Control of repulsive core regularization

B. G. Giraud* and E. Osnes†
Service de Physique Théorique, Centre d'Etudes Nucléaires de Saclay, BP No. 2 - 91190, Gif-sur-Yvette, France
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It is shown that when a potential $V(r)$, which is local and repulsive for small r and singular at $r = 0$, is truncated into a potential bounded by $V(\epsilon)$ where ϵ is small, the corresponding two-body binding energy $E(\epsilon)$ differs from the exact binding energy by less than (const) $\times [V(\epsilon)]^{-1/2}$. A variational procedure is derived from this result.

[NUCLEAR STRUCTURE Singular repulsion cutoff parameter is variational parameter for modified Rayleigh-Ritz principle.]

I. INTRODUCTION

The two-body potentials which are most often used in molecular and nuclear physics contain frequently, in addition to the usual centrifugal barrier, a strong repulsion at short distances. Such potentials, which are implicitly assumed to be local in the repulsive inner region, are chosen either as infinite hard-core potentials [$V(r) = +\infty$ when $0 < r < a$] or as singular potentials which tend faster towards $+\infty$ than the centrifugal barrier $l(l+1)r^{-2}$ when $r \to 0$ [more explicitly, $r^2V(r) \to +\infty$].

Powerful methods[1-3] have been developed for the handling of the radial (two-body) Schrödinger equation containing such singular potentials. Several among them consist in making a specific choice of a family of regular potentials V_ϵ which depend on a regularizing parameter $\epsilon > 0$ in such a way that $V_\epsilon \to V$ when $\epsilon \to 0$. The solutions of the Schrödinger equation for V_ϵ must then be shown to converge,[2,4] in some sense, to the solutions of the initial Schrödinger equation (that containing V) when $\epsilon \to 0$.

The purpose of the present paper is to estimate, in a special case of broad interest, the behavior of the two-body ground state binding energy $E(\epsilon)$ corresponding to V_ϵ. It will be shown that $E(\epsilon)$ can become nearly constant when ϵ becomes small with respect to a typical range parameter of the problem. Physically, this amounts to proving that a singular repulsive potential may be truncated into a regular potential without inducing big changes in binding energies.

The special case under consideration is defined in Sec. II, where a few preliminary results are also established. Then Sec. III is devoted to an estimate of the behavior of $E(\epsilon)$ as a function of ϵ. A short discussion and a very simple numerical application are given in Sec. IV. Generalization and improvement of our results are proposed in Sec. V. A variational procedure is established in Sec. VI and we conclude in Sec. VII.

II. DEFINITIONS AND PRELIMINARY RESULTS

The (real) potential considered here is assumed to be local and positive in an "inner region" defined by $0 < r < a$, where r is the distance between the two particles under consideration. The radial Schrödinger equation reads, with proper units ($\hbar^2/2\mu = 1$),

$$-\psi''(r) + [V(r) - E]\psi(r) = -\int_a^\infty W(r, r')\psi(r')dr' , \tag{1}$$

where ψ, ψ'', V, and W are, respectively, the radial wave function (multiplied by r, as usual), its second derivative, the local part of the potential (including the centrifugal barrier, if any), and the nonlocal part of the potential. It is assumed that $W(r, r') = 0$ when $r < a$ (and $r' < a$), in order to restrict the Schrödinger equation to a purely *differential* equation in the inner region. Only that region will be considered in the following, and the right-hand side of Eq. (1) will thus be replaced by zero. It is further assumed that, in the inner region, except for $r = 0$ where $V(r)$ is defined as diverging, the positive function $V(r)$ is finite and has a first derivative $V'(r)$. There is no need at the present stage to assume that $V(r)$ is a monotonically decreasing function of r in the whole inner region, although this additional property is fulfilled in many physical situations.

