Treatment of the charge-independent pairing Hamiltonian without violation of conservation laws

F. Andreozzi
Istituto di Fisica Teorica dell'Università, Istituto Nazionale di Fisica Nucleare, Napoli, Italy

A. Covello*
The Niels Bohr Institute, University of Copenhagen, Copenhagen, Denmark and Istituto Nazionale di Fisica Nucleare, Napoli, Italy

E. E. Maqueda† and R. P. J. Perazzo†
Comisión Nacional de Energía Atómica, Av. Libertador 8250, 1429 Buenos Aires, Argentina
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An equations-of-motion method is proposed for treating neutron-proton pairing correlations conserving both nucleon number N and isospin T. This method makes it possible to find approximate solutions to the charge-independent pairing Hamiltonian by a step-by-step procedure. It is shown that for the degenerate model the exact results are obtained.

[NUCLEAR STRUCTURE Charge-independent treatment of pairing correlations.]

I. INTRODUCTION

In Ref. 1 an equations-of-motion method was proposed to treat pairing correlations in systems with an even number of identical nucleons. Its main advantage lies in the conservation of the number of particles. This feature extends the range of validity of the method beyond that of the BCS approximation, as it makes it possible to find physical solutions for all values of the strength G so that one can pass continuously from the superfluid to the normal phase.

The problem of treating the charge-independent pairing Hamiltonian by means of a BCS-like approximation has been extensively investigated.[2] This approximation breaks both number and isospin conservation and it has been shown[3,4] that, as a consequence, the quasiparticle ground state has no neutron-proton pairing correlations, reducing to a product of two BCS wave functions for the neutrons and the protons separately. One is therefore led to seek an improvement on the BCS treatment trying to take into account the residual interaction between the quasiparticles.[4] Alternatively, one may try to devise an approximation scheme which avoids the relaxation of conservation laws.[5]

In this paper, we present a generalization of the method of Ref. 1 to the more complicated case of systems of unlike nucleons interacting through a charge-independent pairing interaction.

The practical importance of taking into account the isospin degree of freedom is related to the fact that it plays a major role in the description of the 0^+ states in the region around $A = 56$.[6] In the last few years a number of works have successfully reproduced energies and two-particle transition rates in this region by using both collective[7-9] and microscopic[10,11] models.

II. FORMULATION OF THE METHOD

We consider a system of N nucleons (N even) moving in a set of single-particle orbits and interacting through a charge-independent pairing force, which is effective only for $J=0$, $T=1$ pairs. The model Hamiltonian is written

$$H=\sum_j \epsilon_j \hat{N}(j) - G\sum_{jj'\mu} A_\mu^\dagger(j)A_\mu(j'), \tag{1}$$

where the ϵ_j are the single-particle energies, and

$$\hat{N}(j)=\sum_m \sum_t a_{jmt}^\dagger a_{jmt}, \tag{2}$$

$$A_\mu^\dagger(j)=\sum_{m>0}\sum_{tt'}\langle \tfrac{1}{2}t\tfrac{1}{2}t'\,|\,1\mu\rangle a_{jmt}^\dagger a_{j\bar{m}t'}^\dagger\,. \tag{3}$$

In the latter expression t is the z component of the isospin, $t=+\frac{1}{2}$ $(-\frac{1}{2})$ for neutron (proton) states, and the barred suffix refers to a time-reversed single-particle state. The operators $A_\mu^\dagger(j)$ can be regarded as the three components of a vector operator $\underline{A}^\dagger(j)$ in isospace.

