactions have been widely used for investigations of scattering and bound states of nucleons and hyperons.[2] We will be primarily interested in bound states involving nuclei and hyperons and will assume s-wave separable interaction between any pair of particles. Then

$$\langle\vec{k}\,|V_{ij}\,|\vec{k}'\rangle=-\lambda_{ij}g_{ij}(k)g_{ij}(k') \tag{10}$$

for the interaction of the ith particle with the jth one. λ_{ij}'s are the potential strength parameters and g_{ij}'s are the form factors depending only on the modulus of the momentum vectors. Using the potential form given in Eq. (10) the matrix elements of the two-body t matrix can be written as

$$t_{ij}(\vec{k},\vec{k}';z)=\langle\vec{k}\,|t_{ij}(z)\,|\vec{k}'\rangle=g_{ij}(k)\tau_{ij}(z)g_{ij}(k') \tag{11}$$

where

$$\tau_{ij}^{-1}(z)=-\lambda_{ij}^{-1}-\int\frac{g_{ij}^2(q)}{(z-q^2/2\mu_{ij})}\frac{d\vec{q}}{(2\pi)^3}.$$

From the Eqs. (9), (10), and (11) the matrix elements of the type $M_{\alpha,\beta}$ and $N_{\alpha,\beta}$ are expressed in terms of two other functions $X_{\alpha,\beta}$ and $Y_{\alpha,\beta}$ as defined below:

$$X_{\alpha,\beta}(\vec{p},\vec{p}';z)=U_{\alpha,\beta}(\vec{p},\vec{p}';z)(1-\delta_{\alpha\beta})+\sum_{\gamma\neq\alpha}\int U_{\alpha,\gamma}(\vec{p},\vec{p}'';z)\tau_\gamma\left(z-\frac{\mathfrak{m}}{2\mathfrak{m}_\gamma m_\gamma}p''^2\right)X_{\gamma,\beta}(\vec{p}'',\vec{p}';z)\frac{d\vec{p}''}{(2\pi)^3},$$

where

$$U_{\alpha,\beta}(\vec{p},\vec{p}';z)=\frac{g_\alpha((m_\beta/\mathfrak{m}_\alpha)\vec{p}+\vec{p}')g_\beta(\vec{p}+(m_\alpha/\mathfrak{m}_\beta)\vec{p}')}{z-p^2/2\mu_\beta-p'^2/2\mu_\alpha-(m_\alpha/\mathfrak{m}_\beta\mu_\beta)\vec{p}\cdot\vec{p}'};$$

then

$$M_{\alpha,\beta}(\vec{k}\vec{p};\vec{k}'\vec{p}';z)\equiv g_\alpha(k)g_\beta(k')\tau_\alpha(z-(\mathfrak{m}/2\mathfrak{m}_\alpha m_\alpha)p^2)[(2\pi)^3\delta(\vec{p}-\vec{p}')\delta_{\alpha\beta}+\tau_\beta(z-(\mathfrak{m}/2\mathfrak{m}_\beta m_\beta)p'^2)X_{\alpha,\beta}(\vec{p},\vec{p}';z)]$$

$\alpha(\beta$ or $\gamma)=ij,jk,ki$; (12)

$$Y_{\alpha,\beta}(\vec{\kappa},\vec{\kappa}';z)=W_{\alpha,\beta}(\vec{\kappa},\vec{\kappa}';z)(1-\delta_{\alpha\beta})+\int W_{\alpha,\gamma}(\vec{\kappa},\vec{\kappa}'';z)\tau_\gamma\left(z-\frac{\kappa''^2}{2\mu_\alpha}\right)Y_{\gamma,\beta}(\vec{\kappa}'',\vec{\kappa}';z)\frac{d\vec{\kappa}''}{(2\pi)^3},\quad\gamma\neq\alpha;$$

where

$$W_{\alpha,\beta}(\vec{\kappa},\vec{\kappa}';z)=\frac{g_\alpha(\kappa')g_\beta(\kappa)}{z-\kappa'^2/2\mu_\alpha-\kappa^2/2\mu_\beta},$$

then

$$N_{\alpha,\alpha}(\vec{k}\vec{\kappa};\vec{k}'\vec{\kappa}';z)=g_\alpha(k)g_\alpha(k')\tau_\alpha\left(z-\frac{\kappa^2}{2\mu_\beta}\right)\left[(2\pi)^3\delta(\vec{\kappa}-\vec{\kappa}')+\tau_\alpha\left(z-\frac{\kappa'^2}{2\mu_\beta}\right)Y_{\alpha,\alpha}(\vec{\kappa},\vec{\kappa}';z)\right],\quad\beta\neq\alpha;$$