All the above assumptions, namely inner locality, positivity, and existence of the first-order derivative, are satisfied by most practical cases in the literature. It is then useful to consider the family of regular potentials V_ϵ defined by

$$V_\epsilon(r) = \theta(\epsilon - r)V(\epsilon) + \theta(r - \epsilon)V(r)\,, \tag{2}$$

where $0 < \epsilon < a$ and the θ function is the usual step function [$\theta(x) = 0$ or 1 whether $x < 0$ or $x > 0$, respectively]. From Eq. (2) and Fig. 1 it is obvious that, as an operator, V_ϵ converges towards V when $\epsilon \to 0$. A straightforward differentiation of Eq. (2) with respect to ϵ defines the operator $dV_\epsilon/d\epsilon$ by

$$\frac{dV_\epsilon}{d\epsilon}(r) = \theta(\epsilon - r)V'(\epsilon)\,. \tag{3}$$

It is now assumed that Eq. (1) has at least one bound-state solution (normalized to 1) and that the same holds for the regularized equation

$$-\psi_\epsilon''(r) + [V_\epsilon(r) - E(\epsilon)]\psi_\epsilon(r) = -\int_a^\infty W(r,r')\psi_\epsilon(r')\,dr'\,. \tag{4}$$

Since $V_\epsilon \to V$ when $\epsilon \to 0$ and since V is repulsive, then both ψ_ϵ and $E(\epsilon)$ are expected to converge[2,4] towards ψ and E, respectively, except maybe for very pathological potentials. Such a convergence will be admitted here and the point of interest will rather be the behavior of $E(\epsilon)$. Since the normalization condition $\langle \psi_\epsilon \,|\, \psi_\epsilon \rangle = 1$ implies $\langle \psi_\epsilon \,|\, d\psi_\epsilon/d\epsilon \rangle + \langle d\psi_\epsilon/d\epsilon \,|\, \psi_\epsilon \rangle = 0$, it is trivial to show that

$$\frac{dE}{d\epsilon}(\epsilon) \equiv \frac{d}{d\epsilon}\,\langle \psi_\epsilon \,|H_\epsilon\,|\, \psi_\epsilon \rangle = \langle \psi_\epsilon \,|\, \frac{dV_\epsilon}{d\epsilon}\,|\, \psi_\epsilon \rangle\,, \tag{5}$$

where H_ϵ is the Hamiltonian from which Eq. (4) derives. The purpose of the coming section is to prove that the derivative of $E(\epsilon)$ can be controlled as a small number.

III. AN UPPER BOUND FOR THE REGULARIZATION ERROR

From Eqs. (3) and (5) one obtains

$$\frac{dE}{d\epsilon}(\epsilon) = V'(\epsilon)I(\epsilon)\,, \tag{6}$$

where

$$I(\epsilon) = \int_0^\epsilon |\psi_\epsilon(r)|^2 dr\,. \tag{7}$$

We are interested in finding an upper bound $B(\epsilon)$ to $I(\epsilon)$. If the product $|V'(\epsilon)|B(\epsilon)$ can be integrated, then $E(\epsilon)$ is an approximation to E with at most an error given by

$$|E - E(\epsilon)| \leq \Delta E \equiv \int_0^\epsilon |V'(\epsilon')|B(\epsilon')\,d\epsilon'\,. \tag{8}$$

If $V(r)$ is a monotonic function of r (and thus a decreasing function), the left-hand side of inequality (8) can be written as $E - E(\epsilon)$, without the absolute value symbol.

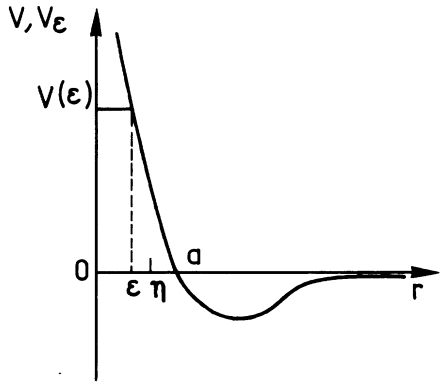

FIG. 1. The regularized potential $V_\epsilon(r)$ (heavy line) and the initial potential $V(r)$ (thin line).