The pair creation operators (3), their Hermitian conjugates, the number operator $\hat{N}(j)$, and the three components of the level isospin $\vec{T}(j)$ can be related[12,13] to the infinitesimal generators of R_5. Their commutation relations are

$$[T_\nu(j), A^\dagger_\mu(j')] = \sqrt{2}\langle 1\mu 1\nu | 1\tau\rangle A^\dagger_\tau(j)\delta_{jj'}, \quad (4a)$$

$$[\hat{N}(j), A^\dagger_\mu(j')] = 2A^\dagger_\mu(j)\delta_{jj'}, \quad (4b)$$

$$[A_\mu(j), A^\dagger_\nu(j')] = \{[\Omega_j - \tfrac{1}{2}\hat{N}(j) - \mu T_0(j)]\delta_{\mu\nu} + T_1(j)\delta_{\mu,\nu-1} - T_{-1}(j)\delta_{\mu,\nu+1}\}\delta_{jj'}, \quad (4c)$$

where $\Omega_j = j + \frac{1}{2}$.

We shall restrict ourselves, in the following, to states with individual level seniorities of zero. The wave function for a system with N particles is then written

$$|N, T, T_z\rangle = \sum_{jT'} c_{jTT'}(N)\{\underline{A}^\dagger(j)|N-2, T'\rangle\}^T_{T_z}, \quad (5)$$

where curly brackets denote isospin coupling:

$$\{\underline{A}^\dagger(j)|N-2, T'\rangle\}^T_{T_z} = \sum_{\mu T'_z}\langle T'T'_z 1\mu | TT_z\rangle A^\dagger_\mu(j)|N-2, T', T'_z\rangle. \quad (6)$$

Some of the definitions of Ref. 1 can be extended to include the isospin degree of freedom. In particular, we now define

$$X_{jTT'}(N) = \langle N, T\|\underline{A}^\dagger(j)\|N-2, T'\rangle, \quad (7)$$

$$e(N, T, T') = E_0(N, T) - E_0(N-2, T'), \quad (8)$$

where $E_0(N, T)$ stands for the energy of the lowest state with N particles, $J=0$, and isospin T.

The equations for the amplitudes $X_{jTT'}$ are obtained by taking reduced matrix elements of the equations of motion for $A^\dagger_\mu(j)$. They are

$$\langle N, T\|[H, \underline{A}^\dagger(j)]\|N-2, T'\rangle = e(N, T, T')X_{jTT'}(N). \quad (9)$$

The commutator $[H, \underline{A}^\dagger(j)]$ is calculated by making use of the commutation relations (4). The left-hand side of Eq. (9) can then be linearized by introducing a complete set of states of the $(N-2)$-particle system, and neglecting all the excited states for a given T under the assumption that the corresponding reduced transfer matrix elements are small. In this way we obtain a homogeneous set of $3n_j$ (n_j is the number of single-particle levels) linear equations in the $3n_j$ unknowns $X_{jTT'}(N)$, namely

$$[e(N, T, T') - 2\epsilon_j]X_{jTT'}(N) = -G\Omega_j \sum_{j'T''}\left\{[1-\rho_{jT'}(N-2)]\delta_{T'T''} + \sqrt{6}(-1)^{T+T'}\begin{Bmatrix} 1 & 1 & 1 \\ T' & T & T'' \end{Bmatrix} t_{jT''T'}(N-2)\right\}X_{j'TT''}(N), \quad (10)$$

where

$$\rho_{jT}(N) = (2T+1)^{-1/2}\left\langle N, T\left\|\frac{\hat{N}(j)}{2\Omega_j}\right\|N, T\right\rangle \quad (11)$$

and

$$t_{jTT'}(N) = \left\langle N, T\left\|\frac{\vec{T}(j)}{\Omega_j}\right\|N, T'\right\rangle. \quad (12)$$

It should be noted that compared with the case of identical particles[1] a new quantity, the isospin matrix $t_{jTT'}$, is introduced in the formalism. This accounts for both the occupation of each level by neutrons and protons and the degree of isospin alignment.

