PHYSICAL REVIEW C VOLUME 16, NUMBER 6 DECEMBER 1977

Relation between cluster and shell-model wave functions

K. T. Hecht*
David M. Dennison Building, University of Michigan, Ann Arbor, Michigan 48109
(Received 26 July 1977)

A method is presented for the expansion of properly antisymmetrized cluster model (or resonating group method) wave functions in terms of translationally invariant shell-model wave functions. By coupling the relative motion (harmonic oscillator) wave functions and the internal cluster functions to resultant SU_3,SU_4 strong coupled functions, the cluster model wave functions can be expanded in terms of standard shell-model wave functions by SU_3, SU_4 recoupling techniques, using readily available recoupling coefficients. As a specific example, the full shell-model expansions are given for $(\alpha + {}^{12}C^{(04)})$ cluster functions for all possible SU_3 symmetries and α-^{12}C relative motion wave functions carrying from 4 to 8 oscillator quanta, hence ^{16}O core excitations up to $4\hbar\omega$.

[NUCLEAR STRUCTURE Shell-model expansions of cluster functions; $\alpha + {}^{12}C$ representations of ^{16}O.]

I. INTRODUCTION

Recent extended shell-model calculations[1,2] and resonating group or cluster model calculations[3-5] for nuclei such as ^{16}O, ^{20}Ne, and ^{44}Ti have renewed interest in the well known relationship between shell-model and resonating group method wave functions.[6] In the extended shell-model calculations of Arima and Tomoda[1] a comparatively rich shell-model basis is augmented by core excitations which are described as α-cluster states with oscillator excitations up to 20 units of $\hbar\omega$, carried by the α-core relative motion wave function. Although introduced to incorporate the physics of α clustering, these states have the additional advantage that they furnish a means of introducing core excitations which are automatically free of spurious center of mass excitations. The recent calculation for ^{16}O by Suzuki,[3] in the framework of the orthogonality condition model,[7] uses a $^{12}C+\alpha$ cluster function basis with core excitations up to 26 units of $\hbar\omega$. Despite its striking success this calculation may overemphasize the α-cluster nature of certain states in ^{16}O. The 1^- state at 7.12 MeV is a particularly interesting example, since its α spectroscopic amplitude is of astrophysical interest. Recent experimental determinations of the α amplitude of the 7.12 MeV state by four-nucleon transfer reactions[8] may be at variance with earlier estimates.[9] This 1^- state is predominantly a 1p-1h state. In the harmonic oscillator shell model the major 1p-1h component of this state is identical with the $\alpha+{}^{12}C$ cluster state corresponding to an oscillator excitation of $1\hbar\omega$. For an accurate determination of its α spectroscopic amplitude, however, both higher cluster-motion and additional small p-h components in its wave function may be important. Moreover, the low energy $^{12}C(\alpha,\gamma)^{16}O$ cross section, which is one of the crucial parameters in the determination of the ^{12}C to ^{16}O ratio in a He-burning star, depends on the *relative* α amplitudes of the 7.12 MeV and the nearby 9.6 MeV 1^- state. More detailed conventional shell-model calculations based on the 1p-1h and 3p-3h excitations[10,11] may be even more inadequate since they fail to account for the wide 1^- α-cluster state at 9.6 MeV whose properties are predicted more successfully by the cluster model calculation of Suzuki.

Detailed calculations which have merged the shell-model and cluster-model pictures have resorted to various approximation techniques. It may therefore be of interest to reexamine the relationship between cluster and shell-model wave functions. It is the purpose of this contribution to present a new method for the expansion of cluster-model wave functions in terms of properly antisymmetrized, normalized, translationally invariant shell-model wave functions which can be carried out explicitly for core excitations up to a few units of $\hbar\omega$ of oscillator excitation. The method uses harmonic oscillator SU_3, SU_4 strong coupled resonating group wave functions and is thus particularly useful for *p* and *sd* shell nuclei. It uses simple SU_3 and SU_4 recoupling techniques, requiring only recoupling coefficients which are readily available. Similar techniques have recently been exploited in the calculation of cluster spectroscopic amplitudes[12] for core excited states in *sd* shell nuclei.

The general method of calculation is outlined in Sec. II. Some of the technical details of the method are reserved for an Appendix. The $\alpha+{}^{12}C$ cluster function basis for ^{16}O is used as the prime example

to illustrate the method. In this basis the internal wave function for ^{12}C is restricted to the SU_3 symmetry (04). The full shell-model expansions for these $\alpha + {}^{12}C$ cluster states are tabulated in Sec. III for all possible SU_3 symmetries and α – core relative motion wave functions carrying from four to eight oscillator quanta, hence ^{16}O core excitations from 0 to 4 units of $\hbar\omega$ of oscillator excitation. These detailed expansions of α-cluster wave functions make possible a study of the approximations inherent in earlier cluster model and mixed shell- and cluster-model calculations, and may be useful in merging earlier p-h shell-model calculations[10] for ^{16}O with the more complicated cluster-model caclulations.

II. METHOD OF CALCULATION

In the resonating group method the nuclear wave function is built in terms of clusters A, B, ... with internal wave functions $\phi_A(\xi_A)$, $\phi_B(\xi_B)$, ..., where ξ_A, ξ_B, ... are intrinsic coordinates of the clusters. The trial wave function for the resonating group method is of the form

$$\mathcal{A}[u(\vec{r}_{AB}, \ldots)_A(\xi_A)\phi_B(\xi_B)\ldots], \tag{1}$$

where $\vec{r}_{AB}$, ... are the relative vectors from the centers of mass of A to B, ..., and $u(\vec{r}_{AB}, \ldots)$ is the wave function describing the relative motion of the cluster. The internal wave functions $\phi(\xi)$ are properly antisymmetrized. The antisymmetrizer $\mathcal{A}$ makes the complete wave function antisymmetric under exchange of nucleons between clusters. It is assumed that the internal wave functions are constructed from single particle harmonic oscillator wave functions $\Phi(\vec{r}_i')$ with $\vec{r}_i' = (m\omega/\hbar)^{1/2}\vec{r}_i$, with one common oscillator frequency ω for all clusters. The relative motion wave function will be expanded in terms of harmonic oscillator wave functions with this same ω. It will then be possible to expand such oscillator cluster wave functions in terms of standard shell-model wave functions. Since the internal wave functions $\phi(\xi)$ are assumed to correspond to oscillator ground state configurations these oscillator cluster wave functions, with excitations restricted to relative motion wave functions, are automatically free of spurious center of mass excitations. In addition, it will be assumed that the internal cluster wave functions $\phi(\xi)$ have good SU_3 and supermultiplet symmetry. (SU_3 quantum numbers will be given by the Elliott labels $(\lambda\mu)$, SU_4 quantum numbers by $[\tilde{f}]$, derived from the U_4 partition numbers conjugate to the space symmetry labels $[f]$.) If the relative motion harmonic oscillator wave function carries Q oscillator quanta ($Q = 2N + L$) it will be described by the SU_3 irreducible representation $(Q0)$. In addition, it will be useful to couple the SU_3 and SU_4 representations of u, $\phi_A(\xi_A)$, $\phi_B(\xi_B)$, ... to resultant total SU_3 and SU_4 symmetries $(\lambda\mu)$, $[\tilde{f}]$. The advantage of this so-called SU_3, SU_4 strong-coupled form of the cluster wave function is related to the fact that these SU_3 strong-coupled functions are eigenfunctions[13] of the exchange kernel, K, associated with the antisymmetrizer.

The $\alpha + {}^{12}C$ cluster function basis for ^{16}O will be used as the prime example to illustrate the technique whereby the cluster states of the form (1) are expanded in terms of standard shell-model wave functions. It is assumed that the ^{12}C cluster function has space symmetry [444], and hence belongs to the scalar SU_4 representation $[\tilde{f}] = [0]$, and is in addition a pure $(0s)^4(0p)^8$ shell-model function of SU_3 symmetry (04). The SU_3, SU_4 strong-coupled cluster function then has the form

$$\Psi_{c1} = \mathcal{A}[\Phi^{(Q0)}(\vec{r}'_{\alpha\text{-}C}) \times \phi_\alpha(\xi_\alpha)^{(00)[0]} \times \phi_{^{12}C}(\xi_{^{12}C})^{(04)[0]}]_a^{(\lambda\mu)[0]}, \tag{2}$$

where the square bracket denotes the SU_3 coupling $(Q0) \times (04) \to (\lambda\mu)$, and the (trivial) SU_4 coupling $[0] \times [0] \to [0]$. The SU_3, SU_4 subgroup labels are designated by a since *any* convenient set of labels can be chosen. (For $[\tilde{f}] = 0$, $S = T = 0$, of course; for any other supermultiplet it is immaterial whether a $KLSJM_J$ or a KLM_LSM_S, or any other basis is chosen.) These SU_3 strong coupled functions can be related to ordinary angular momentum-coupled (SU_3 weak-coupled functions) by straightforward coupling techniques; see, e.g., Eqs. (2) and (3) or Ref. 12. The antisymmetrizer $\mathcal{A}$ is normalized according to

$$\mathcal{A} = \left(\frac{4!12!}{16!}\right)^{1/2}\left(1 + \sum_k (-1)^k P_k\right),$$

where the P_k are permutation operators which exchange at least one pair of nucleons from the α and ^{12}C clusters. With $Q < 4$, the antisymmetrizer annihilates the state (2). For $Q = 4$, the only Pauli-allowed state has $(\lambda\mu) = (00)$ and is the closed shell $(0s)^4(0p)^{12}$ ground state for ^{16}O. For $Q = 7$, only the state $(\lambda\mu) = (74)$ is Pauli forbidden; for $Q \geq 8$, all possible $(\lambda\mu)$ are allowed. Normalization factors for the states (2) have been given by Horiuchi[13] for Q values up to $Q = 19$. The shell-model expansions for the 1^-, 2^-, 3^- states with $Q = 5$, $(\lambda\mu) = (21)$, were first given in a weak-coupled basis among the detailed examples of the pioneering work of Kanellopoulos and Wildermuth.[6]

The relative motion harmonic oscillator function $\Phi^{Q0}(\vec{r}_{\alpha\text{-}C})$ is a function of the dimensionless coordinate

$$\vec{r}'_{\alpha\text{-}C} = \left(\frac{4 \times 12}{16} \times \frac{m\omega}{\hbar}\right)^{1/2} \vec{r}_{\alpha\text{-}C},$$

where m = nucleon mass, and $\vec{r}_{\alpha\text{-}C}$ is the position vector from the center of mass of the α particle to the center of mass of the ^{12}C cluster.