As will be shown below, the bound $B(\epsilon)$ is obtained from the fact that Eq. (4) is a pure differential equation in the inner region. Indeed, when $0 < r < \epsilon$, one obtains from Eqs. (2) and (4)

$$\psi_\epsilon''(r) = k^2(\epsilon)\psi_\epsilon(r)\,, \tag{9}$$

where $k(\epsilon) = [V(\epsilon) - E(\epsilon)]^{1/2}$. Hence, when $0 < r < \epsilon$, the wave function is given by

$$\psi_\epsilon(r) = A\sinh[k(\epsilon)r]\,, \tag{10}$$

where A is a suitable (and positive) normalization factor and ψ_ϵ has been chosen as real and positive near the origin. Furthermore, when $\epsilon < r < a$, it is clear that the curvature of $\psi_\epsilon(r)$ serves to increase ψ_ϵ, because Eq. (4) reads

$$\psi_\epsilon''(r) = [V_\epsilon(r) - E(\epsilon)]\psi_\epsilon(r)\,, \tag{4'}$$

which means that $\psi_\epsilon''(r)$ and $\psi_\epsilon(r)$ have the same, fixed (positive) sign in the whole inner region since $V_\epsilon(r)$ and, *a fortiori*, $V_\epsilon(r) - E(\epsilon)$, are positive.

The first-order derivative $\psi_\epsilon'(r)$ is thus a monotonically increasing function in the whole inner region, which yields, when $\epsilon < r < a$,

$$\psi_\epsilon'(r) > \psi_\epsilon'(\epsilon) = Ak(\epsilon)\cosh[k(\epsilon)\epsilon] \tag{11}$$

and

$$\psi_\epsilon(r) > A\{\sinh[k(\epsilon)\epsilon] + (r - \epsilon)k(\epsilon)\cosh[k(\epsilon)\epsilon]\}\,. \tag{12}$$

The obvious meaning of Eq. (12) is that $\psi_\epsilon(r)$ is larger, when $\epsilon < r < a$, than the linear extrapolation which is provided by Eq. (10). These two results, Eq. (10) and inequality (12), are closely connected to the restriction of Eqs. (1) and (4) to pure differential equations, and provide the basis for the proof of an upper bound, which follows. An illustration of the argument is given in Fig. 2.

Since $\psi_\epsilon(r)$ obviously extends beyond $r = a$, the normalization condition for this wave function gives

$$1=\int_0^\infty \psi_\epsilon^2(r)\,dr>\int_0^\epsilon \psi_\epsilon^2(r)\,dr+\int_\epsilon^a \psi_\epsilon^2(r)\,dr$$
$$>A^2\left(\int_0^\epsilon \sinh^2[k(\epsilon)r]\,dr+\int_\epsilon^a \{\sinh[k(\epsilon)\epsilon]+(r-\epsilon)k(\epsilon)\cosh[k(\epsilon)\epsilon]\}^2 dr\right).$$

After explicitly performing the integrals which are present in inequality (13), one gets

$$A^2<\frac{1}{i(\epsilon)+j(\epsilon)}, \tag{14}$$

where

$$i(\epsilon)=[4k(\epsilon)]^{-1}\{\sinh[2k(\epsilon)\epsilon]-2k(\epsilon)\epsilon\} \tag{15}$$

and

$$j(\epsilon)=(a-\epsilon)\sinh^2[k(\epsilon)\epsilon]+k(\epsilon)\frac{(a-\epsilon)^2}{2}\sinh[2k(\epsilon)\epsilon]+k^2(\epsilon)\frac{(a-\epsilon)^3}{3}\cosh^2[k(\epsilon)\epsilon]. \tag{16}$$

Now, the quantity $I(\epsilon)$ under study is, as defined by Eq. (7), nothing but $A^2 i(\epsilon)$. Therefore, Eq. (14) yields

$$I(\epsilon)<\left[1+\frac{j(\epsilon)}{i(\epsilon)}\right]^{-1}, \tag{17}$$

which provides the desired upper bound. Actually, from Eqs. (15) and (16), it is trivial to simplify inequality (17) to

$$I(\epsilon)<\tfrac{3}{2}[k(\epsilon)(a-\epsilon)]^{-3}<B(\epsilon)\equiv\tfrac{3}{2}(a-\epsilon)^{-3}[V(\epsilon)]^{-3/2}. \tag{18}$$

The final result is thus, provided $0<\epsilon\leq\eta$, where η is any arbitrary positive number smaller than a $(0<\eta<a)$,