$$N_{\alpha,\beta}(\vec{k}\vec{\kappa};\vec{k}'\vec{\kappa}';z)=g_\alpha(k)g_\beta(k')\tau_\alpha\left(z-\frac{\kappa^2}{2\mu_\beta}\right)\tau_\beta\left(z-\frac{\kappa'^2}{2\mu_\alpha}\right)Y_{\alpha,\beta}(\vec{\kappa},\vec{\kappa}';z),\quad\text{for }\alpha\neq\beta,\quad\alpha(\beta\text{ or }\gamma)=ij,kl. \tag{13}$$

In the above equations $X_{\alpha,\beta}$ and $Y_{\alpha,\beta}$ are the off-shell amplitudes for the scattering of a particle by a composite system of two particles and the scattering of two particles in the presence of a pair of other particles interacting between themselves, respectively.

We can now factor out the k dependence explicitly from the functions Ψ and χ of Eqs. (8) and introduce the functions Q and R as follows:

$$\Psi^{na}(\vec{k},\vec{p},\vec{q}) = -\frac{g_{ij}(k)}{\mathfrak{z} - k^2/2\mu_{ij} - p^2/2\mu_{ij,k} - q^2/2\mu_{ijk,l}}\; Q^{na}(\vec{p},\vec{q})$$

for

$$(n;ijk) = (1;123),(2;234),(3;341),(4;412)$$

and

$$\chi^{nb}(\vec{k},\vec{\kappa},\vec{s}) = -\frac{g_{ij}(k)}{\mathfrak{z} - k^2/2\mu_{ij} - \kappa^2/2\mu_{kl} - s^2/2\mu_{ij,kl}}\; R^{nb}(\vec{\kappa},\vec{s})$$

for

$$(n;ijkl) = (1;1234),(2;3412),(3;2341). \tag{14}$$

Q and R then satisfy the following set of integral equations:

$$\begin{aligned}
Q^{1a}(\vec{p},\vec{q}) = \tau_{12}\left(z_1 - \frac{p^2}{2\mu_{12,3}}\right) \int \Bigg\{ & 2\mathcal{K}_{12,31}\left(\vec{p},\frac{m_b}{2m_a+m_b}\vec{q}+\vec{q}\,';z_1\right) Q^{1a}\left(\vec{q}+\frac{m_b}{2m_a+m_b}\vec{q}\,',\vec{q}\,'\right) \\
&+\left[\mathcal{K}_{12,12}\left(\vec{p},\frac{m_a}{2m_a+m_b}\vec{q}+\vec{q}\,';z_1\right)+\mathcal{K}_{12,23}\left(\vec{p},\frac{m_a}{2m_a+m_b}\vec{q}+\vec{q}\,';z_1\right)\right] \\
&\times Q^{2a}\left(\vec{q}+\frac{m_b}{2m_b+m_a}\vec{q}\,',\vec{q}\,'\right)+4\mathcal{K}_{12,31}\left(\vec{p},-\frac{2m_a}{2m_a+m_b}\vec{q}+\vec{q}\,';z_1\right)R^{1b}(\vec{q}-\tfrac{1}{2}\vec{q}\,',\vec{q}\,') \\
&+\left[\mathcal{K}_{12,12}\left(\vec{p},-\frac{m_a+m_b}{2m_a+m_b}\vec{q}+\vec{q}\,';z_1\right)\right. \\
&\left.+\mathcal{K}_{12,23}\left(\vec{p},-\frac{m_a+m_b}{2m_a+m_b}\vec{q}+\vec{q}\,';z_1\right)\right] \\
&\times R^{3b}\left(\vec{q}-\frac{m_b}{m_a+m_b}\vec{q}\,',\vec{q}\,'\right)\Bigg\}\frac{d\vec{q}\,'}{(2\pi)^3},
\end{aligned} \tag{15a}$$