The basic method whereby the cluster wave function (2) is expanded in terms of standard shell-model wave functions makes use of an expansion of harmonic oscillator wave functions. The harmonic oscillator wave function $\Phi^{(Q0)}(\vec{\rho}')$ in a variable $\vec{\rho}'$ is expanded in terms of harmonic oscillator functions $\Phi^{(q0)}(\vec{r}_i')$, with $\vec{r}_i' = (m\omega/\hbar)^{1/2}\vec{r}_i$, or $\Phi^{(q0)}(\vec{R}_k')$, where the dimensionless variables $\vec{\rho}'$ and $\vec{r}_i'$, (or $\vec{R}_k'$), are related by unitary transformations. If the variable $\vec{\rho}'$ is labeled as the first variable in the unitary transformation from $\vec{r}_i'$ to $\vec{\rho}' = \vec{\rho}_1', \ldots,$ then

$$\vec{\rho}' = \sum_i \vec{r}_i' u_{i1} . \tag{3a}$$

In our example,

$$\vec{\rho}' = \vec{r}_{\alpha\text{-}C}' = \frac{1}{4\sqrt{3}}(\vec{r}_1' + \vec{r}_2' + \cdots \vec{r}_{12}') - \frac{\sqrt{3}}{4}(\vec{r}_{13}' + \vec{r}_{14}' + \vec{r}_{15}' + \vec{r}_{16}') . \tag{3b}$$

Or, alternately

$$\vec{\rho}' = \vec{r}_{\alpha\text{-}C}' = \frac{1}{\sqrt{12}}\vec{R}_1' + \frac{1}{\sqrt{6}}\vec{R}_2' - \frac{\sqrt{3}}{2}\vec{R}_3' \tag{4a}$$

with

$$\vec{R}_1' = \tfrac{1}{2}(\vec{r}_1' + \cdots + \vec{r}_4'), \quad \vec{R}_2' = \frac{1}{\sqrt{8}}(\vec{r}_5' + \cdots + \vec{r}_{12}'), \quad \vec{R}_3' = \tfrac{1}{2}(\vec{r}_{13}' + \cdots + \vec{r}_{16}') . \tag{4b}$$

The basic expansion of the harmonic oscillator function $\Phi^{(Q0)}(\vec{\rho}')$ has its simplest form in an SU_3-coupled representation. Under the transformation (3)

$$\phi_{LM}^{(Q0)}\left(\sum_i \vec{r}_i' u_{i1}\right) = \sum_{\substack{q_1 \cdots q_A \\ (\Sigma q_i = Q)}} \left(\frac{Q!}{q_1! q_2! \cdots q_A!}\right)^{1/2} \times \prod_{i=1}^{A} (u_{i1})^{q_i} [\cdots [[\Phi^{(q_1 0)}(\vec{r}_1') \times \Phi^{(q_2 0)}(\vec{r}_2')]^{(q_1+q_2,0)} \times \Phi^{(q_3 0)}(\vec{r}_3')]^{(q_1+q_2+q_3,0)} \times \cdots \Phi^{(q_A 0)}(\vec{r}_A')]_{LM}^{(Q0)} \tag{5}$$

a result given by Kramer[14] which makes use of the transformation properties of the A-particle harmonic oscillator function under the direct product group $U(A) \times U(3)$, and which has been transcribed here into SU_3-coupled language. The square brackets denote SU_3 coupling. Since all SU_3 couplings in Eq. (5) are "stretched," the order of the couplings can be rearranged at will (without change of phase). It will also be useful to express harmonic oscillator functions in terms of polynomials[14] in the oscillator creation operators, $\vec{\eta}_i$, acting on the oscillator ground state

$$\Phi_{nlm}(\vec{r}_i') = P_{lm}^{(q0)}(\vec{\eta}_i)|0\rangle \tag{6}$$

with $q = 2n + l$. For oscillator functions in the single particle variables, $\vec{r}_i'$, the creation operator $\vec{\eta}_i$ is a function of the coordinate and momentum components of particle i. For oscillator functions of variables such as $\vec{R}_k'$ the corresponding oscillator creation operator $\vec{\eta}_k$ is the analogous function of the coordinate $\vec{R}_k'$ and conjugate momentum $\vec{P}_k$. The action of a polynomial creation operator on an oscillator function in the same variable is given by the relation

$$[P^{(q0)}(\vec{\eta}_i)\Phi^{(q'0)}(\vec{r}_i')]_{LM}^{(\lambda\mu)} = \delta_{(\lambda\mu)(q+q',0)}\left(\frac{(q+q')!}{q!\,q'!}\right)^{1/2}\Phi_{LM}^{(q+q',0)}(\vec{r}_i') . \tag{7}$$

Two methods have been used for the expansion of cluster-model functions of the form (2) in terms of standard shell-model components. In the most direct method the relative motion function is expanded in terms of single particle harmonic oscillator functions by successive application of Eq. (5) to the transformations (4a) and (4b) whereby the Q oscillator quanta carried by the relative motion function are partitioned first into Q_1, Q_2, Q_3 quanta associated with the degrees of freedom $\vec{R}_1'$, $\vec{R}_2'$, $\vec{R}_3'$ and secondly into q_1, q_2, ..., q_A quanta associated with the single particle shell-model functions. The bookkeeping procedure for this partitioning process is somewhat complicated (see Ref. 23). A less direct, but somewhat simpler method makes use of a chain calculation which is particularly useful for the calculation of specific shell-model components of a cluster function of moderate core excitation. This technique makes use of a "cluster-like" wave function which is obtained from the true cluster

function by replacing the relative motion function $\Phi^{(Q0)}(\vec{r}'_{\alpha\text{-}C})$ in the cluster function by a function $\Phi^{(Q0)}(\vec{R}'_{\alpha})$, in the coordinates of the last four particles only, and by multiplying the internal function for the core cluster by a wave function of 0s excitation for the center of mass motion of the core, $\Phi^{(00)}(\vec{R}'_{C})$

$$\Psi_{\text{cluster-like}} = \mathcal{A}[\,\Phi^{(Q0)}(\vec{R}'_{\alpha}) \times \phi^{(00)[0]}(\xi_{\alpha}) \times \phi(\xi_{12_C})^{(04)[0]}\Phi^{(00)}(\vec{R}'_{12})\,]^{(\lambda\mu)[0]}_{a}. \tag{8}$$

The $(\alpha + {}^{12}C)$ cluster function is again used as the specific example, so that

$$\vec{R}'_{\alpha} = \frac{1}{\sqrt{4}}(\vec{r}'_{13} + \cdots + \vec{r}'_{16}), \quad \vec{R}'_{C} = \vec{R}'_{12} = \frac{1}{\sqrt{12}}(\vec{r}'_{1} + \cdots + \vec{r}'_{12}), \quad \vec{R}'_{\text{c.m.}} = \frac{1}{\sqrt{16}}(\vec{r}'_{1} + \cdots + \vec{r}'_{16}). \tag{9}$$

The application of transformation (5) to the 2×2 unitary transformation

$$\vec{R}'_{\alpha} = -(\tfrac{12}{16})^{1/2}\vec{r}'_{\alpha\text{-}C} + (\tfrac{4}{16})^{1/2}\vec{R}'_{\text{c.m.}}, \quad \vec{R}'_{12} = (\tfrac{4}{16})^{1/2}\vec{r}'_{\alpha\text{-}C} + (\tfrac{12}{16})^{1/2}\vec{R}'_{\text{c.m.}}. \tag{10}$$

can be used to expand $\Phi^{(Q0)}(\vec{R}'_{\alpha})$ [and trivially $\Phi^{(00)}(\vec{R}'_{12})$] in terms of proper relative motion functions $\Phi^{(Q0)}(\vec{r}'_{\alpha\text{-}C})$ and center of mass excitation operators $P(\vec{\eta}_{\text{c.m.}})$

$$\Phi^{(Q0)}_{LM}(\vec{R}'_{\alpha}) = \sum_{\substack{Q_1 \\ Q_1+Q_2=Q}} \left(\frac{Q!}{Q_1!Q_2!}\right)^{1/2} (\tfrac{4}{16})^{Q_1/2}[-(\tfrac{12}{16})^{1/2}]^{Q_2}\,[\,P^{(Q_1 0)}(\vec{\eta}_{\text{c.m.}}) \times \Phi^{(Q_2 0)}(\vec{r}'_{\alpha\text{-}C})]^{(Q0)}_{LM}. \tag{11}$$