The condition for the system (10) to admit nontrivial solutions leads to a determinantal equation for the energies $e(N, T, T')$. From the normalization condition $\langle N, T, T_z|N, T, T_z\rangle = 1$, we obtain

$$\sum_{jT'} c_{jTT'}(N)X_{jTT'}(N) = (2T+1)^{1/2}. \quad (13)$$

The coefficients $c_{jTT'}(N)$ defined by (5) and the amplitudes $X_{jTT'}(N)$ are in turn related by

$$X_{jTT'}(N) = (2T+1)^{1/2}\sum_{j'T''} c^*_{j'TT''}(N)d^{TT'T''}_{jj'}(N-2), \quad (14)$$

with

$$d^{TT'T''}_{jj'}(N) = \sum_K (-)^{T+T'+K+1}(2K+1)^{1/2}\begin{Bmatrix} T' & 1 & T \\ 1 & T'' & K \end{Bmatrix} \times\langle N, T''\|\{\tilde{\underline{A}}(j')\times\underline{A}^\dagger(j)\}^K\|N, T'\rangle, \quad (15)$$

where the multiplication sign indicates the tensor product and $\tilde{A}_\mu(j) = (-)^{1+\mu}A_{-\mu}(j)$.

As is clear from Eqs. (10) and (14), the calculation of the energies and wave functions for the N-particle system requires knowledge of the matrices $\rho_{jT}(N-2)$, $t_{jTT'}(N-2)$, and $d^{TT'T''}_{jj'}(N-2)$. If these matrices can in turn be expressed in terms of the $c_{jTT'}(N-2)$ and $X_{jTT'}(N-2)$, then the calculation can be carried out through a step-by-step procedure. To this end, we rewrite the definitions (11) and (12) making use of (5). Next we resort to the commutation relations (4) and employ a closure procedure similar to the one that leads to Eq. (10). In this way we obtain

$$\rho_{jT}(N)=\frac{1}{(2T+1)^{1/2}}\left[\frac{1}{\Omega_j}\sum_{T'}c_{jTT'}(N)X_{jTT'}(N)+\sum_{j'T'}c_{j'TT'}(N)X_{j'TT'}(N)\rho_{jT'}(N-2)\right], \tag{16}$$

$$t_{jTT'}(N)=(2T'+1)^{1/2}\left[\frac{\sqrt{6}}{\Omega_j}\sum_{T''}(-)^{T'+T''}\begin{Bmatrix}T'' & 1 & T'\\ 1 & T & 1\end{Bmatrix}c_{jT'T''}(N)X_{jTT''}(N)\right.$$
$$\left.+\sum_{j'T''T'''}(-)^{T'+T'''}\begin{Bmatrix}1 & T'' & T'''\\ 1 & T & T'\end{Bmatrix}c_{j'T'T''}(N)X_{j'TT'''}(N)t_{jT'''T''}(N-2)\right]. \tag{17}$$

In order to make use of a closure procedure for the right-hand side of Eq. (15), the commutation relations (4) are more conveniently recast into the tensor coupled form

$$\{\underline{\tilde{A}}(j')\times A^\dagger(j)\}_Q^K-(-)^K\{\underline{A}^\dagger(j)\times\underline{\tilde{A}}(j')\}_Q^K=(\sqrt{3}\,[\Omega_j-\tfrac{1}{2}\hat{N}(j)]\,\delta_{K0}+\sqrt{2}\,T_Q(j)\delta_{K1})\delta_{jj'}. \tag{18}$$

Then we obtain

$$d_{jj'}^{TT'T''}(N)=\frac{1}{\Omega_j}[1-\rho_{jT'}(N)]\delta_{T'T''}\delta_{jj'}+(-)^{T+T'}\frac{\sqrt{6}}{\Omega_j}\begin{Bmatrix}T & 1 & T'\\ 1 & T'' & 1\end{Bmatrix}t_{jT''T'}(N)\delta_{jj'}$$
$$+\sum_{T'''}(-)^{T'+T''}\begin{Bmatrix}1 & T'' & T'''\\ 1 & T' & T\end{Bmatrix}X_{jT''T'''}(N)X_{j'T'T'''}(N). \tag{19}$$

The matrix $d_{jj'}^{TT'T''}$ bears the exclusion principle effects and is therefore crucial in providing a wave function with the proper symmetry.