$$\begin{aligned}
Q^{2a}(\vec{p},\vec{q}) = 2\tau_{23}\left(z_2 - \frac{p^2}{2\mu_{23,4}}\right) \int \Bigg\{ & 2\mathcal{K}_{23,23}\left(\vec{p},\frac{m_a}{2m_b+m_a}\vec{q}+\vec{q}\,';z_2\right) Q^{3a}\left(\vec{q}+\frac{m_a}{2m_b+m_a}\vec{q}\,',\vec{q}\,'\right) \\
&+\left[\mathcal{K}_{23,34}\left(\vec{p},\frac{m_b}{2m_b+m_a}\vec{q}+\vec{q}\,';z_2\right)+\mathcal{K}_{23,42}\left(\vec{p},\frac{m_b}{2m_b+m_a}\vec{q}+\vec{q}\,';z_2\right)\right] \\
&\times Q^{4a}\left(\vec{q}+\frac{m_a}{2m_a+m_b}\vec{q}\,',\vec{q}\right)+4\mathcal{K}_{23,23}\left(\vec{p},-\frac{2m_b}{2m_b+m_a}\vec{q}+\vec{q}\,';z_2\right)R^{2b}(\vec{q}-\tfrac{1}{2}\vec{q}\,',\vec{q}\,') \\
&+\left[\mathcal{K}_{23,34}\left(\vec{p},-\frac{m_a+m_b}{2m_b+m_a}\vec{q}+\vec{q}\,';z_2\right)+\mathcal{K}_{23,42}\left(\vec{p},-\frac{m_a+m_b}{2m_b+m_a}\vec{q}+\vec{q}\,';z_2\right)\right] \\
&\times R^{3b}\left(\vec{q}-\frac{m_a}{m_a+m_b}\vec{q}\,',\vec{q}\,'\right)\Bigg\}\frac{d\vec{q}\,'}{(2\pi)^3},
\end{aligned} \tag{15b}$$

$$Q^{3a}(\vec{p},\vec{q}) = \tfrac{1}{2}\tau_{34}\left(z_3 - \frac{p^2}{2\mu_{34,1}}\right)\left[\tau_{23}\left(z_2-\frac{p^2}{2\mu_{23,4}}\right)\right]^{-1}\mathcal{Q}^{2a}(\vec{p},\vec{q}), \tag{15c}$$

$$Q^{4a}(\vec{p},\vec{q}) = 2\tau_{41}\left(z_4 - \frac{p^2}{2\mu_{41,2}}\right)\left[\tau_{12}\left(z_1-\frac{p^2}{2\mu_{12,3}}\right)\right]^{-1}\mathcal{Q}^{1a}(\vec{p},\vec{q}), \tag{15d}$$

$$\begin{aligned}
R^{1b}(\vec{\kappa},\vec{s}) = \tau_{12}\left(z_5 - \frac{\kappa^2}{2\mu_{34}}\right)\int\Bigg[& \mathcal{V}_{12,34}(\vec{\kappa},\tfrac{1}{2}\vec{s}+\vec{q}\,';z_5)Q^{1a}\left(-\vec{s}-\frac{2m_a}{2m_a+m_b}\vec{q}\,',\vec{q}\,'\right) \\
&+\mathcal{V}_{12,12}(\vec{\kappa},\tfrac{1}{2}\vec{s}+\vec{q}\,';z_5)Q^{3a}\left(-\vec{s}-\frac{2m_b}{2m_b+m_a}\vec{q}\,',\vec{q}\,'\right)\Bigg]\frac{d\vec{q}\,'}{(2\pi)^3},
\end{aligned} \tag{15e}$$

$$R^{2b}(\vec{\kappa},\vec{s}) = \tau_{34}\left(z_6 - \frac{\kappa^2}{2\mu_{12}}\right)\left[\tau_{12}\left(z_5 - \frac{\kappa^2}{2\mu_{34}}\right)\right]^{-1}\mathcal{R}^{1b}(\vec{\kappa},\vec{s}) , \tag{15f}$$

$$R^{3b}(\vec{\kappa},\vec{s}) = 2\tau_{23}\left(z_7 - \frac{\kappa^2}{2\mu_{41}}\right)\int\Big[\mathcal{V}_{23,23}\left(\vec{\kappa}, \frac{m_a}{m_a+m_b}\vec{s}+\vec{q}';z_7\right)Q^{2a}\left(-\vec{s} - \frac{m_a+m_b}{2m_b+m_a}\vec{q}',\vec{q}'\right) + \mathcal{V}_{23,23}\left(\vec{\kappa},\frac{m_b}{m_a+m_b}\vec{s}+\vec{q}';z_7\right)Q^{4a}\left(-\vec{s}-\frac{m_a+m_b}{2m_a+m_b}\vec{q}',\vec{q}'\right)\Big]\frac{d\vec{q}'}{(2\pi)^3} . \tag{15g}$$

The functions $\mathcal{Q}^{2a}$, $\mathcal{Q}^{1a}$, and $\mathcal{R}^{1b}$ occurring in Eqs. (15c), (15d), and (15f) have their forms similar to the right-hand sides of Eqs. (15b), (15a), and (15e), respectively, with necessary replacements of the subscripts of the $\mathcal{K}_{\alpha,\beta}$'s, $\mathcal{V}_{\alpha,\beta}$'s, and z_i's as has been done for the $M_{\alpha,\beta}$'s, $N_{\alpha,\beta}$'s, and z_i's in Eqs. (8c), (8d), and (8f) [see also the explanation given below Eqs. (8)].