With this transformation the projection of the "cluster-like" function onto a specific shell-model component $|(\Psi^{\gamma}_{\text{sm}})^{(\lambda\mu)}\rangle$ can be related to the overlap of the true cluster function, Eq. (2), with this same shell-model component; (γ is a short-hand symbol which denotes the full configuration and SU_3, SU_4 quantum numbers of each major shell component). Thus

$$\begin{aligned}\langle (\Psi^{\gamma}_{\text{sm}})^{(\lambda\mu)}_{a} \,|\, \Psi^{[(Q0)\times(04)](\lambda\mu)}_{\text{cluster-like}}\rangle &= [-(\tfrac{12}{16})^{1/2}]^{Q}\,\langle (\Psi^{\gamma}_{\text{sm}})^{(\lambda\mu)}_{a} \,|\, \Psi^{[(Q0)\times(04)](\lambda\mu)}_{\text{true cluster}}\rangle \\ &+ \sum_{\substack{Q_1=1 \\ Q_1+Q_2=Q}} \left(\frac{Q!}{Q_1!Q_2}\right)^{1/2} (\tfrac{4}{16})^{Q_1/2}[-(\tfrac{12}{16})^{1/2}]^{Q_2} \sum_{(\lambda'\mu')} U((Q_1\,0)(Q_2 0)(\lambda\mu)(04);(Q0)(\lambda'\mu')) \\ &\times \langle (\Psi^{\gamma}_{\text{sm}})^{(\lambda\mu)}_{a} \,|\, [\,P^{(Q_1 0)}(\vec{\eta}_{\text{c.m.}}) \times \mathcal{A}[\,\Phi^{(Q_2 0)}(\vec{r}'_{\alpha\text{-}C}) \times \phi^{(00)}(\xi_{\alpha})\phi^{(04)[0]}(\xi_{12})\,]^{(\lambda'\mu')}\,]^{(\lambda\mu)[0]}\rangle .\end{aligned} \tag{12}$$

The square brackets in the last term again denote SU_3 (and the trivial SU_4) coupling, and a transformation has been made from the

$$[\,[\,P^{(Q_1 0)}(\vec{\eta}_{\text{c.m.}}) \times \Phi^{(Q_2 0)}(\vec{r}'_{\alpha\text{-}C})\,]^{(Q0)} \times \phi^{(04)}(\xi_{12})\,]^{(\lambda\mu)}$$

to the recoupled

$$[\,P^{(Q_1 0)}(\vec{\eta}_{\text{c.m.}}) \times [\,\Phi^{(Q_2 0)}(\vec{r}'_{\alpha\text{-}C}) \times \phi^{(04)}(\xi_{12})\,]^{(\lambda'\mu')}\,]^{(\lambda\mu)}$$

basis by means of an SU_3 Racah [6-$(\lambda\mu)$] recoupling transformation. The U coefficient is an SU_3 Racah coefficient in unitary form,[12,15] free of multiplicity labels. By a simple symmetry property it is identical with the coefficient $U((04)(Q_2 0)(\lambda\mu)(Q_1 0);(\lambda'\mu')(Q0))$ for which an algebraic expression is given in Ref. 12. More complicated SU_3 U coefficients, including SU_3 couplings which may not be multiplicity-free, are needed for other parts of the calculation process and are available through the computer code of Akiyama and Draayer.[16]

The cluster-like function has a nonzero overlap only with shell-model states of the type

$$(\Psi^{\gamma}_{\text{sm}})^{(\lambda\mu)} = \Psi((0s)^4[(0p)^{8+n_1}(\lambda'\mu')[\tilde{f}'][(sd)^{n_2}(pf)^{n_3}\ldots](\lambda''\mu'')[\tilde{f}'']\,]^{(\lambda\mu)[0]})$$

with $\sum_i n_i = 4$, $Q = \ln_1 + 2n_2 + 3n_3 + \ldots$, and $n_1 \le 4$, and with p shell SU_3, SU_4 representations $(\lambda'\mu') = (0, 4-n_1)$, $[\tilde{f}'] = [1^{n_1}]$, and $(\lambda''\mu'') = (Q-n_1, 0) = (2n_2 + 3n_3 + \cdots, 0)$, $[\tilde{f}''] = [1^{4-n_1}]$. For all other shell-model components, γ, the left hand side of Eq. (12) is zero. The nonzero overlaps of $\Psi^{\gamma}_{\text{sm}}$ with the cluster-like function become

$$\begin{aligned}\langle \Psi((0s)^4[(0p)^{8+n_1}(0, 4-n_1)\times [(sd)^{n_2}(pf)^{n_3}_{a}](Q-n_1, 0)]^{(\lambda\mu)[0]} \,|\, \Psi^{[(Q0)\times(04)](\lambda\mu)}_{\text{cluster-like}}\rangle \\ = \left[\left(\frac{Q!}{(1!)^{n_1}(2!)^{n_2}(3!)^{n_3}\cdots}\right)^{1/2}(\tfrac{1}{2})^{Q}\right]\left(\frac{4!}{\prod_i (n_i!)}\right)^{1/2}\left(\frac{(8+n_1)!}{8!\,n_1!}\right)^{1/2} \\ \times U((04)(n_1 0)(\lambda\mu)(Q-n_1, 0);(0, 4-n_1)(Q0))\left(\frac{\dim_{s_8}[44]}{\dim_{s_{8+n_1}}[44n_1]}\right)^{1/2}.\end{aligned} \tag{13}$$

To derive Eq. (13) it is convenient to express the cluster-like function in terms of

$$\Psi_{\text{cluster-like}} = \mathcal{A}\mathcal{A}'[\phi^{(00)[0]}(1,2,3,4)\phi^{(04)[0]}(5,\ldots,12)[\Phi^{(Q0)}(\vec{R}'_\alpha)\phi_\alpha^{(00)[0]}(13,\ldots,16)]]^{(\lambda\mu)[0]}, \tag{14}$$

where the antisymmetrizer $\mathcal{A}'$ [normalized as in Eq. (2)] makes the ^{12}C internal function antisymmetric under exchange of nucleons between the $(0s)^4$ cluster containing nucleons numbered $1,\ldots,4$ and the $(0p)^8$ cluster containing nucleons 5 to 12. The reordering of the SU_3 coupling from $(Q0)\times(04) \rightarrow (\lambda\mu)$ to $(04)\times(Q0)\rightarrow(\lambda\mu)$, with $\lambda+\mu-Q-4=$ even, introduces no change of sign (see Appendix A). The transformation (5) applied to the function $\Phi^{(Q0)}(\vec{R}'_\alpha=\frac{1}{2}(\vec{r}'_{13}+\cdots+\vec{r}'_{16}))$ gives $(4!/n_1!n_2!n_3!\ldots)$ $\times$ (factor in the first square bracket) where it is assumed that nucleons numbered $13,\ldots 13+n_1$ have been placed in the p shell, $\ldots$, by the transformation (5). In the overlap (13) the antisymmetrizers $\mathcal{A}$, $\mathcal{A}'$ can be made to act to the left to yield the factor $[16!/4!8!4!]^{1/2}$. The overlap between the totally antisymmetric shell-model function on the left and the cluster functions inside the square bracket of (14) then yield the factor

$$\{[4!(8+n_1)!n_2!n_3!\ldots]/16!\}^{1/2} \times [8+n_1 \rightarrow (8+n_1)\ p \text{ shell cfp}],$$

after a recoupling transformation from the $[\phi^{(04)}\times[\Phi^{(n_1 0)}\times\Phi^{(Q-n_1,0)}]^{(Q0)}]^{(\lambda\mu)}$ to the $[[\phi^{(04)}\times\Phi^{(n_1 0)}]^{(0,4-n_1)}\times\Phi^{(Q-n_1,0)}]^{(\lambda\mu)}$ basis in which the p-shell nucleons are coupled to $(\lambda'\mu')=(0,4-n_1)$. This recoupling transformation is effected by the SU_3 U coefficient which is of the simple form encountered previously. The $8+n_1\rightarrow(8+n_1)$ p-shell coefficient of fractional parentage (cfp) is an SU_3, SU_4-reduced multiparticle cfp which for the p shell is given by the simple symmetric group dimension ratio shown and has the phase +1 (see Appendix A). To calculate the overlap between a true cluster function and a specific shell-model component, Ψ^γ_{sm}, the last term of Eq. (12) must still be evaluated. Note that the antisymmetrizer $\mathcal{A}$ commutes with the totally symmetric center of mass motion excitation operator so that $\mathcal{A}$ can be made to act directly on the cluster function. The last term of Eq. (12) thus involves the action of the center of mass excitation operator on a properly antisymmetrized true cluster function with $Q_2<Q$. In most cluster-model studies wave functions with relative motion excitations up to some maximum number of quanta Q are usually of interest. The cluster functions with $Q_2<Q$ will thus have been expanded in terms of shell-model components in an earlier step of the calculational process. (For a more direct method not dependent on such a chain calculation, see Ref. 23). In the $\alpha+^{12}$C cluster system, the cluster state with $Q=7$, $(\lambda\mu)=(41)$, for example, can be decomposed into shell-model components by Eq. (12) if the shell-model decomposition of cluster states with $Q_2=6$ and $(\lambda\mu)=(42)$, (31), and $Q_2=5$, $(\lambda\mu)=(21)$ has been evaluated in an earlier step of the calculational process. The minimum Pauli-allowed Q value makes no contribution in this example, since the only state with $Q_2=4$ has $(\lambda\mu)=(00)$ and cannot be connected to $(\lambda\mu)=(41)$ by $P^{(30)}(\vec{\eta}_{\text{c.m.}})$. The spurious states generated by the center of mass motion excitation operator $P^{(10)}(\vec{\eta}_{\text{c.m.}})$ acting on the true cluster states with $Q_2=6$, $(\lambda\mu)=(42)$, (31) and $P^{(20)}(\vec{\eta}_{\text{c.m.}})$ acting on the state with $Q_2=5$, $(\lambda\mu)=(21)$, when projected onto a shell-model component, give the projection of the spurious center of mass motion of the "cluster-like" function with $Q=7$ onto this specific shell-model component. The matrix elements of $P^{(Q_1 0)}(\vec{\eta}_{\text{c.m.}})$ connecting specific shell-model components can be evaluated by SU_3, SU_4 recoupling techniques. For this purpose it is again convenient to expand $P^{(Q_1 0)}(\vec{\eta}_{\text{c.m.}})$ in terms of single-particle operators $P^{(q0)}(\vec{\eta}_i)$, $i=1,\ldots,A$, by transformation (5). To illustrate this SU_3, SU_4 recoupling process a specific example is worked out in Appendix B.

