Threshold charged-pion photoproduction and radiative pion capture. II. $\pi^- d \to \gamma nn$*

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(Received 22 February 1977)

The transition operator for threshold charged-pion photoproduction and radiative pion capture is utilized in its nonrelativistic limit to calculate the absolute rate for the $\pi^- d \to \gamma nn$ reaction from a $1s$ atomic orbital in the impulse approximation. Sensitivity to assumptions concerning the initial state of the $\pi^- d$ system and the short-range character of the n-n wave function in the final state is discussed, as is the relevance of the absolute rate to the determination of the n-n scattering length from this reaction.

[NUCLEAR REACTIONS $^2H(\pi^-, \gamma)2n$; calculated capture rate; n-n scattering length; impulse approximation.]

I. INTRODUCTION

The usefulness of the $\pi^- d \to \gamma nn$ reaction in the determination of the n-n scattering length was discussed previously.[1] In that paper, only the shape of the spectrum was utilized, as has been the practice in all of the experiments to date.[2,3] In the present work, we attempt to estimate the absolute rate for the $\pi^- d \to \gamma nn$ reaction relative to the $\pi^- p \to \gamma n$ rate. The latter rate can be inferred[4] from the very precisely measured Panofsky ratio[5] and the known value[6] for the pion-nucleon scattering length combination $|a_1 - a_3|$. We discuss the sensitivity of our rate calculation to variations in the model assumptions, both in the initial state and the final state. We also discuss the relevance of the absolute normalization of the spectrum to the extraction of the n-n scattering length.

In Sec. II we outline briefly the model which we use. In Sec. III we present results for the absolute rate of the radiative capture reaction, and we discuss the uncertainties, including the dependence upon the n-n scattering length and effective range. We consider the absolute normalization for the photon spectrum in the kinematically incomplete experiment in Sec. IV. In Sec. V we discuss the absolute normalization for the coincidence geometries. We state our conclusions in Sec. VI.

II. THEORETICAL MODEL

We assume that the transition operator discussed in Ref. 4 is adequate to describe, in an impulse approximation treatment, the $\pi^- d \to \gamma nn$ reaction at rest for a composite deuteron. Neglecting for the moment the $(k/M)^2$ type corrections, that operator can be expressed as

$$\hat{O}^- = \frac{2\pi i}{\sqrt{k m_\pi}} \bar{A} \tau^- \vec{\sigma} \cdot \vec{\epsilon} \left[1 + \frac{1}{2M}(\vec{p}_p - \vec{p}_n) \cdot \hat{k}\right], \quad (1)$$

where $\bar{A}^2 = 23.0 \times 10^{-4}$ fm^2, $\vec{\epsilon}$ is the photon polarization, $\vec{\sigma}$ operates on the proton spinor, m_π is the pion mass, M is the nucleon mass, k is the photon energy, τ^- is the isospin operator transforming a proton into a neutron, and $\vec{p}_p$ and $\vec{p}_n$ are the momenta of the proton before and after the transition. We should mention that the $\sqrt{1/k}$ from the boson normalization of the photon is approximated as $\sqrt{1/m_\pi}$ by some authors; such an approximation is not serious near threshold, where $E_\gamma \approx m_\pi$, but it is not valid in a rate calculation which is an integral over the entire spectrum.

Thus, our perturbing Hamiltonian takes the simple form

$$H' = \hat{O}^- e^{i\vec{k}\cdot\vec{r}/2}, \quad (2)$$

where the $\vec{r}/2$ describes the position of the interacting proton. For completeness we quote the expression for the absolute rate:

$$\tau_{1s}^{-1} = \frac{c}{(2\pi)^4} \int d^3k \sum_{\text{pol}} \overline{\sum_{\substack{\text{deut}\\ \text{spin}}}} |\mathfrak{M}(k)|^2 \frac{4}{a_B^3} Mp, \quad (3)$$

where the Bohr radius in the pion wave function is $(\alpha = 1/137.0)$

$$a_B = \frac{1}{\alpha \mu_{\pi d}}, \quad \mu_{\pi d} = \frac{2m_\pi M}{2M + m_\pi},$$

and p is the n-n relative momentum in the final state. The matrix element is

$$\mathfrak{M}(k) = \int dr\, \psi_d(r) \frac{\tilde{\phi}_\pi(r)}{\phi^1(0)} H' \psi_{nn}(r),$$

where the pion and nucleon-nucleon wave functions are discussed below. We note that our computer code did limit to the proper proton rate as the deuteron binding was taken to zero.