The kernels of the integral Eqs. (15) are given by

$$\mathcal{K}_{ij,ij}(\vec{p},\vec{p}';z) = U_{ij,jk}(\vec{p},\vec{p}';z) + \int X_{ij,ij}(\vec{p},\vec{p}'';z)\tau_{ij}\left(z - \frac{p''^2}{2\mu_{ij,k}}\right)U_{ij,jk}(\vec{p}'',\vec{p}';z)\frac{d\vec{p}''}{(2\pi)^3} , \tag{16a}$$

$$\mathcal{K}_{ij,jk(ki)}(\vec{p},\vec{p}';z) = \int X_{ij,jk(ki)}(\vec{p},\vec{p}'';z)\tau_{jk(ki)}\left(z - \frac{p''^2}{2\mu_{jk,i(ki,j)}}\right)U_{jk(ki),ki(ij)}(\vec{p}'',\vec{p}';z)\frac{d\vec{p}''}{(2\pi)^3} , \tag{16b}$$

$$\mathcal{V}_{ij,ij(kl)}(\vec{\kappa},\vec{\kappa}';z) = W_{ij,kl}(\vec{\kappa},\vec{\kappa}':z)\delta_{ij,ij(kl)} + \int Y_{ij,ij(kl)}(\vec{\kappa},\vec{\kappa}'';z) \times \tau_{ij(kl)}\left(z - \frac{\kappa''^2}{2\mu_{kl(ij)}}\right)W_{ij(kl),kl(ij)}(\vec{\kappa}'',\vec{\kappa}';z)\frac{d\vec{\kappa}''}{(2\pi)^3} . \tag{17}$$

In Eqs. (16), (ijk) takes the values (123), (234), (341), and (412) for the kernels occurring in Eqs. (15a), (15b), (15c), and (15d), respectively. In Eqs. (17) $(ijkl)$ takes the values (1234), (3412), and (2341) for the kernels occurring in Eqs. (15e), (15f), and (15g), respectively.

The partial wave analysis of Eqs. (15) is carried out in the next section.

IV. ANGULAR MOMENTUM ANALYSIS

Let us assume that the four particles being considered are in a state characterized by the quantum numbers L and M for the total orbital angular momentum and its third component, respectively. The spins of the particles are not taken into account at present. Therefore, for s-wave two-body interactions, we can write

$$\vec{L} = \vec{l}_p + \vec{l}_q = \vec{l}_s ,$$

where $\vec{l}_p$, $\vec{l}_q$, and $\vec{l}_s$ are the angular momenta corresponding to the angular variables of the momentum vectors $\vec{p}$, $\vec{q}$, and $\vec{s}$, respectively.

Each of the equations (15) depends on two momentum vectors and the functions therein can be expanded[13] in terms of the spherical harmonics associated with the directions of the momenta. By this method the set of Eqs. (15) can be reduced to the following set of equations for the partial wave components of Q and R,