The technique whereby cluster-model functions are expanded in terms of standard shell-model components depends on three technical details, (1) SU_3, SU_4 recoupling transformations, (2) the reordering of SU_3-, SU_4-coupled functions, and (3) the uncoupling or recoupling of open shell functions through SU_3-, SU_4-reduced fractional parentage coefficients. Since it is important to use a consistent set of phases, these three steps of the calculational process will be discussed in some detail in Appendix A.

III. SHELL-MODEL EXPANSIONS OF THE (α + ^{12}C) CLUSTER FUNCTIONS

With the techniques outlined in Sec. II the full expansions of the $(\alpha+^{12}C^{(04)})$ cluster functions can now be projected onto all possible shell-model components. The method leads naturally to shell-model components expressed in terms of SU_3, SU_4 strong-coupled functions. Whenever results are available for all possible $(\lambda\mu)$, for a particular Q, these SU_3, SU_4 strong-coupled functions can be transformed to ordinary angular momentum coupled functions (SU_3 weak-coupled functions) by

straightforward SU_3 coupling techniques, (see Eqs. (2) and (3) of Ref. 12). Results for the expansions of the antisymmetrized $[\Phi^{(Q0)}_{\text{rel.}} \times (\alpha + {}^{12}\text{C})^{(04)}]^{(\lambda\mu)}$ cluster functions are given in Table I for $Q \leq 8$, hence for ^{16}O core excitations up to $4\hbar\omega$. Results could be extended to higher Q values. For small values of $(\lambda\mu)$, however, the number of shell-model components becomes very large, and a full shell-model expansion may then no longer be of interest. The shell-model components in Table I are SU_3, SU_4 strong-coupled functions. For the sake of compactness, however, SU_4 labels are omitted whenever they follow unambiguously from the SU_3 labels and the fact that the resultant total SU_4 label is $[\tilde{f}]=[0]$. For the state

$$|(0s)^4[(0p)^{10}(02)[(sd)^1(20)(pf)^1(30)](\lambda_2\mu_2)](\lambda\mu)\rangle,$$

e.g., the SU_4 representation for the $(0p)^{10}$ configuration, $[\tilde{f}_{10}]=[1^2]$, follows from $(\lambda_{10}\mu_{10})=(02)$; the two-particle configuration $(sd)^1(pf)^1$ must thus be coupled to SU_4 symmetry $[\tilde{f}_2]=[1^2]$, a spatially symmetric two-particle state, to yield the required $[\tilde{f}_{10}]\times[\tilde{f}_2]\rightarrow[0]$. The expansion of the $\alpha+{}^{12}$C cluster wave function of Eq. (2) in terms of shell-model components gives $(1/N)$ times a normalized shell-model functions. Table I lists the normalized shell-model functions along with the normalization coefficients N of the cluster wave functions. The values of $(1/N)$ have previously been calculated by Horiuchi.[13]

TABLE I. Shell-model expansions of $\alpha+{}^{12}$C cluster wave functions for ^{16}O. State vectors are given in SU_3, SU_4 strong-coupled form. Square brackets denote both SU_3 and SU_4 coupling. SU_4 representation labels $[\tilde{f}]$ are omitted whenever they follow (unambiguously) from the SU_3 labels $(\lambda\mu)$ and the fact that the final resultant $[\tilde{f}]=[0]$.

$Q=4$

$(\lambda\mu)=(00)\quad (1/N)=(\frac{40}{9})^{1/2}$

$|(0s)^4(0p)^{12}(00)\rangle$

$Q=5$

$(\lambda\mu)=(21)\quad (1/N)=-(\frac{112}{81})^{1/2}$

$|(0s)^4[(0p)^{11}(01)(sd)^1(20)](21)\rangle$

$Q=6$

$(\lambda\mu)=(42)\quad (1/N)=(\frac{20}{27})^{1/2}$

$|(0s)^4[(0p)^{10}(02)(sd)^2(40)](42)\rangle$

$(\lambda\mu)=(31)\quad (1/N)=(\frac{32}{81})^{1/2}$

$\{(\frac{7}{64})^{1/2}|(0s)^4[(0p)^{11}(01)(pf)^1(30)](31)\rangle$
$+(\frac{9}{128})^{1/2}|(0s)^4[(0p)^{10}(02)(sd)^2(40)](31)\rangle$
$-(\frac{105}{128})^{1/2}|(0s)^4[(0p)^{10}(10)(sd)^2(21)](31)\rangle\}$

$(\lambda\mu)=(20)\quad (1/N)=(\frac{112}{81})^{1/2}$

$\{(\frac{27}{64})^{1/2}|(0s)^4[(0p)^{11}(01)(pf)^1(30)](20)\rangle$
$+(\frac{1}{32})^{1/2}|(0s)^4[(0p)^{10}(02)(sd)^2(40)](20)\rangle$
$-(\frac{5}{64})^{1/2}|(0s)^4[(0p)^{10}(02)(sd)^2(02)](20)\rangle$
$+(\frac{15}{64})^{1/2}|(0s)^4[(0p)^{10}(10)(sd)^2(21)](20)\rangle$
$+(\frac{15}{64})^{1/2}|[(0s)^3[(0p)^{12}(00)(sd)^1(20)]](20)\rangle\}$

$Q=7$

$(\lambda\mu)=(63)\quad (1/N)=-(\frac{40}{81})^{1/2}$

$|(0s)^4[(0p)^9(03)(sd)^3(60)](63)\rangle$

TABLE I. (*Continued*)

$(\lambda\mu) = (52) \quad (1/N) = (\frac{4}{27})[\frac{55}{3}]^{1/2}$

$\{(-\frac{1}{4})|(0s)^4[(0p)^9(03)(sd)^3(60)](52)\rangle$

$+(\frac{25}{32})^{1/2}|(0s)^4[(0p)^9(11)(sd)^3(41)](52)\rangle$

$-(\frac{5}{32})^{1/2}|(0s)^4[(0p)^{10}(02)[(sd)^1(20)(pf)^1(30)](50)](52)\rangle\}$

$(\lambda\mu) = (41) \quad (1/N) = (\frac{80}{81})^{1/2}$

$\{-(\frac{1}{80})^{1/2}|(0s)^4[(0p)^9(03)(sd)^3(60)](41)\rangle$

$+(\frac{3}{160})^{1/2}|(0s)^4[(0p)^9(03)(sd)^3(22)](41)\rangle$

$+(\frac{5}{16})(\frac{15}{11})^{1/2}|(0s)^4[(0p)^9(11)(sd)^3(41)](41)\rho=1\rangle$

$-\left(\frac{15}{32\times 11}\right)^{1/2}|(0s)^4[(0p)^9(11)(sd)^3(41)](41)\rho=2\rangle$

$+(\frac{15}{256})^{1/2}|(0s)^4[(0p)^9(11)(sd)^3(22)](41)\rangle$

$-(\frac{9}{40})^{1/2}|(0s)^4[(0p)^{10}(02)[(sd)^1(20)(pf)^1(30)](50)](41)\rangle$

$-(\frac{9}{160})^{1/2}|(0s)^4[(0p)^{10}(02)[(sd)^1(20)(pf)^1(30)](31)](41)\rangle$

$-(\frac{25}{128})^{1/2}|(0s)^4[(0p)^{10}(10)[(sd)^1(20)(pf)^1(30)](50)](41)\rangle$

$+(\frac{27}{128})^{1/2}|[(0s)^3[(0p)^{11}(01)(sd)^2(40)](41)](41)\rangle$

$-(\frac{3}{64})^{1/2}|(0s)^4[(0p)^{11}(01)(sdg)^1(40)](41)\rangle\}$

$(\lambda\mu) = (30) \quad (1/N) = (\frac{8}{9})^{1/2}$

$\{-(\frac{1}{240})^{1/2}|(0s)^4[(0p)^9(03)(sd)^3(60)](30)\rangle$

$-(\frac{7}{1920})^{1/2}|(0s)^4[(0p)^9(03)(sd)^3(22)](30)\rangle$

$-(\frac{5}{96})^{1/2}|(0s)^4[(0p)^9(11)(sd)^3(41)](30)\rangle$

$-(\frac{7}{96})^{1/2}|(0s)^4[(0p)^9(11)(sd)^3(22)](30)\rangle$

$+(\frac{7}{384})^{1/2}|(0s)^4[(0p)^9(11)(sd)^3(11)](30)\rangle$

$+(\frac{35}{384})^{1/2}|(0s)^4[(0p)^9(00)(sd)^3(30)](30)\rangle$

$-(\frac{45}{512})^{1/2}|(0s)^4[(0p)^{10}(02)[(sd)^1(20)(pf)^1(30)](50)](30)\rangle$

$-(\frac{35}{256})^{1/2}|(0s)^4[(0p)^{10}(02)[(sd)^1(20)(pf)^1(30)](31)](30)\rangle$

$-(\frac{7}{512})^{1/2}|(0s)^4[(0p)^{10}(02)[(sd)^1(20)(pf)^1(30)](12)](30)\rangle$

$-(\frac{35}{256})^{1/2}|(0s)^4[(0p)^{10}(10)[(sd)^1(20)(pf)^1(30)](31)](30)\rangle$