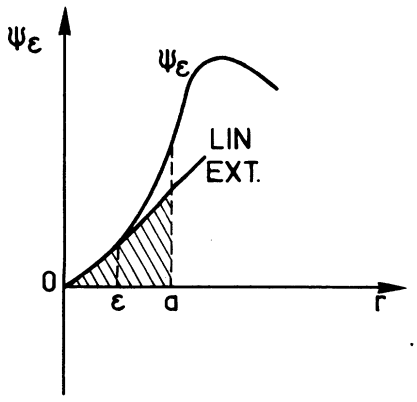

FIG. 2. The areas defined by the square of $\psi_\epsilon(r)$ when $0<r<\epsilon$ and the square of the linear extrapolation of $\psi_\epsilon(r)$ from $r=\epsilon$ when $\epsilon<r<a$ add up to less than 1.

$$\left|\frac{dE}{d\epsilon}(\epsilon)\right|<C\,|V'(\epsilon)\,|[V(\epsilon)]^{-3/2}, \tag{19}$$

where the constant $C\equiv\frac{3}{2}(a-\eta)^{-3}$ is a parameter depending on the physical system under consideration.

In the special case where $V(r)$ is a monotonically decreasing function of r, insertion of Eq. (19) into Eq. (8) yields, for $\epsilon=\eta$,

$$\Delta E=3(a-\eta)^{-3}[V(\eta)]^{-1/2}. \tag{20}$$

IV. DISCUSSION

It will be noticed that the mathematical argument which has been used in Secs. II and III is not restricted to singular potentials. The basic inequality (19) holds whether V diverges faster than r^{-2} or not. Even when V is finite, inequality (19) is valid. Furthermore, inequality (19) holds whether one deals with s waves or higher partial waves (the centrifugal barrier being included in V).

It will also be noticed that all bound states, and not just the ground state, fulfill the properties leading to inequality (19). This is because the argument employed in Secs. II and III makes use of simple properties of the differential equation, Eqs. (4) or (4′), which make no difference between ground state and excited bound states.

The result which has been obtained appears therefore to be of rather general value. However, it is really useful only when $V(r)$ diverges fast enough, as suggested by inequality (20). A brief numerical estimate is in order. Typically for nuclear physics, in the case of a cluster model with two heavy ions, one may consider parameter values as $a=2\eta=6$ fm, $V(\eta)=400$ MeV, $\hbar^2/2\mu=4$ fm^2 MeV, leading to $\Delta E=0.05$ MeV, which is a gratifying accuracy. For lighter ions, however, typical values such as $a=2\eta=2$ fm, $V(\eta)=400$ MeV, $\hbar^2/2\mu=10$ fm^2 MeV, yield $\Delta E=5$ MeV, which is not a sufficient accuracy for the regularization procedure.

V. FURTHER DISCUSSION

A first question to be raised is whether a smaller upper bound than that given by inequalities (19) and (20) can be found. It is known that a singular repulsive potential $V(r)$ generates a wave function $\psi(r)$ which decreases near the origin as $\exp\{-[V(r)]^{1/2}\}$. Such an exponential decrease of

the wave function indicates that inequality (20), although fairly satisfactory since it contains $a[V(\epsilon)]^{-1/2}$ decrease, might still be very weak and could be improved towards a stronger (exponential?) decrease. Indeed, a stronger decrease would help in reducing the orders of magnitude of ΔE which were estimated at the end of the previous section and it would thus allow a stronger cutoff of the potential.

Since accurate numerical procedures are available for solving Eq. (4′), it is interesting to use them and obtain "exact numerical" values of $E - E(\epsilon)$. Just to give an example, we have considered a two-nucleon s-wave with a potential

$$V(r) = A\left[\frac{1}{r^6} - B\frac{\exp(-\nu/r)}{r}\right],$$

with $A = 2.8$ MeV fm^6, $B = 31.8$ fm^{-5}, $\nu = 0.7$ fm^{-1}, and $\hbar^2/2\mu = 41.54$ MeV fm^2. Besides being singular this potential is fairly realistic, since it has a Yukawa-like tail taken from the one-pion-exchange potential. As shown by Table I, the discrepancy $E - E(\epsilon)$ does not exceed a few MeV or tenths of an MeV for reasonable cutoff. This is a much more gratifying order of magnitude than that obtained at the end of the previous section. Other numerical examples which have been tested give similar accuracy.