$$\begin{aligned} Q^{1a}_{l_p l_q L}(p,q) = \frac{1}{2\pi^2}\tau_{12}\left(z_1 - \frac{p^2}{2\mu_{12,3}}\right)\Bigg[&\frac{2m_a+m_b}{m_b}\sum_{l_p' l_q'}\int_0^\infty dq'\int_{|q-[m_b/(2m_a+m_b)]q'|}^{q+[m_b/(2m_a+m_b)]q'} dp'\,\frac{p'q'}{q} \\ &\times \mathcal{K}_{(12,31)l_p}(p,p_1';z_1)I^{1(L)}_{l_p l_q, l_p' l_q'}(q,q';\mathfrak{x}_1)Q^{1a}_{l_p' l_q' L}(p',q') \\ &+\frac{2m_b+m_a}{2m_b}\sum_{l_p' l_q'}\int_0^\infty dq'\int_{|q-[m_b/(2m_b+m_a)]q'|}^{q+[m_b/(2m_b+m_a)]q'} dp'\,\frac{p'q'}{q} \\ &\times\{\mathcal{K}_{(12,12)l_p}(p,p_2';z_1) + \mathcal{K}_{(12,23)l_p}(p,p_2';z_1)\}I^{2(L)}_{l_p l_q, l_p' l_q'}(q,q';\mathfrak{x}_2)Q^{2a}_{l_p' l_q' L}(p',q') \\ &+4\int_0^\infty ds'\int_{|q-(1/2)s'|}^{q+(1/2)s'} d\kappa'\,\frac{\kappa' s'}{q}\mathcal{K}_{(12,31)l_p}(p,p_3';z_1)J^{1(L)}_{l_p l_q}(q,s';\mathfrak{y}_1)R^{1b}_L(\kappa',s') \\ &+\frac{m_a+m_b}{2m_b}\int_0^\infty ds'\int_{|q-[m_b/(m_a+m_b)]s'|}^{q+[m_b/(m_a+m_b)]s'} d\kappa'\,\frac{\kappa' s'}{q} \\ &\times\{\mathcal{K}_{(12,12)l_p}(p,p_4';z_1) + \mathcal{K}_{(12,23)l_p}(p,p_4';z_1)\}J^{2(L)}_{l_p l_q}(q,s';\mathfrak{y}_2)R^{3b}_L(\kappa',s')\Bigg] , \end{aligned} \tag{18a}$$

$$Q^{2a}_{l_p l_q L}(p,q)=\frac{1}{\pi^2}\tau_{23}\left(z_2-\frac{p^2}{2\mu_{23,4}}\right)\Bigg\{\frac{2m_b+m_a}{m_a}\sum_{l_p' l_q'}\int_0^\infty dq'\int_{|q-[m_a/(2m_b+m_a)]q'|}^{q+[m_a/(2m_b+m_a)]q'}dp'\,\frac{p'q'}{q}$$
$$\times\mathcal{K}_{(23,23)l_p}(p,p_5';z_2)I^{3(L)}_{l_p l_q, l_p' l_q'}(q,q';\mathfrak{x}_3)Q^{3a}_{l_p' l_q' L}(p',q')$$
$$+\frac{2m_a+m_b}{2m_a}\sum_{l_p' l_q'}\int_0^\infty dq'\int_{|q-[m_a/(2m_a+m_b)]q'|}^{q+[m_a/(2m_a+m_b)]q'}dp'\,\frac{p'q'}{q}$$
$$\times\{\mathcal{K}_{(23,34)l_p}(p,p_6';z_2)+\mathcal{K}_{(23,42)l_p}(p,p_6';z_2)\}I^{4(L)}_{l_p l_q, l_p' l_q'}(q,q';\mathfrak{x}_4)Q^{4a}_{l_p' l_q' L}(p',q')$$
$$+4\int_0^\infty ds'\int_{|q-(1/2)s'|}^{q+(1/2)s'}d\kappa'\,\frac{\kappa's'}{q}\mathcal{K}_{(23,23)l_p}(p,p_7';z_2)J^{3(L)}_{l_p l_q}(q,s';\mathfrak{y}_3)R^{2b}_L(\kappa',s')$$
$$+\frac{m_a+m_b}{2m_a}\int_0^\infty ds'\int_{|q-[m_a/(m_a+m_b)]s'|}^{q+[m_a/(m_a+m_b)]s'}d\kappa'\,\frac{\kappa's'}{q}$$
$$\times[\mathcal{K}_{(23,34)l_p}(p,p_8';z_2)+\mathcal{K}_{(23,42)l_p}(p,p_8';z_2)]\,J^{4(L)}_{l_p l_q}(q,s';\mathfrak{y}_4)R^{3b}_L(\kappa',s')\Bigg\}, \quad (18b)$$

$$Q^{3a}_{l_p l_q L}(p,q)=\tfrac{1}{2}\tau_{34}\left(z_3-\frac{p^2}{2\mu_{34,1}}\right)\left[\tau_{23}\left(z_2-\frac{p^2}{2\mu_{23,4}}\right)\right]^{-1}\mathfrak{Q}^{2a}_{l_p l_q L}(p,q), \quad (18c)$$

$$Q^{4a}_{l_p l_q L}(p,q)=2\tau_{41}\left(z_4-\frac{p^2}{2\mu_{41,2}}\right)\left[\tau_{12}\left(z_1-\frac{p^2}{2\mu_{12,3}}\right)\right]^{-1}\mathfrak{Q}^{1a}_{l_p l_q L}(p,q), \quad (18d)$$