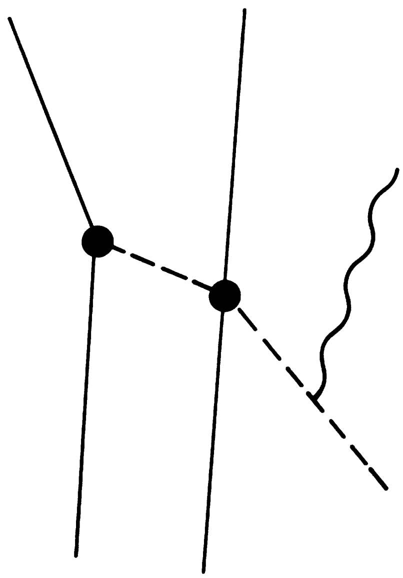

FIG. 1. Diagram of a process not included in the present calculation, as discussed in Sec. II.

In assuming an impulse approximation operator, we are clearly neglecting diagrams of the type shown in Fig. 1. However, such far-off-shell processes should not contribute significantly to the peak region of the spectrum, and the corrections should be of the same order of magnitude as the corrections to the second order on-shell pion scattering discussed below, which are estimated to be less than 6%. It is true, though, that our estimates of the absolute magnitude of the peak of the spectrum and our estimates of the shape of the spectrum near the peak should be more reliable than our estimates of the total rate.

Because we are dealing with an isotope of hydrogen, the Stark mixing experienced by the pionic atom insures that the capture occurs from an s orbital.[7] The atomic effects come from differences in the atomic radial wave function over a distance comparable to the nuclear radius. Since s-state atomic wave functions are essentially constant over the nucleus for principal quantum numbers $n \leq 12$, the effect upon spectral shapes is small. However, the variation in the normalization constant [i.e., the value of $\phi_\pi^n(0)$] with n will affect the absolute rate. It is because we do not know in general the distribution of principal quantum numbers for the atomic systems leading to capture (i.e., it differs markedly from liquid to gas and with pressure in a gas) that we shall compute a rate for capture from the 1s orbital. In this case, the principal modification of the wave function of the bound pion from its constant value over the nucleus is due to the presence of the neutron in the deuteron. For this modification a Born approximation estimate was made using an optical potential consisting of a π-n amplitude folded with the neutron density within the deuteron. Only s-wave π-n scattering was considered, since it dominates near zero energy.[1] The pion wave function modification, for π^-n scattering length $a_{\pi n}$, is

$$\frac{\tilde{\phi}_\pi(r)}{\phi_\pi^1(0)} = 1 + a_{\pi n}\left[\frac{1}{r}\int_0^r r'^2\rho(r')dr' + \int_r^\infty r'\rho(r')dr'\right]. \quad (4)$$

This leads to a maximum reduction of some 15% at the origin, decreasing with increasing r.

The nuclear model employed in our analysis is described in detail in Ref. 1. We review below the salient points and discuss briefly the additional model variations that are considered here.