A second question is whether a similar result could be developed for nonlocal potentials. It is clear that a potential function $v(r, r')$ which would be positive for $r < a$ (and $r' < a$) and which would have a small and *finite* nonlocality ($v(r, r') = 0$ if $|r - r'| > \lambda$, where $\lambda \ll a$), would preserve the main properties of positivity which have been used in Secs. II and III.

A final problem to be solved is whether a generalization of inequality (19) can be found for the many-body problem with, say, N identical particles. The Hamiltonian is now, in a familiar notation,

$$H = \sum_{i=1}^{N} T_i + \sum_{i<j}^{N} V(r_{ij}), \tag{21}$$

and becomes, after regularization,

$$H_\epsilon = \sum_{i=1}^{N} T_i + \sum_{i<j}^{N} V_\epsilon(r_{ij}). \tag{22}$$

The analog of Eq. (5) reads

$$\frac{dE}{d\epsilon}(\epsilon) = \sum_{i<j}^{N} \langle \psi_\epsilon \left| \frac{dV_\epsilon}{d\epsilon}(r_{ij}) \right| \psi_\epsilon \rangle, \tag{23}$$

which, because of the symmetry of V_ϵ and the antisymmetry of ψ_ϵ in the particle coordinates, reduces to

$$\frac{dE}{d\epsilon}(\epsilon) = \tfrac{1}{2} N(N-1) \langle \psi_\epsilon \left| \frac{dV_\epsilon}{d\epsilon}(r_{12}) \right| \psi_\epsilon \rangle, \tag{24}$$

where only the contribution of *one* particle pair appears. One may conjecture that the matrix element on the right-hand of Eq. (24) is again bounded by the same upper bound as that shown by inequality (19). A most likely conjecture is thus, with $0 < \epsilon < \eta < a$,

$$\frac{dE}{d\epsilon}(\epsilon) < \tfrac{3}{2}(a-\eta)^{-3}\, \tfrac{1}{2} N(N-1) \left|V'(\epsilon)\right| [V(\epsilon)]^{-3/2}. \tag{25}$$

A rigorous proof of this conjecture would demand an analysis of the properties of partial derivative equations in a region of strongly repulsive potentials. Only a physical, and intuitive, argument is offered here: since the particles strongly repel each other at short distances, the occurrence of more than two particles in a narrow neighborhood is statistically unlikely, and therefore one may suspect that an approximation, which assumes that the contribution to Eq. (23) of each pair is the same as if the pair were independent, is likely to be valid. If furthermore saturation properties hold for Eq. (23), as they do for the total binding energy, the coefficient $\frac{1}{2}N(N-1)$ in inequality (25) might even be too large and might have to be replaced by a smaller (linear?) coefficient.

TABLE I. Discrepancy $E - E(\epsilon)$ for the potential chosen in Sec. V. The "exact" binding energy is $E = -2.22$ MeV. There is only one bound state.

$V(\epsilon)$ (MeV)	ϵ (fm)	$E - E(\epsilon)$ (MeV)
800	0.372	0.91
400	0.405	1.5
200	0.435	2.2
100	0.459	2.8
50	0.476	3.3

VI. VARIATIONAL PROCEDURE

In this section, it is assumed that, when $0 < r < \eta$, the potential $V(r)$ is a monotonic decreasing function of r. The operator H_ϵ [defined by Eq. (22)] is then a decreasing function of ϵ, and the eigenvalue $E(\epsilon)$ corresponding to the ground state is also a decreasing function of ϵ. If ϕ is any trial wave function for the ground state of the N-body problem, a most traditional variational principle reads

$$E(\epsilon) = \inf_{\phi} \frac{\langle \phi | H_\epsilon | \phi \rangle}{\langle \phi | \phi \rangle} . \tag{26}$$

Since $E(\epsilon)$ decreases when ϵ increases from 0 to η, the ground-state energy E for the initial Hamiltonian defined by Eq. (21) is

$$E = \sup_\epsilon E(\epsilon) , \quad 0 < \epsilon < \eta , \tag{27}$$

provided, of course, that convergence is assumed.