$-(\frac{21}{256})^{1/2}|(0s)^4[(0p)^{11}(01)(sdg)^1(40)](30)\rangle$

$+(\frac{21}{512})^{1/2}|[(0s)^3[(0p)^{11}(01)(sd)^2(40)](30)](30)\rangle$

$-(\frac{105}{512})^{1/2}|[(0s)^3[(0p)^{11}(01)(sd)^2(21)](30)](30)\rangle$

$-(\frac{7}{128})^{1/2}|[(0s)^3[(0p)^{12}(00)(pf)^1(30)](30)](30)\rangle\}$

$Q = 8$

$(\lambda\mu) = (84) \quad (1/N) = (\frac{280}{729})^{1/2}$

$|(0s)^4[(0p)^8(04)(sd)^4(80)](84)\rangle$

$(\lambda\mu) = (73) \quad (1/N) = (\frac{280}{729})^{1/2}$

$\{(\frac{3}{16})|(0s)^4[(0p)^8(04)(sd)^4(80)](73)\rangle$

$-(\frac{195}{256})^{1/2}|(0s)^4[(0p)^8(12)(sd)^4(61)](73)\rangle$

$+(\frac{13}{64})^{1/2}|(0s)^4[(0p)^9(03)](sd)^2(40)(pf)^1(30)](70)](73)\rangle\}$

TABLE I. (*Continued*)

$(\lambda\mu) = (62) \quad (1/N) = (40 \times 43/3^7)^{1/2}$

$$\frac{1}{8(43)^{1/2}}\{(\tfrac{75}{7})^{1/2}|(0s)^4[(0p)^8(04)(sd)^4(80)](62)\rangle$$
$$-(\tfrac{165}{14})^{1/2}|(0s)^4[(0p)^8(04)(sd)^4(42)](62)\rangle$$
$$-(49\times\tfrac{165}{34})^{1/2}|(0s)^4[(0p)^8(12)(sd)^4(61)](62)\rho=1\rangle$$
$$+(45\times\tfrac{32}{17})^{1/2}|(0s)^4[(0p)^8(12)(sd)^4(61)](62)\rho=2\rangle$$
$$-(\tfrac{165}{2})^{1/2}|(0s)^4[(0p)^8(12)(sd)^4(42)](62)\rangle$$
$$+(165)^{1/2}|(0s)^4[(0p)^8(20)(sd)^4(42)](62)\rangle$$
$$+73/(14)^{1/2}|(0s)^4[(0p)^9(03)[(sd)^2(40)(pf)^1(30)](70)](62)\rangle$$
$$+(33\times\tfrac{32}{7})^{1/2}|(0s)^4[(0p)^9(03)[(sd)^2(40)(pf)^1(30)](51)](62)\rangle$$
$$+10(\tfrac{55}{7})^{1/2}|(0s)^4[(0p)^9(11)[(sd)^2(40)(pf)^1(30)](70)](62)\rangle$$
$$+(\tfrac{165}{14})^{1/2}|(0s)^4[(0p)^9(11)[(sd)^2(40)(pf)^1(30)](51)](62)\rangle$$
$$+(\tfrac{165}{2})^{1/2}|(0s)^4[(0p)^9(11)[(sd)^2(21)(pf)^1(30)](51)](62)\rangle$$
$$+(\tfrac{11}{2})^{1/2}|(0s)^4[(0p)^{10}(02)(pf)^2(60)](62)\rangle$$
$$+(27\times11)^{1/2}|(0s)^4[(0p)^{10}(02)[(sd)^1(20)(sdg)^1(40)](60)](62)\rangle$$
$$+(81\times\tfrac{11}{2})^{1/2}|[(0s)^3[(0p)^{10}(02)(sd)^3(60)](62)](62)\rangle\}$$

$(\lambda\mu) = (51) \quad (1/N) = (40 \times 43/3^7)^{1/2}$

$$\frac{1}{64[43]^{1/2}}\{19(\tfrac{3}{7})^{1/2}|(0s)^4(0p)^8(04)(sd)^4(80)](51)\rangle$$
$$+(\tfrac{165}{14})^{1/2}|(0s)^4[(0p)^8(04)(sd)^4(42)](51)\rangle$$
$$-109\left(\frac{165}{2\times263}\right)^{1/2}|(0s)^4[(0p)^8(12)(sd)^4(61)](51)\rho=1\rangle$$
$$+21(5\times\tfrac{77}{263})^{1/2}|(0s)^4[(0p)^8(12)(sd)^4(61)](51)\rho=2\rangle$$
$$-5\times3^3\left(\frac{33}{14\times19}\right)^{1/2}(\tfrac{33}{14}\times19)^{1/2}|(0s)^4[(0p)^8(12)(sd)^4(42)](51)\rho=1\rangle$$
$$+10(3\times\tfrac{77}{19})^{1/2}|(0s)^4[(0p)^8(12)(sd)^4(42)](51)\rho=2\rangle$$
$$-3(165)^{1/2}|(0s)^4[(0p)^8(12)(sd)^4(23)](51)\rangle$$
$$-15(\tfrac{33}{7})^{1/2}|(0s)^4[(0p)^8(12)(sd)^4(31)](51)\rangle$$
$$-25(\tfrac{66}{7})^{1/2}|(0s)^4[(0p)^8(20)(sd)^4(42)](51)\rangle$$
$$+15(\tfrac{22}{7})^{1/2}|(0s)^4[(0p)^8(20)(sd)^4(31)](51)\rangle$$
$$-45(\tfrac{11}{2})^{1/2}|(0s)^4[(0p)^8(01)(sd)^4(50)](51)\rangle$$
$$+68(\tfrac{11}{7})^{1/2}|(0s)^4[(0p)^9(03)[(sd)^2(40)(pf)^1(30)](70)](51)\rangle$$
$$+53(\tfrac{33}{14})^{1/2}|(0s)^4[(0p)^9(03)[(sd)^2(40)(pf)^1(30)](51)](51)\rangle$$
$$+8(33)^{1/2}|(0s)^4[(0p)^9(03)[(sd)^2(40)(pf)^1(30)](32)](51)\rangle$$
$$-(330)^{1/2}|(0s)^4[(0p)^9(03)[(sd)^2(02)(pf)^1(30)](32)](51)\rangle$$
$$+25(\tfrac{11}{2})^{1/2}|(0s)^4[(0p)^9(11)[(sd)^2(40)(pf)^1(30)](70)](51)\rangle$$
$$-(\tfrac{63}{2})(\tfrac{165}{46})^{1/2}|(0s)^4[(0p)^9(11)[(sd)^2(40)(pf)^1(30)](51)](51)\rho=1\rangle$$
$$+(\tfrac{205}{2})(\tfrac{33}{46})^{1/2}|(0s)^4[(0p)^9(11)[(sd)^2(40)(pf)^1(30)](51)](51)\rho=2\rangle$$
$$+(165)^{1/2}/2|(0s)^4[(0p)^9(11)[(sd)^2(40)(pf)^1(30)](32)](51)\rangle$$
$$+5(\tfrac{33}{2})^{1/2}|(0s)^4[(0p)^9(11)[(sd)^2(02)(pf)^1(30)](32)](51)\rangle$$
$$+(\tfrac{409}{2})\left(\frac{165}{14\times23}\right)^{1/2}(\tfrac{165}{14}\times23)^{1/2}|(0s)^4[(0p)^9(11)[(sd)^2(21)(pf)^1(30)](51)](51)\rho=1\rangle$$
$$+(\tfrac{55}{2})(\tfrac{231}{46})^{1/2}|(0s)^4[(0p)^9(11)[(sd)^2(21)(pf)^1(30)](51)](51)\rho=2\rangle$$
$$+(\tfrac{9}{2})(55)^{1/2}|(0s)^4[(0p)^9(11)[(sd)^2(21)(pf)^1(30)](32)](51)\rangle$$

TABLE I. (*Continued*)

$$+15\left(\tfrac{66}{7}\right)^{1/2}|(0s)^4[(0p)^9(11)[(sd)^2(21)(pf)^1(30)](40)](51)\rangle$$
$$-25\left(\tfrac{33}{2}\right)^{1/2}|(0s)^4[(0p)^9(00)[(sd)^2(21)(pf)^1(30)](51)](51)\rangle$$
$$+\left(\tfrac{17}{2}\right)(55)^{1/2}|(0s)^4[(0p)^{10}(02)(pf)^2(60)](51)\rangle$$
$$+\left(\tfrac{5}{2}\right)(33)^{1/2}|(0s)^4[(0p)^{10}(10)(pf)^2(41)](51)\rangle$$
$$+6(330)^{1/2}|(0s)^4[(0p)^{10}(02)[(sd)^1(20)(sdg)^1(40)](60)](51)\rangle$$
$$+\left(\tfrac{15}{2}\right)(55)^{1/2}|(0s)^4[(0p)^{10}(02)[(sd)^1(20)(sdg)^1(40)](41)](51)\rangle$$
$$+5(154)^{1/2}|(0s)^4[(0p)^{10}(10)[(sd)^1(20)(sdg)^1(40)](60)](51)\rangle$$
$$-\left(\tfrac{35}{2}\right)(11)^{1/2}|(0s)^4[(0p)^{10}(10)[(sd)^1(20)(sdg)^1(40)](41)](51)\rangle$$
$$+9(55)^{1/2}|[(0s)^3[(0p)^{10}(02)(sd)^3(60)](51)](51)\rangle$$
$$+\left(\tfrac{45}{2}\right)(33)^{1/2}|[(0s)^3[(0p)^{10}(02)(sd)^3(41)](51)](51)\rangle$$
$$-\left(\tfrac{45}{2}\right)(33)^{1/2}|[(0s)^3[(0p)^{10}(10)(sd)^3(41)](51)](51)\rangle$$
$$-\left(\tfrac{27}{2}\right)(55)^{1/2}|[(0s)^3[(0p)^{11}(01)[(sd)^1(20)(pf)^1(30)](50)[\tilde{f}]=[1^2]](51)](51)\rangle$$
$$+\left(\tfrac{15}{2}\right)(33)^{1/2}|[(0s)^3[(0p)^{11}(01)[(sd)^1(20)(pf)^1(30)](50)[\tilde{f}]=[2]](51)](51)\rangle$$
$$+7(33)^{1/2}|(0s)^4[(0p)^{11}(01)(pfh)^1(50)](51)\rangle\}$$

$$(\lambda\mu)=(40)\quad (1/N)=\left(40\times\frac{19}{3^6}\right)^{1/2}$$

(This state has 47 shell-model components. Amplitudes are available from the author.)

