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Off-shell Jost function for a superposition of Yukawa potentials†

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Integral representations for the s-wave, off-shell Jost function and half-off-shell T matrix for a superposition of Yukawa potentials are derived. A representation for the form factor or vertex function associated with a bound state or resonance produced by such a potential is also derived. The analytic structure of the various functions is discussed. The applicability of the form factor result to the $^1S_0$ virtual bound state of the two-nucleon system is pointed out, and a calculation of the form factor for the $^1S_0$ Reid potential is presented.

[NUCLEAR REACTIONS Off-shell Jost function, T matrix, and form factor for a superposition of Yukawa potentials; application to $^1S_0$ Reid potential.]

I. INTRODUCTION

The off-shell Jost function[1] is obtained from an irregular solution of an inhomogeneous form of the Schrödinger equation. In this equation two momenta k and q appear, where k is an on-shell momentum, related to the energy by $E=k^2$ (our units are such that $\hbar^2/2m=1$), and q is an off-shell momentum. When $q=k$ the inhomogeneous equation goes over into the Schrödinger equation, and the off-shell Jost function becomes the ordinary Jost function.[2] The half-off-shell T matrix can be expressed directly in terms of the off-shell Jost function.[1] A concise and thorough discussion of the T matrix, and its use in few and many body theories, is given in the review article by Srivastava and Sprung.[3]

Analytic expressions for the s-wave, off-shell Jost function have been derived for the square well,[1] exponential,[1] Hulthén,[1] Morse,[4] and Woods-Saxon[4] potentials. A momentum space formulation of the off-shell Jost function has been developed,[5] and various integral representations have been obtained for it.[5,6] It has been shown that it is an analytic function of q^2 except for a right hand cut, and that the discontinuity across this cut is directly related to a function which plays an essential role in Kowalski's[7] generalized Sasakawa method.

Here we shall derive two integral representations for the s-wave, off-shell Jost function produced by a superposition of Yukawa potentials. The method employed here is closely related to the technique developed by Martin[2,8] for studying the ordinary Jost function of such a potential. The representations obtained are similar in spirit to those of Brayshaw.[9] One of the representations shows clearly that the off-shell Jost function for this potential is an analytic function in the upper half of the complex q plane and possesses two branch cuts in the lower half of this plane. This representation leads directly to an integral formula for the half-off-shell T matrix, which reveals clearly its analytic structure in the momentum q. From this formula for the T matrix, we shall obtain an expression for the vertex function or form factor associated with the bound states or resonances produced by a superposition of Yukawa potentials. A similar representation for the bound state form factor has been obtained previously by other authors.[10] Our result shows clearly that the vertex function is an analytic function of q^2 except for a left hand cut, and that the position of the cut and its low q^2 discontinuity depend only on the tail of the potential, the position of the bound state or resonance, and its effective range. Numerical results are given for the form factor associated with the $^1S_0$ virtual bound state of the two nucleon system produced by the Reid potential.[11] Knowledge of this function, which has not been calculated previously, is necessary for constructing the unitary pole approximation[12] to the $^1S_0$ T matrix.

II. OFF-SHELL JOST FUNCTION

According to Eqs. (2.12), (2.15), and (2.20) of Ref. 1, the s-wave, off-shell Jost function is given by

$$f(k,q)=f(k,q,0)\,, \tag{1}$$

where $f(k,q,r)$ is the irregular solution of the equation

$$\left[k^2+\frac{d^2}{dr^2}-V(r)\right]f(k,q,r)=(k^2-q^2)e^{iqr}\,, \tag{2}$$

which satisfies the boundary condition

$$f(k,q,r)\underset{r\to\infty}{\sim}e^{iqr}\,. \tag{3}$$

The half-off-shell T matrix is given by[1] the expression

$$T(k,q;s)=\frac{f(k,q)-f(k,-q)}{\pi i q f(k)}\,, \quad s=k^2+i\epsilon\,, \tag{4}$$

where

$$f(k)=f(k,k) \tag{5}$$

and is the ordinary Jost function.[2] The function $f(k,q,r)$ is related to the irregular solutions[2] of the radial Schrödinger equation by