$$R^{1b}_L(\kappa,s)=\frac{1}{2\pi^2}\tau_{12}\left(z_5-\frac{\kappa^2}{2\mu_{34}}\right)\Bigg[\frac{2m_a+m_b}{4m_a}\sum_{l_p' l_q'}\int_0^\infty dq'\int_{|s-[2m_a/(2m_a+m_b)]q'|}^{s+[2m_a/(2m_a+m_b)]q'}dp'\,\frac{p'q'}{s}\mathcal{V}^{(L)}_{12,34}(\kappa,\kappa_1';z_5)$$
$$\times K^{1(L)}_{l_p' l_q'}(s,q';\mathfrak{z}_1)Q^{1a}_{l_p' l_q' L}(p',q')$$
$$+\frac{2m_b+m_a}{4m_b}\sum_{l_p' l_q'}\int_0^\infty dq'\int_{|s-[2m_b/(2m_b+m_a)]q'|}^{s+[2m_b/(2m_b+m_a)]q'}dp'\,\frac{p'q'}{s}$$
$$\times\mathcal{V}^{(L)}_{12,12}(\kappa,\kappa_2';z_5)K^{2(L)}_{l_p' l_q'}(s,q';\mathfrak{z}_2)Q^{3a}_{l_p' l_q' L}(p',q')\Bigg], \quad (18e)$$

$$R^{2b}_L(\kappa,s)=\tau_{34}\left(z_6-\frac{\kappa^2}{2\mu_{12}}\right)\left[\tau_{12}\left(z_5-\frac{\kappa^2}{2\mu_{34}}\right)\right]^{-1}\mathfrak{R}^{1b}_L(\kappa,s), \quad (18f)$$

$$R^{3b}_L(\kappa,s)=\frac{1}{\pi^2}\tau_{23}\left(z_7-\frac{\kappa^2}{2\mu_{41}}\right)\Bigg[\frac{2m_b+m_a}{2(m_b+m_a)}\sum_{l_p' l_q'}\int_0^\infty dq'\int_{|s-[(2m_a+m_b)/(2m_b+m_a)]q'|}^{s+[(m_a+m_b)/(2m_b+m_a)]q'}dp'\,\frac{p'q'}{s}$$
$$\times\mathcal{V}^{(L)}_{23,23}(\kappa,\kappa_3';z_7)K^{3(L)}_{l_p' l_q'}(s,q';\mathfrak{z}_3)Q^{2a}_{l_p' l_q' L}(p',q')$$
$$+\frac{2m_a+m_b}{2(m_a+m_b)}\sum_{l_p' l_q'}\int_0^\infty dq'\int_{|s-[(m_a+m_b)/(2m_a+m_b)]q'|}^{s+[(m_a+m_b)/(2m_a+m_b)]q'}dp'\,\frac{p'q'}{s}\mathcal{V}^{(L)}_{23,23}(\kappa,\kappa_4';z_7)$$
$$\times K^{4(L)}_{l_p' l_q'}(s,q';\mathfrak{z}_4)Q^{4a}_{l_p' l_q' L}(p',q')\Bigg]. \quad (18g)$$

Expressions for $\mathfrak{Q}^{2a}_{l_p l_q L}$, $\mathfrak{Q}^{1a}_{l_p l_q L}$, and $\mathfrak{R}^{1b}_L$ in Eqs. (18c), (18d), and (18f) can be obtained from those of $Q^{2a}_{l_p l_q L}$, $Q^{1a}_{l_p l_q L}$ and R^{1b}_L in Eqs. (18b), (18a), and (18e) respectively, by changing the subscripts of the partial wave components $\mathcal{K}_{(\alpha,\beta)l_p}$'s, $\mathcal{V}^{(L)}_{\alpha,\beta}$'s, and z_i's according to the earlier prescription, given below Eqs. (15).