To represent the deuteron in the initial state we use as a standard the reduced radial wave function $[\psi_d(r)]$ generated from the Reid soft-core (RSC) potential,[8] including the D state. To test the sensitivity of the rate to this model assumption, we have studied modifications of the following types: (i) alteration of the deuteron D-state probability; (ii) distortion of the radial shape by means of unitary transforms of the class

$$\begin{aligned} U &= 1 - 2|g\rangle\langle g|, \\ \langle r|g\rangle &= Cr(1-\beta r)e^{-\alpha r}, \end{aligned} \quad (5)$$

where C is a normalization constant (see, for example, Ref. 9); (iii) alteration of the radial function by requiring a node at $r = 0.4$ fm as well as at the origin [referred to as the forbidden-state (FS) wave function].[10]

The largest uncertainty in the calculation resides in the description of the reduced radial n-n wave function in the final state $[\psi_{nn}(r)]$. The short-range character of the N-N interaction is reflected in the small-distance behavior of the scattering state coordinate-space wave function. In order to easily incorporate the effects of the strong repulsion believed to be present in the N-N force as well as to maintain control over the asymptotic region so that the effect of varying a_{nn} could be studied, we have utilized the procedure of Picker, Redish, and Stephenson[11] (PRS). A local potential is assumed to be known beyond some distance R (we used typically the RSC potential beyond 1.4 fm); the phase shift is specified; the Schrödinger equation is integrated inward to obtain $\psi_{nn}(r)$ in the interval (R, ∞). The interior of $\psi_{nn}(r)$ is then parametrized by a low order polynomial satisfying specified boundary conditions at the origin and matched to the external wave function and a given number of its derivatives at R. For our standard model, a fifth order polynomial was used: the magnitude and first two derivatives were set to zero at the origin and matched at R. As in Ref. 1, the exterior radial wave function was approximated by the Born term of the integral equation described as Eq. (19) of Ref. 11, an approximation sufficiently accurate for our purposes. We assume in our standard model that all partial waves higher than $l=0$ are plane waves. For s-waves we determine the phase shift as a function of energy using an effective range expansion below $E_{nn}=20$ MeV and the phase shifts of MacGregor-Arndt-Wright, MAW X[12] above that energy.

We test the sensitivity of the rate to the final-state model assumptions by modifying $\psi_{nn}(r)$ in various ways: (i) adding a term of the form $\eta r^3(r-R)^3/p$ to the parametrization of the interior of the PRS wave function and varying η by matching it to a third derivative at R; (ii) applying the same rank-two unitary transformation to the scattering state as was described for the bound state; (iii) making the forbidden state (FS) modification to the scattering wave function (i.e., requiring a node at $r=0.4$ fm as well as at the origin).

III. ABSOLUTE RATE

Using the transition operator defined in Eq. (1) and the code utilized in our previous spectral shape analysis,[1] we can easily make an impulse approximation estimate of the ratio of the rates for radiative capture of a pion from the 1s state by the deuteron and hydrogen. Neglecting the $(k/M)^2$ corrections discussed in Ref. 4 (in particular those of the anomalous moments), we obtain

$$\frac{\tau^{-1}(\pi^- d\to\gamma nn)}{\tau^{-1}(\pi^- p\to\gamma n)}=0.87.$$

In Table I we present the percentage variations in the impulse approximation rate estimates [without $(k/M)^2$ corrections] for the $\pi^- d\to\gamma nn$ reaction obtained by varying the model assumptions discussed in Sec. II. First, we point out the insensitivity of the rate to reasonable uncertainties in the n-n scattering length and effective range. Variation of a_{nn} from -15 to -20 fm or r_{nn} from 2.6 to 3.0 fm changes the rate by less than 1%. Thus the capture rate is not sensitive to these scattering parameters, especially considering the estimated uncertainty in the actual theoretical rate. Likewise, altering the initial-state deuteron wave function, final-state n-n wave function, or both, by imposing the forbidden-state model condition of an additional node at 0.4 fm, has little or no effect upon the rate. Additionally, improving the PRS final-state n-n wave function by determining η through matching of a third derivative instead of setting it to zero shows no significant effect upon the rate. It is clear that omitting the D-state nature of the deuteron can produce a noticeable effect; however, such a model is not considered realistic and is only included here to demonstrate that point. The model change to $[\tilde{\phi}_\pi(r)/\phi_\pi^1(0)]\equiv 1$ indicates that the first order correction to the hydrogenic pion atomic wave function, a correction that is included in our standard model, is only 18%. The next order correction is therefore expected to be less than 4%, and the entire error in our use of this first order (in the rescattering correction) wave function should be less than 6%. Likewise, the off-shell effects which we have neglected should also be less than this 6%; i.e., a correction to the second order absorption process which is included in our standard model result.