Insertion of Eq. (26) into Eq. (27) readily gives

$$E = \sup_\epsilon \left[\inf_{\phi} \frac{\langle \phi | H_\epsilon | \phi \rangle}{\langle \phi | \phi \rangle} \right] , \quad 0 < \epsilon < \eta , \tag{28}$$

which clearly means that one must *first* minimize with respect to the trial wave functions and *then* maximize with respect to the regularization parameter. As shown more explicitly in Fig. 3, the double variational principle defined by Eq. (28) is of little value, for it may lead to very poor upper bounds of E. One might then suggest using Eq. (26), rather than Eq. (28), provided ϵ is taken to be any "reasonable" value ϵ_0, to be selected by a suitable criterion which has yet to be defined. As shown in Fig. 3, it is clear, however, that the upper bound of $E(\epsilon_0)$ can be anything, and might even be a lower bound of E. (When the cutoff is nonmonotonic, there may exist a value ϵ_0 which gives the exact result, as for the separation distance method of Moszkowski and Scott.[5] But this ϵ_0 does not seem to follow from a variational principle.)

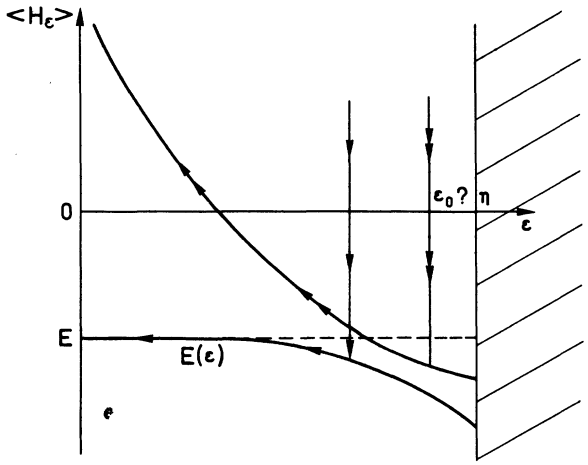

FIG. 3. The lower curve is the plot of the ground-state energy $E(\epsilon)$ of the Hamiltonian H_ϵ. The upper curve is what can be obtained at best from the minimization of $\langle H_\epsilon \rangle$ with respect to trial functions ϕ which, in practice, span only a very limited subspace of the Hilbert space. The trajectory with single arrows is that suggested ideally by Eq. (28). The trajectory with double arrows shows how, in practice, a minimization with respect to ϕ might give an unexpected lower bound to the exact energy and how it becomes meaningless to maximize with respect to ϵ.

It is possible to transform Eq. (28) into a useful variational principle, provided a counter term $F(\epsilon)$ is added to $E(\epsilon)$. Let $M(\epsilon)$ be an upper bound to the modulus of the derivative $dE(\epsilon)/d\epsilon$ and define

$$F(\epsilon) = \int_0^\epsilon M(\epsilon')\, d\epsilon' . \tag{29}$$

Since, under the condition of convergence of $E(\epsilon)$ towards E when $\epsilon \to 0$, which was admitted above, one may write

$$E = E(\epsilon) - \int_0^\epsilon \frac{dE}{d\epsilon}(\epsilon')\, d\epsilon' , \tag{30}$$

the very definition of $M(\epsilon)$ as an upper bound of $-(dE/d\epsilon)(\epsilon)$ yields

$$E \leq E(\epsilon) + F(\epsilon) . \tag{31}$$

Furthermore, $F(\epsilon)$ vanishes when $\epsilon \to 0$, and Eq. (31) can be written as

$$E = \inf_\epsilon [E(\epsilon) + F(\epsilon)] , \quad 0 < \epsilon < \eta . \tag{32}$$

Insertion of Eq. (26) into Eq. (32) yields

$$E = \inf_{\epsilon, \phi} \left[\frac{\langle \phi | H_\epsilon | \phi \rangle}{\langle \phi | \phi \rangle} + F(\epsilon) \right] , \quad 0 < \epsilon < \eta , \tag{33}$$

which means that one must minimize twice: (i) with respect to the trial function, and (ii) with respect to the regularization parameter. Both minimizations can be dealt with in an arbitrary order, which is an important advantage of Eq. (33) with respect to Eq. (28).