p_i''s and κ_i''s occurring in the set of Eqs. (18) can be expressed as follows:

$$p_1'=\left[p'^2-\frac{4m_a(m_a+m_b)}{(2m_a+m_b)^2}(q^2-q'^2)\right]^{1/2},$$

$$p_2'=\left[\frac{m_a(2m_b+m_a)}{m_b(2m_a+m_b)}p'^2-\frac{2m_a(m_a+m_b)^2}{m_b(2m_a+m_b)^2}q^2+\frac{2(m_a+m_b)^2}{(2m_a+m_b)(2m_b+m_a)}q'^2\right]^{1/2},$$

$$p_3' = \left[\frac{4m_a}{2m_a+m_b}\,\kappa'^2 - \frac{4m_a(m_a+m_b)}{(2m_a+m_b)^2}\,q^2 + \frac{m_a+m_b}{2m_a+m_b}\,s'^2\right]^{1/2},$$

$$p_4' = \left[\frac{(m_a+m_b)^2}{m_b(2m_a+m_b)}\,\kappa'^2 - \frac{2m_a(m_a+m_b)^2}{m_b(2m_a+m_b)^2}\,q^2 + \frac{m_a+m_b}{2m_a+m_b}\,s'^2\right]^{1/2},$$

$$\kappa_1' = \left[\frac{2m_a+m_b}{4m_a}\,p'^2 - \frac{m_a+m_b}{4m_a}\,s^2 + \frac{m_a+m_b}{2m_a+m_b}\,q'^2\right]^{1/2},$$

$$\kappa_3' = \left[\frac{m_a(2m_b+m_a)}{(m_a+m_b)^2}\,p'^2 - \frac{2m_a m_b}{(m_a+m_b)^2}\,s^2 + \frac{2m_b}{2m_b+m_a}\,q'^2\right]^{1/2};$$

p_5', p_6', p_7', p_8', κ_2', and κ_4' can be obtained by the replacement $m_a \leftrightarrow m_b$ in the expressions for p_1', p_2', p_3', p_4', κ_1', and κ_3', respectively.

The functions $I^{i(L)}_{l_p l_q, l_p' l_q'}$, $J^{i(L)}_{l_p l_q}$, and $K^{i(L)}_{l_p' l_q'}$ are defined by the expressions

$$I^{i(L)}_{l_p l_q, l_p' l_q'}(q,q';\mathfrak{x}_i) = (4\pi)^{3/2}(2l_q'+1)^{-1/2}\sum_{\mu\mu'}(-1)^{l_q+l_q'-\mu-\mu'}(l_p L\mu, -\mu|l_q 0)$$
$$\times(l_p' L\mu', -\mu|l_q'\,\mu'-\mu)Y^*_{l_p\mu}(\vartheta_i,0)Y_{l_p'\mu'}(\vartheta_i',0)Y_{l_q'\mu-\mu'}(\theta_i,0),$$

$$J^{i(L)}_{l_p l_q}(q,s';\mathfrak{y}_i) = (4\pi)(2L+1)^{-1/2}\sum_{\mu}(-1)^{l_p-\mu}(l_p L\mu, -\mu|l_q 0)Y^*_{l_p\mu}(\beta_i,0)Y_{L\mu}(\beta_i',0),$$

$$K^{i(L)}_{l_p' l_q'}(s,q';\mathfrak{z}_i) = (4\pi)(2l_q'+1)^{-1/2}\sum_{\mu'}(-1)^{l_p'-\mu'}(l_p' L\mu', 0|l_q'\mu')Y_{l_p'\mu'}(\sigma_i,0)Y^*_{l_q'\mu'}(\sigma_i',0),$$

where

$$\mathfrak{x}_1 \equiv \cos\theta_1 = \frac{1}{qq'}\left[\frac{2m_a+m_b}{2m_b}\,(p'^2-q^2)\,\frac{m_b}{2(2m_a+m_b)}\,q'^2\right],$$

$$\cos\vartheta_1 = \frac{[m_b/(2m_a+m_b)]q + q'\mathfrak{x}_1}{\{[m_b^2/(2m_a+m_b)^2]q^2 + q'^2 + [2m_b/(2m_a+m_b)]qq'\mathfrak{x}_1\}^{1/2}},$$

$$\cos\vartheta_1' = \frac{q + [m_b/(2m_a+m_b)]q'\mathfrak{x}_1}{\{q^2 + [m_b^2/(2m_a+m_b)^2]q'^2 + [2m_b/(2m_a+m_b)]qq'\mathfrak{x}_1\}^{1/2}},$$

$$\mathfrak{x}_2 \equiv \cos\theta_2 = \frac{1}{qq'}\left[\frac{2m_b+m_a}{2m_b}\,(p'^2-q^2) - \frac{m_b}{2(2m_b+m_a)}\,q'^2\right],$$

$$\cos\vartheta_2 = \frac{[m_a/(2m_a+m_b)]q + q'\mathfrak{x}_2}{\{[m_a^2/(2m_a+m_b)^2]q^2 + q'^2 + [2m_a/(2m_a+m_b)]qq'\mathfrak{x}_2\}^{1/2}},$$