The rate ratio quoted above was based upon the transition operator defined in Eq. (1). If one assumes that the $(k/M)^2$ corrections to that operator just scale for the proton and deuteron rates, then the ratio can be combined with the inferred proton rate[4] of $(4.94\pm0.10)\times10^{14}$ sec^{-1} to obtain an estimate of the actual deuteron capture rate

$$\tau^{-1}(\pi^- d\to\gamma nn)=(4.3\pm0.5)\times10^{14}\ \text{sec}^{-1}.$$

The conservative error estimate quoted is com-

TABLE I. Percentage variation in the impulse approximation rate for $\pi^- d\to\gamma nn$ at rest for different model modifications. The standard model with which comparison is made utilized $a_{nn}=-16.0$ fm and $r_{nn}=2.8$ fm.

Model change	$\Delta\tau^{-1}$ (%)	Model change	$\Delta\tau^{-1}$ (%)	Model change	$\Delta\tau^{-1}$ (%)
$a_{nn}=-15$	-0.2	FS for ψ_d	-0.1	$\alpha=5,\ \beta=2$ for ψ_d	-2.0
$a_{nn}=-17$	$+0.2$	FS for ψ_{nn}	-0.1	$\alpha=5,\ \beta=2$ for ψ_{nn}	-1.9
$a_{nn}=-18$	$+0.4$	FS for both	$+0.1$	$\alpha=5,\ \beta=2$ for both	-0.1
$a_{nn}=-20$	$+0.7$	η from ψ_{nn}'''	$+0.8$	$\alpha=3,\ \beta=2$ for ψ_d	-37.4
$r_{nn}=2.6$	-0.6	$P_D\equiv 0$	$+7.2$	$\alpha=3,\ \beta=2$ for ψ_{nn}	-32.7
$r_{nn}=2.8$	$+0.4$	$\frac{\tilde{\phi}_\pi(r)}{\phi_\pi^1(0)}\equiv 1$	17.9	$\alpha=3,\ \beta=2$ for both	-4.0
				$\alpha=3,\ \beta=4$ for both	-3.9

prised primarily of the N-N and off-shell π-N related uncertainties discussed above as well as the uncertainty in the scaling of the model dependent $(k/M)^2$ contributions.

The more interesting variations are perhaps those produced by introducing unitary transformations of the N-N wave functions. Short-range alteration of either the initial or final state (i.e., requiring the transformed function to heal to the original function inside of 2 fm) as is illustrated by the $\alpha = 5$, $\beta = 2$ example does not significantly (< 2%) alter the rate. It is also clear that, when the same unitary transformation is made in both the initial and final state, the rate is essentially unchanged. Hence, the rate is not particularly sensitive to short-range uncertainties in the model wave functions. Larger changes in the rate can be achieved if longer-range modification of either the initial-state deuteron wave function or the final-state n-n wave function is allowed, as is illustrated by the $\alpha = 3$, $\beta = 2$ result. For these cases the rate is reduced by some 35%. However, when the same long-range unitary transformation is applied simultaneously to the initial and final state, the change in the rate is less than 5%. This result is not surprising since $\frac{1}{2}k$ in Eq. (2) is at most $\frac{1}{3}$ fm^{-1}, such that $U^\dagger H' U \approx H'$. Therefore, we can say that rate measurements are able to rule out long-range unitary transformations *if* they are postulated to exist in only the $T = 1$ or $T = 0$ channel but not both.