$$f(\pm k,r)=f(k,\pm k,r)\,. \tag{6}$$

For the potential we assume the form

$$V(r)=\int da\,\rho(a)e^{-ar}, \quad \rho(a)=0 \quad \text{for } a<a_0\,, \tag{7}$$

which reduces to an ordinary Yukawa potential if $\rho(a)$ is a constant. Following Martin,[2,8] it is natural to look for a solution of (2) in the form

$$f(k,q,r)=\left[1+\int da\, s(a,k,q)e^{-ar}\right]e^{iqr}\,. \tag{8}$$

Insertion of (8) into (2) leads to the following equation for $s(a,k,q)$:

$$[k^2+(iq-a)^2]s(a,k,q) = \rho(a)+\int_{a_0}^{a-a_0} da'\,\rho(a-a')s(a',k,q)\,, \tag{9}$$

where the integral term is absent for $a<2a_0$. It should be noted that (9) is really a recursion relation in that $s(a,k,q)$ can be determined on the interval $na_0<a<(n+1)a_0$ from its values on the interval $a_0<a<na_0$. In particular, we have

$$s(a,k,q)=0, \quad a<a_0\,,$$

$$s(a,k,q)=\frac{\rho(a)}{k^2+(iq-a)^2}\,, \quad a_0<a<2a_0\,,$$

$$s(a,k,q)=\frac{\rho(a)}{k^2+(iq-a)^2}+\int_{a_0}^{a-a_0} da'\,\frac{\rho(a-a')\rho(a')}{[k^2+(iq-a)^2][k^2+(iq-a')^2]}\,, \quad 2a_0<a<3a_0\,, \tag{10}$$

etc. According to (1) and (8), the off-shell Jost function is given by

$$f(k,q)=1+\int_{a_0}^{\infty} da\, s(a,k,q)\,. \tag{11}$$

Martin's result[2,8] for the on-shell Jost function is obtained by setting $q=k$.

Another representation for the off-shell Jost function can be obtained from the relation[5]

$$f(k,q)=1+\int_0^{\infty} dr\, e^{iqr}V(r)\Phi(k,r)\,, \tag{12}$$

where $\Phi(k,r)$ is the regular solution of the Schrödinger equation, which satisfies the boundary condition[2]

$$\lim_{r\to 0} r^{-1}\Phi(k,r)=1\,. \tag{13}$$

This function can be expressed in terms of the irregular solutions $f(\pm k,r)$ by means of the relation[2]

$$\Phi(k,r)=\frac{1}{2ik}\,[f(-k)f(k,r)-f(k)f(-k,r)]\,. \tag{14}$$

Inserting (14) into (12), and using (2) and (8) with $q=\pm k$, we find

$$f(k,q)=1+\frac{1}{2ik}\int_{a_0}^{\infty} da\left(f(-k)\,\frac{a(a-2ik)s(a,k)}{a-iq-ik} - f(k)\,\frac{a(a+2ik)s(a,-k)}{a-iq+ik}\right), \tag{15}$$

where

$$s(a,\pm k)=s(a,k,\pm k)\,. \tag{16}$$

This representation shows that $f(k,q)$ is an analytic function of q except for the branch cuts given by

$$q=\pm k-ia, \quad a_0\leq a<\infty\,. \tag{17}$$

From (4) and (15) we obtain

$$T(k,q;s)=\frac{1}{\pi i k f(k)}\int_{a_0}^{\infty} da\left(f(-k)\,\frac{a(a-2ik)s(a,k)}{q^2+(a-ik)^2} - f(k)\,\frac{a(a+2ik)s(a,-k)}{q^2+(a+ik)^2}\right) \tag{18}$$

which shows that the half-off-shell T matrix has cuts in the q^2 plane given by

$$q^2=-(a\pm ik)^2, \quad a_0\leq a<\infty\,. \tag{19}$$

It is by now well known[13,14] that the off-shell T matrix has a separable residue at the bound state and resonance energies given by