Since the recoupling transformations used in the expansion of the properly antisymmetrized cluster function introduce many shell-model components with 12-particle cores other than $(0s)^4(0p)^8(04)$, a properly antisymmetrized cluster function has many pieces which do not have a $^{12}C(\lambda\mu)=(04)$ + four particle $(\lambda\mu)=(Q0)$ parentage. Only the cluster-like shell-model components $(0s)^4(0p)^{8+n}(\lambda\mu)=(0,4-n)(sd,pf,\ldots)^{Q-n}(\lambda\mu)=(Q-n,0)$ can contribute to the α-spectroscopic amplitudes. For small values of $(\lambda\mu)$ these often comprise a relatively small percentage of the whole wave function. With the exception of the most deformed states for a particular core excitation up to $Q=8$, viz., the states with $(\lambda\mu)=(21)$ for $Q=5$, (42) for $Q=6$, (63) for $Q=7$, and (84) for $Q=8$, which are equivalent to single one-component shell-model functions (Table I) and 100% cluster-like, the remaining states often have surprisingly small cluster-like percentages in their wave functions. In the cluster states with $Q=6$ and $(\lambda\mu)=(31)$ and (20), the cluster-like components make up only 18% and 45% of these wave functions. For states with $Q=7$, the cluster-like components comprise only 21.9%, 28.4%, and 17.4% of the $(\lambda\mu)=(52)$, (41), and (30) cluster states, respectively. The largest component (78%) of the $(\lambda\mu)=(52)$ state, e.g., is the shell-model state $|(0s)^4[(0p)^9(11)(sd)^3(41)](52)\rangle$ which has a non-cluster-like (sd) configuration and a p-shell configuration with zero parentage to the cluster configuration $(0p)^8(04)$. The effects of antisymmetrization can thus be seen to be of crucial importance for $\alpha+^{12}C$ cluster states with $Q\leq 8$, and approximation techniques which ignore the antisymmetrization process may be very poor for cluster states with important low-Q components.

The shell-model decompositions of Table I also make it possible to compare these α-cluster states with other core excitations in ^{16}O, e.g., the giant $E2$ excitations which may have to be incorporated into a full description of the ^{16}O spectrum, particularly for accurate predictions of the $E2$ transition probabilities. Table I shows that the α-cluster states of ^{16}O have large components with single particle excitations of $2\hbar\omega$. The state with $Q=6$, $(\lambda\mu)=(20)$, e.g., has $s^4p^{11}pf^1$ and $s^3p^{12}sd^1$ components of 42.2% and 23.4% which correspond to excitations in ^{16}O in which a single particle has been lifted from the $0p$ to the pf shell and from the $0s$ to the sd shell, respectively. The overlap of this (20) α-cluster state with a giant $E2$ excitation based on the ^{16}O closed shell configuration is thus large, (0.808), with important implications for the α-breakup of this giant $E2$ excitation. Giant $E2$ excitations based on higher core-excited states, in particular the np-nh core excitations of the largest possible intrinsic deformation, the shell-model states with $(\lambda\mu)=(21)$, (42), (63), and (84) for $n=1$, 2, 3, and 4, also have large over-

laps with the α-cluster states of Table I. For example, the α-cluster state with $Q=7$ and $(\lambda\mu)=(41)$ has an overlap of 0.816 with the giant $E2$ excitation with the same SU_3 symmetry based on the 1p-1h state with $(\lambda\mu)=(21)$. For the states with $Q=8$ $(\lambda\mu)=(62)$, $Q=9$ $(\lambda\mu)=(83)$, and $Q=10$ $(\lambda\mu)=(10,4)$ the corresponding overlaps are 0.818, 0.821, and 0.832, (see Ref. 24).

Since the cluster wave functions are free of spurious center of mass excitations, states with $Q=5$, 6, 7, 8 give a translationally invariant shell-model basis for p-h excitations in ^{16}O. Since states for all possible $(\lambda\mu)$ have been constructed, it is now possible to reexamine the weak-coupling model[21,22] for ^{16}O in terms of 0^+, 2^+, 4^+ four-hole states (^{12}C) and 0^+, 2^+, ..., 8^+ four-particle states (^{20}Ne), e.g., without the uncertainties introduced by the use of p-h excitations contaminated by spurious components.

The inclusion of a select set of properly antisymmetrized cluster states may give a convenient way of introducing a few of the most physically relevant low-lying core excitations into the shell-model description of many nuclei. These core excitations are then automatically free of spurious center of mass contamination. The techniques presented in this investigation make it possible to treat these complicated core excitations on an equal footing with the shell-model states of the simpler ground state configurations. The detailed expansion of the $\alpha+^{12}$C cluster wave functions for ^{16}O make it possible to test some of the approximations made in orthogonality condition model calculations. These wave functions should also be useful in combining earlier p-h shell-model calculations for ^{16}O with the much more ambitious cluster-model calculations.

APPENDIX A. TECHNICAL DETAILS OF THE SU_3,SU_4 RECOUPLING PROCESS

It is important to use a consistent set of phases for the unitary group coupling and recoupling coefficients. This can be achieved by interpreting both U_3 and U_4 recoupling coefficients for multiplicity-free couplings in terms of permutation group recoupling coefficients.[17,18] The permutation group phase conventions of Kramer[18] have been adopted. These are consistent with the SU_3 phase conventions of Draayer and Akiyama,[16] as well as the ordinary SU_2 phase conventions. However, some of the SU_4 phase conventions used earlier[15,19] must be modified.

The three steps of the calculational process will be discussed in terms of the present phase conventions.

1. Reordering of SU_3,SU_4 coupling

In the present unified phase convention the change in the order of a unitary group coupling from $\Gamma_1\times\Gamma_2\to\Gamma_3$ to $\Gamma_2\times\Gamma_1\to\Gamma_3$ is particularly simple, *provided* the coupling $\Gamma_1\times\Gamma_2\to\Gamma_3$ is multiplicity free (the representation Γ_3 occurs only once in the Kronecker product $\Gamma_1\times\Gamma_2$). In the case of multiplicity-free couplings the reordering introduces the phase factor

$$[\Psi^{\Gamma_2}\times\Psi^{\Gamma_1}]^{\Gamma_3}_a=(-1)^{\phi(\Gamma_1)+\phi(\Gamma_2)-\phi(\Gamma_3)}[\Psi^{\Gamma_1}\times\Psi^{\Gamma_2}]^{\Gamma_3}_a \tag{15}$$

with

$$\phi(\Gamma)=\frac{1}{2}\sum_{i<j}^{N}(f_i-f_j)$$

for all $U(N)$. The f_i are the partition numbers which indicate the number of squares in the ith row of the Young tableau which characterizes the irreducible representation of $U(N)$. For $U(2)$, with $\phi=\frac{1}{2}(f_1-f_2)=j$, the phase factor of Eq. (15) is the usual $(-1)^{j_1+j_2-j_3}$. For $U(3)$, $\phi(\lambda,\mu)=\lambda+\mu$; and the phase factor becomes $(-1)^{\lambda_1+\mu_1+\lambda_2+\mu_2-\lambda_3-\mu_3}$ which is consistent with Ref. 16. For couplings $(\lambda_1\mu_1)(\lambda_2\mu_2)\to(\lambda_3\mu_3)$ with d-fold occurrences of $(\lambda_3\mu_3)$, $d>1$, the states of Ref. 16 are based on the Biedenharn-Louck conventions, with multiplicity labels denoted by ρ. With $d>1$, the coupled states of Ref. 16 have no simple symmetry property under interchange of the order of representations $(\lambda_1\mu_1)$ and $(\lambda_2\mu_2)$; but the coupling coefficients fit consistently into the above scheme. With $d>1$, however, reordering of representations $(\lambda_1\mu_1)$ and $(\lambda_2\mu_2)$ must be handled with care. For the representations $[\tilde f_1\tilde f_2\tilde f_3\tilde f_4]$ of $U(4)$ the phase factors ϕ have the values

$$\phi=\tfrac{3}{2}\tilde f_1+\tfrac{1}{2}\tilde f_2-\tfrac{1}{2}\tilde f_3-\tfrac{3}{2}\tilde f_4\,. \tag{16}$$

2. SU_3 and SU_4 recoupling coefficients

SU_3 Racah and 9-$(\lambda\mu)$ recoupling coefficients of the type introduced in Refs. 12, 15, and 16 are consistent with the above phase conventions. The SU_4 recoupling for α-cluster functions usually contain the scalar SU_4 representation $[0]\equiv[1^4]$ and are of a particularly simple kind. Their magnitudes are often given by simple SU_4 dimension ratios. Their phases are fixed by the equivalent permutation group recoupling coefficients [see Eqs. (3.17)–(3.19) of Ref. 17]. The basic phases can be fixed by the signs of the recoupling coefficients of the form

$$U([f'][1^{n_1}][f][1^{n_2}];\,[f'']1^n])$$
$$=\left(\frac{n_1!\,n_2!}{n!}\right)^{1/2}\prod_{i,j}\epsilon_{ji}(1-1/\tau_{ij})^{1/2} \tag{17}$$