The rate estimate here differs in several aspects from that quoted in Ref. 13. First, our value[4] of $0.034 m_\pi^{-1}$ for $\bar{A}$ is somewhat larger than their $0.032 m_\pi^{-1}$. Second, our estimate of the rate reduction due to pion scattering effects in the initial state is larger; we assume a double scattering model in which there is a $\pi^- n$ scattering prior to absorption of the pion by the proton, and their

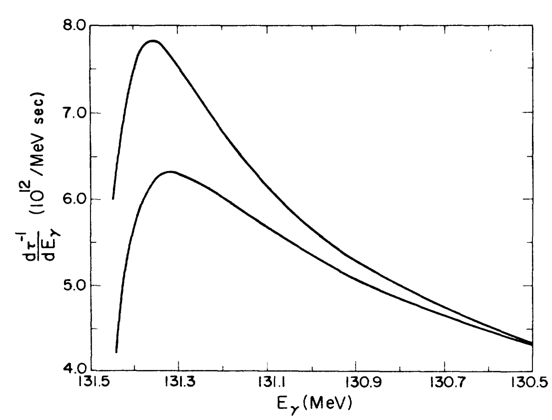

FIG. 2. Energy spectrum for detection of the photon only; upper curve is for $a_{nn} = -20$ fm and lower curve is for $a_{nn} = -15$ fm.

optical model assumption averages the initial-state scattering over both $\pi^- n$ and $\pi^- p$, which tend to cancel. Third, we have included a $(k/M)^2$ type correction, which further increases our calculated rate compared to theirs. The differences due to the first and second points are of opposite sign and essentially cancel. Even considering the difference due to the third point, there is a 7% discrepancy between our rate estimate and that of Ref. 13. This is possibly due to treatment of three-body phase space, since our correction of the static Kroll-Ruderman result would not contain a $1/(1 + m_\pi/2M)$ factor.

IV. KINEMATICALLY INCOMPLETE GEOMETRIES

Because the only kinematically incomplete experiment which uniquely relates the n-n relative momentum (and therefore the scattering length) to the measured spectrum is that in which the photon is detected, we shall limit discussion to that geometry.[2] In this configuration, the scattering length affects the spectrum primarily within the first few hundred keV of threshold, the region which is sensitive to $p \leq 20$ MeV/c. For that reason, we are interested only in the first MeV of the spectrum.

In Fig. 2 we compare absolute spectra in our impulse approximation for n-n scattering length assumptions of -15 and -20 fm. One can see that the differences occur primarily within the first 200 keV of the spectrum. An absolute measurement of better than 1% would be required to extract a_{nn} to an accuracy of 0.5 fm. The theoretical uncertainties in the absolute normalization are considerably greater than this 1%.

A shape analysis of the spectrum is clearly required, if one is to extract useful information about a_{nn} from the kinematically incomplete geometry. The peak position is sensitive to a_{nn}, as is the shape. However, the accuracy required to pin down the peak is almost prohibitive. If one were to consider the shift in the peak position as a function of a_{nn}, one would need an experimental resolution of better than 10 keV at a photon energy of 130 MeV in order to determine a_{nn} to an uncertainty of 0.5 fm. The resolution required in an analysis of the shape of the first MeV away from threshold is not 10 keV but more like 100 keV. Even though such a resolution would be difficult to achieve, such a shape analysis would be free of a sizable uncertainty in the theoretical estimate of the absolute magnitude.

V. COINCIDENCE GEOMETRIES

Kinematically complete experiments would appear to be better suited to the task of making useful absolute measurements. At least one neutron is detected, and the required percentage resolution is more easily achieved for such a low energy neutron than is possible for the photon of 130 MeV. Various coincidence geometries are available. One may detect the photon and one neutron.[14] One may detect both neutrons, either in a symmetric configuration in which the detectors are equidistant from the target[3] or in an asymmetric configuration in which the detectors are at different distances.[15] Of course, in these latter experiments the photon may be used as a start pulse.

The absolute rate is, however, not particularly useful in the analysis of any of these kinematically complete experiments. The absolute measurement cannot be justified because of the uncertainties in the theoretical calculation, with which comparison must be made. Shape analysis of the spectrum is a better approach, where it seems possible to extract a_{nn} to an uncertainty of less than 0.3 fm.[1] The codes exist to analyze data from any of the above mentioned geometries as it becomes available.