As illustrated by Fig. 4, the variational principle

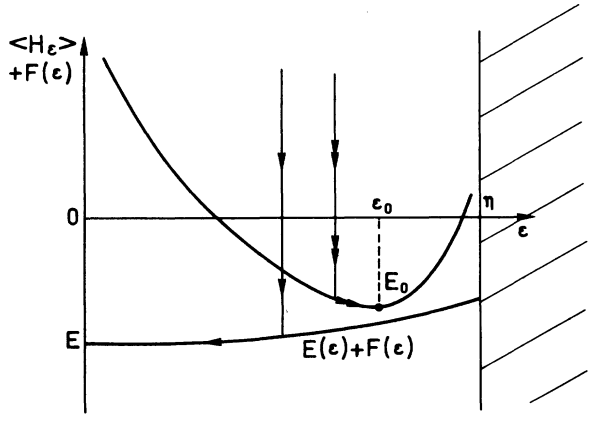

FIG. 4. The lower and upper curves show the result of an *ideal* and *practical* minimization, respectively, of $\langle H_\epsilon \rangle + F(\epsilon)$ with respect to ϕ. Then minimization with respect to ϵ gives as ideal and practical termination points, respectively, the values E and E_0 which are close to each other and satisfy $E < E_0$. An optimal cutoff parameter ϵ_0 is thus defined, while it had to be guessed in Fig. 3.

defined by Eq. (33) is much more convenient than that defined by Eq. (28). However, the value ϵ_0 for which the minimum is reached does not necessarily define an obvious best "effective" interaction V_{ϵ_0}, since it is the sum of the expectation value $\langle H_\epsilon \rangle$ and the counter term $F(\epsilon)$ which has been minimized. Nonetheless, the value ϵ_0 is likely to define a reasonable effective interaction. Similarly, the wave function ϕ_0 for which the minimum is reached might be a tolerable approximation to the exact ground state.

If the conjecture of inequality (25) is verified, Eq. (33) becomes

$$E = \inf_{\epsilon,\phi} \left\{ \frac{\langle \phi | H_\epsilon | \phi \rangle}{\langle \phi | \phi \rangle} + 3(a-\eta)^{-3} \frac{N(N-1)}{2} [V(\epsilon)]^{-1/2} \right\}. \quad (34)$$

One may also conjecture that a reasonable counter term is obtained by multiplying by $\frac{1}{2} N(N-1)$ the "exact numerical" two-body counter term, such as that shown in Table I for the potential considered in Sec. V. If there are several bound states, there will be as many two-body counter terms, and one might select the largest one.

VII. CONCLUSION

The main result of this paper is the possibility of having a variational principle with regularized Hamiltonians H_ϵ derived from an initially singular, local interaction V which is repulsive at short distances. Such a result has been achieved through the introduction of an additive term $F(\epsilon)$ which (over)compensates for the shift of the binding energy due to the regularization of potential.

An estimate of the counter term has been rigorously derived in the case of the two-body problem. It leads to a reasonable conjecture for the case of the N-body problem.

While most of the literature of singular potentials deals with two-body scattering and some of the literature with two-body bound states, there seems to be only few papers in which the study of N-body bound states generated by singular potentials is undertaken along the same mathematical lines as those used for the two-body problem. It is well known[1] that the influence of a singular potential on the behavior of phase-shifts at high energies is very strong. As suggested by the present work, and especially the fast decrease of $[V(\epsilon)]^{-1/2}$ when $\epsilon \to 0$ in inequalities (19) and (25), the binding energies may be less affected, and the problem of bound states with a repulsive singular potential seems to be reducible to the one with a regular (truncated) potential, at the cost of a small error only.

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*Also Ministère de l'Industrie et de la Recherche, Direction des Mines, Paris, France.

† Permanent address: Institute of Physics, University of Oslo, Oslo 3, Norway. Also Department of Physics, State University of New York at Stony Brook, New York 11794 [supported by USERDA contract No. AT (11-1)-3001].

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