$$\cos\vartheta_2' = \frac{q + [m_b/(2m_b+m_a)]q'\mathfrak{x}_2}{\{q^2 + [m_b^2/(2m_b+m_a)^2]q'^2 + [2m_b/(2m_b+m_a)]qq'\mathfrak{x}_2\}^{1/2}},$$

$$\mathfrak{y}_1 \equiv \cos\beta_1 = \frac{1}{qs'}\left[-\kappa'^2 + \tfrac{1}{4}s'^2 + q^2\right],$$

$$\cos\beta_1' = \frac{-[2m_a/(2m_a+m_b)]q + s'\mathfrak{y}_1}{\{[4m_a^2/(2m_a+m_b)^2]q^2 + s'^2 - [4m_a/(2m_a+m_b)]qs'\mathfrak{y}_1\}^{1/2}},$$

$$\mathfrak{y}_2 \equiv \cos\beta_2 = \frac{1}{qs'}\left[\frac{m_a+m_b}{2m_b}\,(-\kappa'^2+q^2) + \frac{m_b}{2(m_a+m_b)}\,s'^2\right],$$

$$\cos\beta_2' = \frac{-[(m_a+m_b)/(2m_a+m_b)]q + s'\mathfrak{y}_2}{\{[(m_a+m_b)^2/(2m_a+m_b)^2]q^2 + s'^2 - [2(m_a+m_b)/(2m_a+m_b)]qs'\mathfrak{y}_2\}^{1/2}},$$

$$\mathfrak{z}_1 \equiv \cos\sigma_1 = \frac{1}{sq'}\left[\frac{2m_a+m_b}{4m_a}\,(p'^2-s^2) - \frac{m_a}{2m_a+m_b}\,q'^2\right],$$

$$\cos\sigma_1' = \frac{-\{s + [2m_a/(2m_a+m_b)]q'\mathfrak{z}_1\}}{\{s^2 + [4m_a^2/(2m_a+m_b)^2]q'^2 + [4m_a/(2m_a+m_b)]sq'\mathfrak{z}_1\}^{1/2}},$$

$$\mathfrak{z}_3 \equiv \cos\sigma_3 = \frac{1}{sq'}\{[(2m_b+m_a)/2(m_a+m_b)](p'^2-s^2) - [(m_a+m_b)/2(2m_b+m_a)]q'^2\},$$

$$\cos\sigma_3' = \frac{-\{s+[(m_a+m_b)/(2m_b+m_a)]q'\mathfrak{z}_3\}}{\{s^2+[(m_a+m_b)^2/(2m_b+m_a)^2]q'^2+[2(m_a+m_b)/(2m_b+m_a)]sq'\mathfrak{z}_3\}^{1/2}} .$$

By the mutual replacement of m_a and m_b in the above expressions for $\cos\theta_1$, $\cos\vartheta_1$, $\cos\vartheta_1'$, $\cos\theta_2$, $\cos\vartheta_2$, $\cos\vartheta_2'$, $\cos\beta_1$, $\cos\beta_1'$, $\cos\beta_2$, $\cos\beta_2'$, $\cos\sigma_1$, $\cos\sigma_1'$, $\cos\sigma_3$, $\cos\sigma_3'$, one can get the corresponding expressions for $\cos\theta_3$, $\cos\vartheta_3$, $\cos\vartheta_3'$, $\cos\theta_4$, $\cos\vartheta_4$, $\cos\vartheta_4'$, $\cos\beta_3$, $\cos\beta_3'$, $\cos\beta_4$, $\cos\beta_4'$, $\cos\sigma_2$, $\cos\sigma_2'$, $\cos\sigma_4$, $\cos\sigma_4'$, respectively.

V. DISCUSSION

The infinite set of coupled integral equations (18) in two variables are the relevant equations for investigating the bound states of a four-particle system consisting of two distinct pairs of identical particles. For a system consisting of nucleons and hyperons, we have short range forces to deal with and it will be sufficient to consider a finite number of equations from this set. Since we are considering s wave interaction between any two particles, we note, in particular, that for a four-body state with $L=0$, contributions can come from states with l_p and l_q equal. In this case, it will not be a bad approximation to consider the Eqs. (18) with $l_p=l_q=0$ only. Even then, we have to solve a set of seven coupled integral equations in two variables while in case of four identical particles[13] the corresponding number of equations is two. In practice, therefore, one has to make some further approximations for the three-particle amplitudes. In a future work we shall use one such approximation to our Eqs. (18) for studying the bound states of some physical systems.

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