The step-by-step procedure outlined above has to be started with the obvious initial values

$$\rho_{jT}(0)=0,\quad t_{jTT'}(0)=0,$$
$$d_{jj'}^{T00}(0)=\Omega_j\delta_{jj'}\delta_{T1}. \tag{20}$$

It is to be noted that for $N=2$ our procedure is exact. In this case Eq. (10) reduces to the well-known[2] eigenvalue equation

$$\sum_{j,m>0}\frac{G}{2\epsilon_j-E}=1 \tag{21}$$

and

$$c_{j10}(2)=\frac{X_{j10}(2)}{\sqrt{3}\,\Omega_j}. \tag{22}$$

III. DEGENERATE CASE

We come now to consider the special case of degenerate single-particle levels $\epsilon_j=0\;\forall j$. In this case there is only one state of seniority $v=0$ with given isospin T for each nucleon number, and therefore the closure procedure is exact. In this section we show that our method, once applied to the degenerate model, leads indeed to the exact results for the energies and two-particle amplitudes which can be obtained analytically by other methods.[12-14] From now on we drop the orbit index j.

A. Energy spectrum

In the degenerate case the exact values of $\rho_T(N)$ and $t_{TT'}(N)$ are given by

$$\rho_T(N)=\frac{N}{2\Omega}, \tag{23}$$

$$t_{TT'}(N)=\frac{1}{\Omega}[T(T+1)(2T+1)]^{1/2}\delta_{TT'}. \tag{24}$$

Then Eq. (10) becomes

$$e(N,T,T')=-G\{\Omega-\tfrac{1}{2}N+2+\tfrac{1}{2}[T'(T'+1)-T(T+1)]\}, \tag{25}$$

and the energy $E_0(N,T)$ of the N-particle system can be easily derived. One obtains

$$E_0(N,T)=-\tfrac{1}{2}G[N(\Omega+\tfrac{3}{2}-\tfrac{1}{4}N)-T(T+1)]\}, \tag{26}$$

which is the exact result[14] for states with seniority and reduced isospin of zero.

B. Two-particle amplitudes

A general proof can be given by induction. Since our method is exact for $N=2$, we have to derive the X's for $N+2$ assuming that they coincide with the exact results for the N-particle system which can be obtained using group properties of R_5.[12] The values of the isospin T_N of the exact $v=0$ states of the N-particle system are restricted to those having the same parity as $\frac{1}{2}N$; hence, there are only two nonzero amplitudes. They are[12]

$$X_{T_N, T_{N-2}}(N)\,|_{T_{N-2}=T_{N-1}} = [\tfrac{1}{2}T_N(2\Omega - \tfrac{1}{2}N - T_N + 2) \times (\tfrac{1}{2}N + T_N + 1)]^{1/2},$$

$$X_{T_N, T_{N-2}}(N)\,|_{T_{N-2}=T_{N}+1} = -[\tfrac{1}{2}(T_N+1)(2\Omega - \tfrac{1}{2}N + T_N + 3) \times (\tfrac{1}{2}N - T_N)]^{1/2}. \quad (27)$$

We first prove that the amplitudes $X_{T_{N+2}, T_N}(N+2)$ are given by the expressions (27) replacing N by $N+2$. To this end, we calculate the values of $d^{T_{N+2}, T_N, T'_N}(N)$ for $T_{N+2}=T$, $T_N = T\pm 1$, and $T'_N = T\pm 1$ inserting (23), (24), and (27) into (19). We obtain

$$d^{T, T-1, T-1}(N) = \frac{T(2\Omega - \frac{1}{2}N - T + 1)(T + \frac{1}{2}N + 2)}{2(2T+1)}, \quad (28a)$$