with $n=n_1+n_2$. The product runs over the n_2 indices i, and the n_1 indices j as follows: The Young tableau for the representation $[f'']\,(\equiv[f''_1 f''_2, \ldots, f''_N])$ is obtained from the tableau for $[f]$ by removing squares from the rows corresponding to labels i, while the tableau for $[f']$ is then obtained from the tableau for $[f'']$ by the further removal of squares from the rows corresponding to labels j. The axial distance between squares corresponding to i and j is τ_{ij}, with $\tau_{ji}=-\tau_{ij}$. The ϵ_{ji} is the sign of the axial distance τ_{ji}; $\epsilon_{ji}=+1$ if the row index for $j\leq$ row index for i, $\epsilon_{ji}=-1$ if the row index for $j>$ row index for i. The above U coefficients for all possible $[f'']$ give one column in a unitary transformation matrix. If this transformation is 2×2, the coefficients in the second column are both positive in the simple case when $n_2=1$. For greater dimensions or more complicated cases, the signs of the remaining coefficients can be obtained from these simple ones by recursion relations. The above phase conventions which fit onto those of Ref. 16 have the following additional advantages: (1) All U coefficients which describe 1-dimensional unitary transformations are always +1; (2) U(4) recoupling coefficients with at most three-rowed tableaux are equivalent to U(3) recoupling coefficients; U(3) recoupling coefficients with at most two-rowed tableaux are equivalent to U(2) recoupling coefficients (including phase). SU_4 9-$[\tilde{f}]$ recoupling coefficients containing only multiplicity-free couplings can be related to SU_4 U coefficients by

$$X_{SU_4}\begin{bmatrix}[f_a] & [f_b] & [f_{ab}]\\ [f_c] & [f_d] & [f_{cd}]\\ [f_{ac}] & [f_{bd}] & [f]\end{bmatrix}=\sum_{[f']}(-1)^{\phi[f_c]+\phi[f_{ab}]-\phi[f']}(-1)^{\phi[f_c]+\phi[f_a]-\phi[f_{ac}]}$$
$$\times U([f_{ab}][f_c][f][f_d];[f'][f_{cd}])U([f_c][f_a][f'][f_b];[f_{ac}][f_{ab}])U([f_{ac}][f_b][f][f_d];[f'][f_{bd}])\,, \tag{18}$$

where the reordering phase factors ϕ are given by Eq. (16). In α-cluster calculations SU_4 X coefficients of the following type are needed most frequently:

$$X_{SU_4}\begin{bmatrix}[1^{n_1}] & [1^{4-n_1}] & [0]\\ [1^{n_2}] & [1^{4-n_2}] & [0]\\ [\tilde{f}] & [\tilde{f}^*] & [0]\end{bmatrix}=(-1)^{\phi[1^{n_1}]+\phi[1^{n_2}]-\phi[\tilde{f}]}U([1^{n_2}][1^{n_1}][1^{n_2}][1^{4-n_1}];[\tilde{f}][0]), \tag{19}$$

where the U coefficient is given in terms of SU_4 dimension factors by

$$U([f_a][f_b][f_a][f_b^*];[f][0])=U([f_b^*][f_b][f_a][f_a];[0][f])=(-1)^{\phi[f_a]+\phi[f_b]-\phi[f]}\left(\frac{\dim_{SU(4)}[f]}{\dim_{SU(4)}[f_a]\dim_{SU(4)}[f_b]}\right)^{1/2}. \tag{20}$$

3. SU_3,SU_4 reduced cfp's

Within a single major shell the overlaps between SU_3-, SU_4-coupled n_1+n_2 particle functions and n-particle functions of definite SU_3 and SU_4 symmetry are given by reduced fractional parentage coefficients. For the $(0p)$ shell these multinucleon cfp's are given by simple permutation group (S_n) dimension ratios. For multiplicity free couplings

$$\langle[\Psi((0p)^{n_1}1,2\ldots,n_1)^{(\lambda_1\mu_1)[\tilde{f}_1]}\times\Psi((0p)^{n_2}n_1+1,n_1+2,\ldots,n_1+n_2)^{(\lambda_2\mu_2)[\tilde{f}_2]}]^{(\lambda\mu)[\tilde{f}]}$$
$$\times|\Psi((0p)^n 1,2,\ldots,n=n_1+n_2)^{(\lambda\mu)[f]}\rangle=(-1)^{\Phi}\left(\frac{\dim_{S_{n_1}}[f_1]\times\dim_{S_{n_2}}[f_2]}{\dim_{S_n}[f]}\right)^{1/2}\quad\text{with }(-1)^{\Phi}=\prod_{s,t}\epsilon_{st}\,, \tag{21}$$

where $[f]$ describe the full three-rowed tableaux which give the space symmetry of the $0p$-shell functions, from which the SU_3 and SU_4 quantum numbers can be read. The phase factor Φ follows the phase conventions of Kramer.[17] It can be expressed most simply in terms of the signs ϵ_{st} of the axial distances corresponding to particles numbered s and t, where s ranges from 1 to $n-1$, with

$s<t$, and t ranges only from n_1+1 to n. The full Young tableau for $[f]=[\lambda+\mu+\nu, \mu+\nu, \nu]$ is labeled by particle indices $1, \ldots, n$ in an allowed manner according to the following recipe: (1) The removal of particle labels $n_1+1, \ldots, n=n_1+n_2$ from the tableau for $[f]$ leaves the tableau for $[f_1]$. (2) Particle labels $1,2\ldots, n_1$ are placed within $[f_1]$ in cardinal order: $1,2\ldots, \lambda_1+\mu_1+\nu_1$ in the first row; $\lambda_1+\mu_1+\nu_1+1,\ldots, \lambda_1+2\mu_1+2\nu_1$ in the second row; $\lambda_1+2\mu_1+2\nu_1+1, \ldots, \lambda_1+2\mu_1+3\nu_1$ in the third row. (3) The particle labels $n_1+1, n_1+2, \ldots, n$ must, in addition, be placed in the tableau for $[f]$ in the following manner: If the tableau $[f]$ is emptied of indices $1,2\ldots, n_1$, a shift up and/or to the left of the labels $n_1+1, n_1+2,\ldots$ in order, leads to a tableau of shape $[f_2]$ with cardinal labeling, $n_1+1, n_1+2,\ldots, n_1+\lambda_2+\mu_2+\nu_2$ in the first row, etc. (4) The path along which n_1+2 is shifted to its final position must not cross the path along which n_1+1 was shifted; similarly for greater indices. This process is illustrated with a few examples in Fig. 1. (Note that once n_1+1 is in place, n_1+2 cannot be shifted to the right or down in its journey to its final position.) Although applications in the present work are limited to multiplicity-free $(0p)$-shell couplings, the above rules for determining the reduced multiparticle p-shell cfp's can be made to include the case of couplings with multiplicities, by properly matching the phases of the SU_4 recoupling coefficients with multiplicities to those of Ref. 16 for the ρ-dependent SU_3 U coefficients.

For the sd shell, the corresponding SU_3-, SU_4-reduced fractional parentage coefficients are given by $[n_1!n_2!/n!]^{1/2}$ times the quadruple-barred reduced matrix elements or $[n_1!n_2!/n!]^{1/2}\times(C$ factor) of Ref. 15. The computer code for these coefficients, including extensions into the fp shell, is also available.[20] (One-particle cfp's previously published by Akiyama[20] are based on a different phase convention.) The computer code[20] calculates the products of the needed reduced cfp's (C factors) multiplied by reduced $SU_4\supset S,T$ Wigner coefficients (D factors). Both C and D factors are tabulated in Ref. 15, the D factors taken from Ref. 19. To be consistent with the present phase conventions, however, the D factors for some of the unitary group couplings (hence also the C factors) need to be multiplied by overall phase factors of -1. In particular, all $n\to n-1$ particle C factors (Table A.4 of Ref. 15) with $n=even$ must be multiplied by -1. Of the $n\to n-2$ particle C factors (Table A.3 of Ref. 15) those connecting space symmetries $[31]\to[11]$, $[32]\to[21]$, $[43]\to[41]$, and their higher SU_4 equivalents, such as $[443]\to[441]$, must be multiplied by -1. Of the $n\to n-3$ particle C factors (Table A.2) those connecting space symmetries $[41]\to[2]$, $[41]\to[11]$, $[42]\to[3]$, $[431]\to[32]$, and $[4422]\to[432]$ must again be multiplied by -1.

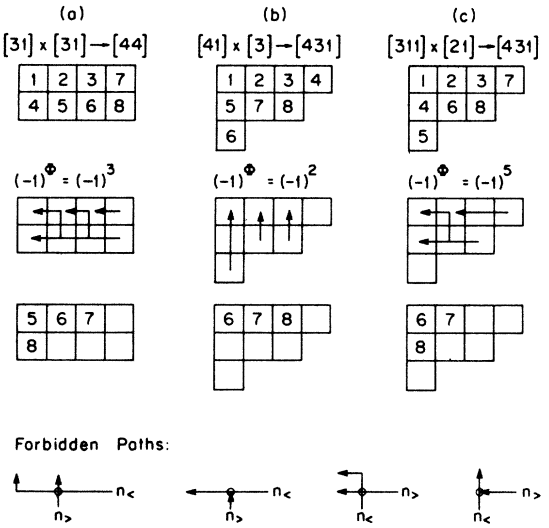

FIG. 1. Illustration of the numbering process for the determination of the phase factor Φ.