$$f(k_0)=0\,, \tag{20}$$

i.e.,

$$T(p,q;s)\xrightarrow[k\to k_0]{}\frac{g(p)g(q)}{2k_0(k-k_0)}\,. \tag{21}$$

The function g is called the vertex function or form factor for the bound state or resonance. Using (18), (20), (21) and (11) with $q=k=k_0$, we find

$$g(q)=\left[-\frac{4ik_0}{\pi(1+ik_0\rho)}\right]^{1/2}\times\int_{a_0}^{\infty} da\,\frac{a(a-2ik_0)s(a,k_0)}{q^2+(a-ik_0)^2}\,, \tag{22}$$

where

$$\rho = 2\int_0^\infty dr[e^{2ik_0r} - f^2(k_0, r)], \tag{23}$$

and is the effective range for the bound state or resonance. In deriving this result we have used Eqs. (3.18) and (3.19) of Ref. 14. We see that for a bound state or virtual bound state (k_0 pure imaginary) $g(q)$ is an analytic function of q^2, except for a left hand cut beginning at $q^2 = -(a_0 \pm |k_0|)^2$.

A useful application of (22) is to the $^1S_0$ virtual bound state of the two nucleon system. From (7), (22), and (23) it follows that the location of the cut and the discontinuity across the low q^2 end of the cut are given by the effective range parameters and the tail of the potential, which presumably is given reliably by one pion exchange. We have carried out a calculation of $g(q)$ for the $^1S_0$ Reid potential,[11] which is given by

$$V(r) = -10.463\, e^{-x}/x - 1650.6\, e^{-4x}/x + 6484.2\, e^{-7x}/x \text{ (MeV)}, \tag{24}$$

$$x = 0.7r \quad (r \text{ in fm}).$$

The scattering length, effective range, and shape parameter for this potential are[11]

$$a = -17.1 \text{ fm}, \quad r_0 = 2.80 \text{ fm}, \quad P = 0.020, \tag{25}$$

respectively. We have found the position of the virtual state by solving the equation

$$k\cot\delta = ik \tag{26}$$

with

$$k\cot\delta = -\frac{1}{a} + \frac{1}{2}r_0k^2 - Pr_0^3k^4. \tag{27}$$

The result is

$$k_0 = -i0.0543 \text{ fm}^{-1}. \tag{28}$$

The approximation (27) is easily seen to be very good, since the third term on the right hand side is very small. For the potential (24) the function $\rho(a)$ is simply a sum of three step functions. Simpson's rule was used to evaluate the integrals in (9) and (22). The results for $g(q)/g(k_0)$ are presented in Table I. According to (20), (22), and (11) with $q = k = k_0$

$$g(k_0) = \pm\left[\frac{4ik_0}{\pi(1+ik_0\rho)}\right]^{1/2} = \pm i0.245 \text{ fm}^{-1/2}, \tag{29}$$

where we have used r_0 for ρ. The sign is of no significance as only g^2 appears in the T matrix. These results should be of use in constructing the unitary pole approximation[12] for the Reid $^1S_0$ T matrix.

TABLE I. Form factor for the $^1S_0$ Reid potential.

q (fm^{-1})	$g(q)/g(k_0)$	q (fm^{-1})	$g(q)/g(k_0)$
0.0	0.998	4.2	−0.347
0.2	0.969	4.4	−0.328
0.4	0.891	4.6	−0.308
0.6	0.783	4.8	−0.285
0.8	0.660	5.0	−0.262
1.0	0.530	5.2	−0.239
1.2	0.400	5.4	−0.216
1.4	0.273	5.6	−0.194
1.6	0.153	5.8	−0.173
1.8	0.042	6.0	−0.153
2.0	−0.056	6.2	−0.134
2.2	−0.141	6.4	−0.116
2.4	−0.212	6.6	−0.100
2.6	−0.269	6.8	−0.086
2.8	−0.312	7.0	−0.073
3.0	−0.344	7.2	−0.061
3.2	−0.364	7.4	−0.051
3.4	−0.375	7.6	−0.042
3.6	−0.377	7.8	−0.034
3.8	−0.372	8.0	−0.027
4.0	−0.362		

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