As an example of the advantage of shape analysis over the absolute rate measurements, let us consider the effect of the unitary transformations discussed in Sec. III upon the scattering length extracted from test spectra generated by using the unitarily transformed wave functions. The analysis of the spectra is done in terms of our standard model. Recall that the effect of the unitary transformations upon the rate was in most cases much greater than the effect of varying a_{nn} from -15 to -20 fm, when the transform was applied to both the initial-state deuteron wave function and the final-state n-n wave function in the same calculation. In Table II we present results for the extracted value of a_{nn} from spectra generated with $a_{nn} = -16.0$ fm. (We have chosen the symmetric, two-neutron geometry for this example; θ_{nn} is the opening angle subtended by the two neutron detectors.) It is clear that for the smaller opening angles of 5° and 10°, which are most sensitive to a_{nn}, the effect of the unitary transformation upon the shape analysis determination of a_{nn} is minimal, in contrast to the effect seen in the absolute rate calculations in Sec. III. The larger effect seen at $\theta_{nn} = 30°$ is due to the fact that the spectrum in that case does not significantly sample $p < 20$ MeV/c and is therefore not as sensitive to the scattering length.[1] In addition, we illustrate what happens when the unitary transformation appears in only the scattering state: the error in the extracted value of a_{nn} is significantly larger, being approximately 0.3 fm at $\theta_{nn} = 5°$. However, the effect is not nearly so large as was previously found for the absolute rate.

TABLE II. The n-n scattering length extracted from spectra calculated with unitarily transformed wave functions having $a_{nn} = -16.0$ fm, $r_{nn} = 2.8$ fm. The Coulomb corrected p-p scattering length was read from Fig. 1 of Sauer (Ref. 9).

α	β \ θ_{nn} (deg)	a_{nn} (−16 fm assumed) 5	10	20	30	a_{pp}
3	1	−16.052	−16.064	−16.123	−16.234	−17
3	2	−16.034	−16.041	−16.086	−16.177	−33
3	3	−16.018	−16.022	−16.043	−16.095	−26
3	4	−16.018	−16.022	−16.046	−16.096	−25
		Transform in scattering state only				
3	2	−15.693	−15.497	−14.951	−13.851	−33

For comparison, we also indicate the effect of such unitary transformations upon the extraction of the Coulomb corrected p-p scattering length from low energy p-p scattering as reported by Sauer.[9] It is clear that the effect of the unitary transformation upon the extraction of a_{nn} from the $\pi^- d \to \gamma nn$ spectrum is much less than the corresponding effect upon the extraction of a_{pp} from low energy p-p scattering. This is due to the fact that the absolute rate for the $\pi^- d \to \gamma nn$ reaction is not involved; i.e., the $\pi^- d \to \gamma nn$ spectrum shape is not sensitive to short-range uncertainties in the N-N wave functions, whereas the Coulomb effect in p-p scattering depends very sensitively upon the short-range character of the N-N scattering wave function.

VI. SUMMARY AND CONCLUSIONS

We have calculated the ratio of rates for radiative capture of a 1s state pion by the deuteron and the proton. We have used this ratio and the inferred experimental rate in hydrogen to estimate

$$\tau^{-1}(\pi^- d \to \gamma nn) = (4.3 \pm 0.5) \times 10^{14}\ \text{sec}^{-1}.$$

Furthermore, we have discussed the reasons that theoretical uncertainties in the absolute normalization make absolute measurements in any of the coincidence geometries less useful than precise determinations of the shape of the spectrum. In particular, we have illustrated the case that short-range unitary transformations of the N-N wave functions have little effect upon the extraction of the n-n scattering length by means of such a shape analysis as opposed to the extraction of a Coulomb corrected p-p scattering length from low energy p-p scattering.

*Work performed under the auspices of the U. S. Energy Research and Development Administration.

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