APPENDIX B. SAMPLE CALCULATION

The calculation of the matrix elements of $P^{(00)}(\vec{\eta}_{c.m.})$ which are needed in the evaluation of Eq. (12) will be illustrated with a simple example, the matrix element of $P^{(20)}(\vec{\eta}_{c.m.})$ between the cluster state with $Q_2=5$, $(\lambda\mu)=(21)$ on the right and a specific shell-model component on the left. The cluster state with $Q_2=5$, $(\lambda\mu)=(21)$ is made up of the single shell-model component $|s^4[p^{11}(01)sd^1(20)](21)\rangle$, multiplied by the norm factor $-(\frac{112}{81})^{1/2}$ (Table I). When expanded according to the transformation (5) the c.m. excitation operator is

$$P^{(20)}(\vec{\eta}_{c.m.})$$
$$=\frac{1}{16}\sum_{i=1}^{16}P^{(20)}(\vec{\eta}_i)$$
$$+\left(\frac{2!}{1!1!}\right)^{1/2}\frac{1}{16}\sum_{i<j}^{16}[\,P^{(10)}(\vec{\eta}_i)\times P^{(10)}(\vec{\eta}_j)\,]^{(20)}. \quad (22)$$

The first term acting on the above shell-model state gives rise to shell-model states of the configurations $s^4p^{11}(sdg)^1$, $s^4p^{10}(sd)^1(pf)^1$, and $s^3p^{11}(sd)^2$, while the second term gives rise to shell-model states of the configurations $s^4p^{10}(sd)^1(pf)^1$, $s^3p^{12}(pf)^1$, $s^3p^{11}(sd)^2$, and $s^4p^9(sd)^3$. Consider the last configuration as a special example. Of the $11!/9!2!$ operators of Eq. (22) which convert the $s^4p^{11}sd^1$ configuration into an $s^4p^9(sd)^3$ configuration, it is sufficient to consider

a single one. With

$$\langle\gamma| = \langle s^4[p^9(\lambda_9\mu_9) = (03)sd^3(\lambda_3\mu_3) = (60)\,](\lambda\mu)|\,, \tag{23}$$

$$\langle\gamma|P^{(20)}(\vec{\eta}_{\text{c.m.}})|s^4[p^{11}(01)sd^1(20)\,](21)\rangle$$

$$= \frac{[2]^{1/2}}{16}\frac{11!}{9!2!}\left(\frac{16!}{4!11!1!}\right)^{1/2}\sum_{\substack{(\bar{\lambda}\bar{\mu})=\\(21)(10)}} U((20)(01)(\lambda\mu)(20);(\bar{\lambda}\bar{\mu})(21))$$

$$\times\langle\gamma|\Psi^{(00)[0]}(s^4 1,2,3,4)[[[P^{(10)}(\vec{\eta}_{14})\times P^{(10)}(\vec{\eta}_{15})\,]^{(20)}\times\Psi^{(01)[1^3]}(p^{11}5,\ldots,15)]^{(\bar{\lambda}\bar{\mu})[1^3]}$$

$$\times\Psi^{(20)[1]}(sd,16)\,]^{(\lambda\mu)[0]}\rangle\,.$$

Here, a (normalized) antisymmetrizer which antisymmetrizes the 4+11+1-particle function of the right hand side has been allowed to act to the left to yield the factor $(16!/4!11!1!)^{1/2}$, and an SU_3 recoupling transformation has been performed to allow the operator to act directly on the p-shell part of the wave function.

By uncoupling particles labeled 14 and 15 from the p^{11} configuration by means of a p-shell reduced cfp, applying a reordering transformation to the coupling $(20)\times(01)\to(\bar{\lambda}\bar{\mu})$, and employing a second SU_3 recoupling transformation

$$[[P^{(10)}(\vec{\eta}_{14})\times P^{(10)}(\vec{\eta}_{15})\,]^{(20)}\times\Psi^{(01)[1^3]}(p^{11}5,\ldots,15)\,]^{(\bar{\lambda}\bar{\mu})[1^3]}$$

$$= \left(\tfrac{2}{11}\right)^{1/2}(-1)^{2+1-\bar{\lambda}-\bar{\mu}}\sum_{(\lambda_2\mu_2)} U((03)(20)(\bar{\lambda}\bar{\mu})(20);(01)(\lambda_2\mu_2))$$

$$\times[\Psi^{(03)[1]}(p^9,5,\ldots,13)\times[[P^{(10)}(\vec{\eta}_{14})\times P^{(10)}(\vec{\eta}_{15})\,]^{(20)}\times\Psi^{(20)[1^2]}(p^2,14,15)]^{(\lambda_2\mu_2)[1^2]}]^{(\bar{\lambda}\bar{\mu})[1^3]}$$

$$+\ldots\,. \tag{24}$$

The parentage expansion includes additional terms (indicated by $+\cdots$) but only the term with $(\lambda_9\mu_9)=(03)$ and cfp $=(\frac{2}{11})^{1/2}$ [Eq. (21)] can make a contribution to the matrix element (23). Next

$$[[P^{(10)}(\vec{\eta}_{14})\times P^{(10)}(\vec{\eta}_{15})\,]^{(20)}\times\Psi^{(20)[1^2]}(p^2,14,15)\,]^{(\lambda_2\mu_2)[1^2]} = \sqrt{2}\times\sqrt{2}X_{SU_3}\begin{bmatrix}(10)&(10)&(20)\\(10)&(10)&(20)\\(20)&(20)&(\lambda_2\mu_2)\end{bmatrix}\Psi^{(\lambda_2\mu_2)}(sd^2,14,15)$$

$$= 2[\,\delta_{(\lambda_2\mu_2)(40)}-\tfrac{1}{2}\delta_{(\lambda_2\mu_2)(02)}]\Psi^{(\lambda_2\mu_2)}(sd^2,14,15)\,, \tag{25}$$

where the $\sqrt{2}$ factors arise through the application of Eq. (7), and the SU_3 9-$(\lambda\mu)$ coefficient with at most two-rowed tableaux is equivalent to a simple SU_2 9-j coefficient (in unitary form) and is thus easily evaluated. One final recoupling transformation from the $[[(03)\times(\lambda_2\mu_2)\,](\bar{\lambda}\bar{\mu})\times(20)\,](\lambda\mu)$ to the $[(03)\times[(\lambda_2\mu_2)\times(20)\,](\lambda_3\mu_3)\,](\lambda\mu)$ scheme is needed to couple the sd particles to three-particle $(\lambda_3\mu_3)$. The overlap with the wave function for the sd^3 configuration of the left hand side of (23) introduces an sd-shell reduced cfp which, for $(\lambda_3\mu_3)=(60)$, has the simple value $(+1)\delta_{(\lambda_2\mu_2)(40)}$. The combination of the various steps yields the matrix element

$$\langle s^4[p^9(03)sd^3(60)\,](\lambda\mu)|P^{(20)}(\vec{\eta}_{\text{c.m.}})|s^4[p^{11}(01)sd^1(20)\,](21)\rangle$$

$$= \frac{[2]^{1/2}}{16}\frac{11!}{9!2!}\left(\frac{16!}{4!11!1!}\right)^{1/2}\left(\frac{4!9!3!}{16!}\right)^{1/2}\left(\frac{2}{11}\right)^{1/2}2\times\sum_{\substack{(\bar{\lambda}\bar{\mu})=\\(21)(10)}}(-1)^{2+1-\bar{\lambda}-\bar{\mu}}U((20)(01)(\lambda\mu)(20);(\bar{\lambda}\bar{\mu})(21))$$

$$\times U((03)(20)(\bar{\lambda}\bar{\mu})(20);(01)(40))$$

$$\times U((03)(40)(\lambda\mu)(20);(\bar{\lambda}\bar{\mu})(60))\,. \tag{26}$$

The needed SU_3 U coefficients are readily available.[16] (In this example, with the final state SU_4 representation [0], all SU_4 recoupling coefficients have the trivial value +1.)

*Supported by the U. S. National Science Foundation.

[1]T. Tomoda and A. Arima, in Proceedings of the International Symposium on Cluster Structure of Nuclei and Transfer Reactions Induced by Heavy Ions, Tokyo, 1972 (unpublished); A. Arima, in Proceedings of the Second International Conference on Clustering Phenomena in Nuclei, College Park, Maryland, 1975 (unpublished).

[2]M. Kamimura, T. Matsuse, and K. Takada, Prog. Theor. Phys. 47, 1537 (1972).

[3]Y. Suzuki, Progr. Theor. Phys. 55, 1751 (1976); 56, 111 (1976).

[4]H. Bando, Nucl. Phys. A261, 269 (1976).

[5]M. LeMere, Y. C. Tang, and D. R. Thompson, Phys. Rev. C 14, 23, 1715 (1976).

[6]K. Wildermuth and W. McClure, *Springer Tracts in Modern Physics* (Springer, Berlin, 1966), Vol. 41; and Th. Kanellopoulos and K. Wildermuth, Nucl. Phys. 14, 349 (1960).

[7]S. Saito, Progr. Theor. Phys. 41, 705 (1969).

[8]J. Janecke, F. D. Becchetti, and F. Milder, Bull. Am. Phys. Soc. 11, 21, 554 (1976); F. D. Becchetti *et al.*, *ibid.* 22; and (unpublished).

[9]P. Dyer and C. A. Barnes, Nucl. Phys. A233, 495 (1974).

[10]P. J. Ellis and T. Engeland, Nucl. Phys. A144, 161 (1970); A181, 368 (1972).

[11]D. J. Millener (private communication); Brookhaven National Laboratory Report No. 21829 (unpublished).

[12]K. T. Hecht, Nucl. Phys. A283, 223 (1977).

[13]H. Horiuchi, Progr. Theor. Phys. 51, 745 (1974).

[14]P. Kramer, Rev. Mex. Fis. 19, 241 (1970).

[15]K. T. Hecht and D. Braunschweig, Nucl. Phys. A244, 365 (1975).

[16]Y. Akiyama and J. P. Draayer, Comp. Phys. Commun. 5, 405 (1973); J. Math. Phys. 14, 405 (1973).

[17]P. Kramer, Z. Phys. 205, 181 (1967).

[18]P. Kramer, Z. Phys. 216, 68 (1968); P. Kramer and T. H. Seligman, *ibid.* 219, 105 (1969).

[19]K. T. Hecht and S. C. Pang, J. Math. Phys. 10, 1571 (1969).

[20]D. Braunschweig (unpublished); Y. Akiyama, Nucl. Data A2, 403 (1966).

[21]A. Arima, H. Horiuchi, and T. Sebe, Phys. Lett. 24B, 129 (1967).

[22]T. Engeland and P. J. Ellis, Phys. Lett. 25B, 57 (1967).

[23]K. T. Hecht, University of Michigan report (unpublished).

[24]K. T. Hecht, International Conference on Nuclear Structure, Tokyo, 1977 (unpublished).