$$d^{T, T+1, T+1}(N) = \frac{(T+1)(2\Omega - \frac{1}{2}N + T + 2)(\frac{1}{2}N - T + 1)}{2(2T+1)} \quad (28b)$$

$$d^{T, T-1, T+1}(N) = d^{T, T+1, T-1}(N) = -\frac{1}{2(2T+1)}[T(T+1)(T+\tfrac{1}{2}N+2)(\tfrac{1}{2}N - T + 1)(2\Omega - \tfrac{1}{2}N - T + 1)(2\Omega - \tfrac{1}{2}N + T + 2)]^{1/2}. \quad (28c)$$

By making use of (14) and (15) we can now calculate the parentage coefficients $c_{TT'}(N+2)$ and the two-particle amplitudes $X_{TT'}(N+2)$. As can be easily verified, the determinant of the system (14),

$$\Delta = d^{T, T-1, T-1}(N)\, d^{T, T+1, T+1}(N) - [d^{T, T-1, T+1}(N)]^2, \quad (29)$$

is equal to zero. This is related to the fact that the same state $|N+2, T, T_z\rangle$ can be obtained in two ways, namely as $\{\underline{A}^\dagger|N, T-1\rangle\}^T_{T_z}$ or $\{\underline{A}^\dagger|N, T+1\rangle\}^T_{T_z}$. Hence, we can make an arbitrary choice of one of the two parentage coefficients, for instance $c_{T, T+1}(N+2)=0$ or $c_{T, T-1}(N+2)=0$, and solve the system (14) supplemented by the normalization condition (13). In particular, by choosing $c_{T, T+1}(N+2)=0$, we obtain

$$c_{T, T-1}(N+2) = \left[\frac{2(2T+1)}{T(2\Omega - \frac{1}{2}N - T + 1)(T + \frac{1}{2}N + 2)}\right]^{1/2}, \quad (30)$$

$$X_{T, T-1}(N+2) = [\tfrac{1}{2}T(2\Omega - \tfrac{1}{2}N - T + 1)(T + \tfrac{1}{2}N + 2)]^{1/2}, \quad (31a)$$

$$X_{T, T+1}(N+2) = -[\tfrac{1}{2}(T+1)(2\Omega - \tfrac{1}{2}N + T + 2) \times (\tfrac{1}{2}N - T + 1)]^{1/2}. \quad (31b)$$

The expressions (31) are precisely the exact ones for the degenerate model and are obviously independent of the arbitrariness in the choice of the coefficients $c_{TT'}(N+2)$.

It is readily shown now that, in the degenerate limit, our wave functions fulfill the requirement $(-)^{N/2+T} = +1$. In fact, the matrix elements $d^{T_{N+2}, T_N, T'_N}(N)$, which are required to calculate $X_{T_{N+2}, T_N = T_{N+2}}(N+2)$ vanish identically. It is in this way evident that the d matrices carry the information necessary to provide wave functions with the proper symmetry.

Finally, it is straightforward to check that inserting (23), (24) and (30), (31) into (16) and (17) one obtains the exact values of $\rho_T(N+2)$ and $t_{TT'}(N+2)$. This completes the proof.

IV. CONCLUDING REMARKS

The method presented in this paper can be applied to the realistic case of N nucleons in nondegenerate levels keeping the amount of numerical work rather limited. Thus, it should prove to be an effective tool to analyze experimental data in terms of the pairing model. In particular, the two-particle transfer amplitudes associated with each j shell are obtained directly and can be used to analyze two-nucleon transfer cross sections. The impossibility of providing such amplitudes is a short-coming of all collective approaches[7-9,11] that have to resort to a microscopic model to procure a valid input to distorted wave Born approximation codes.

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*Present address: Istituto di Fisica Teorica dell'Università, Mostra d'Oltremare, Pad. 19, 80125 Napoli, Italy.

†Fellow of the Consejo Nacional de Investigaciones Científicas y Técnicas, Argentina.

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