Field theoretic treatment of π-nuclear scattering*

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The dynamical assumption that single mesons are emitted or absorbed by nucleons is applied to π-nucleus scattering and leads to a π-nucleus Low equation. With the neglect of multinucleon currents and true meson absorption, the π-nucleus Low equation is shown to be equivalent to a linear equation of the same form as in standard theories except for a striking propagator modification arising from the energy dependence of the driving term. The use of a corresponding linear equation for π-nucleon scattering enables us to derive a field-theoretic multiple-scattering series. It is found that the propagator modification eliminates the Kisslinger singularity, reduces the Lorentz-Lorenz effect and local field corrections. The effect of true meson absorption on π-nucleus elastic scattering is also examined and found to be somewhat important at low ($\sim$50 MeV) energies but negligible (due to strong first-order optical absorption) for energies greater than 100 MeV.

[NUCLEAR REACTIONS Scattering theory, π-nucleon scattering, π-^{16}O scattering, $E=50$–275 MeV.]

I. INTRODUCTION

We consider the problem of how a meson interacts with a nucleus when the basic meson-nucleon interaction is taken as either a single absorption by, or emission from, a nucleon. This is opposed to the standard treatment which assumes that mesons and nucleons interact via a potential and in which a sum over all of the potential interactions with nucleons is carried out. Our different dynamical assumption leads to effects which are not contained in standard treatments. Among these effects are the presence of crossed diagrams and the influence of reactions in which the meson deposits its entire energy into the nucleus on elastic scattering.

The understanding of π-nucleus interactions is being pursued with great vigor, and a complete set of references may be found in Ref. 1. Of particular relevance is the work of Dover and Lemmer[2] (see also Ingraham[3]) and Cammarata and Banerjee.[4] Dover and Lemmer were the first modern workers to use a field theoretic approach, and Cammarata and Banerjee initiated attempts to solve the π-nucleus Low equation.

The present approach can be described as follows. First we obtain a linear equation which is equivalent to the nonlinear π-nucleon Low equation. A linear equation equivalent to the π-nucleus Low equation can also be obtained. The development of the multiple-scattering theory proceeds via the technique of inserting π-nucleon scattering information (as summarized by the linear equation) into the π-nucleus linear equation. A brief report of some of the salient features of this work is given in Ref. 5.

The energy dependence of the π-nucleon and π-nucleus driving terms complicate the procedure of finding equivalent linear equations. In many cases, however, such equations can be obtained.

It is worthwhile to give a brief table of contents. The linear equation equivalent to the π-nucleon Low equation, derived in Ref. 6, serves as essential input, and the principal results of that reference are given in this Introduction. Sections II–V contain the necessary approximations and essential development in which the multiple-scattering series as an expansion in terms of a crossing-symmetric π-nucleon T matrix, which includes π-nucleon inelasticities, is obtained from a basic field theoretic Hamiltonian. Sections VI–VIII contain discussions of the implications of the resulting theory and treatments of various correction terms.

The results of Ref. 6 may be summarized as follows. If one works in the π-nucleon c.m. frame, includes only p-wave scattering and nucleon recoil kinetic energies to order $(1/m)$, the following equations obtain. The driving term of the Low equation is $v(q,p)/z$ where

$$\frac{v(q,p)}{z}=\sum_\alpha \frac{\lambda_\alpha}{z}u_\alpha(q,p)\equiv u_z(q,p), \tag{1.1}$$

where q represents the (relative) momentum of the pion and its isospin index; α is the label for the spin-isospin channel ($\alpha=1$, 2, 3, and 4 refers to the 11, 13, 31, and 33 channels, respectively). The interaction strengths λ_α are $(2/3)f^2/\mu^2[4,1,1,-2]$ where f is the renormalized coupling constant and u is the pion mass. The quantity $u_\alpha(q,p)$ is given by

$$u_\alpha(q,p)=\frac{4\pi}{(4\omega_q\omega_p)^{1/2}}u(q)u(p)P_\alpha(q,p) \tag{1.2}$$

with $u(q)$ representing the π-nucleon form factor, $\omega_q=(q^2+\mu^2)^{1/2}$ and $P_\alpha(q,p)$ are the channel projection operators of Chew and Low.[7] The π-nucleon T matrix t is given by

$$t_z(q,p)=\sum_\alpha h_\alpha(z)u_\alpha(q,p), \tag{1.3}$$

with

$$h_\alpha(W_k)=-e^{i\delta_\alpha}\sin\delta_\alpha[u^2(k)k^3]^{-1}(1+\omega_k/m),$$

where $W_k=(k^2+\mu^2)^{1/2}+k^2/2m$ and m is the nucleon mass. The Low equation is

$$h_\alpha(z)=\frac{\lambda_\alpha}{z}+\frac{1}{\pi}\int_\mu^\infty\frac{dW_k k^3}{(1+\omega_k/m)}\times\left[\frac{|h_\alpha(W_k)|^2}{z-W_k+i\epsilon}-\sum_\beta A_{\alpha\beta}\frac{|h_\beta(W_k)|^2}{z+W_k}\right], \tag{1.4}$$

where $A_{\alpha\beta}$ is the crossing matrix.[7] It may be shown that the solution of the linear equation

$$t_\alpha(z)=\frac{a_\alpha(z)}{z}u_\alpha\left[1+\left(\frac{a_\alpha(h_0)}{a_\alpha(z)}\right)^2\frac{z^2}{h_0^{\,2}}\frac{1}{z-h_0+i\epsilon}t_\alpha(z)\right], \tag{1.5}$$

where $h_0(k)=W_k$ and

$$a_\alpha(z)=\lambda_\alpha-\frac{z}{\pi}\sum_\beta\int_\mu^\infty\frac{dW_k}{(1+\omega_k/m)}A_{\alpha\beta}\frac{|h_\beta(W_k)|^2}{z+W_k}, \tag{1.6}$$

is a solution of (1.4). Equation (1.5) does not necessarily simplify the solution of the Low equation because $a_\alpha(z)$ depends on $t_\alpha(z)$. However, once $h_\alpha(z)$ is obtained, (1.5) provides a useful summary of the π-nucleon scattering information. If one neglects the crossed term [third term of Eq. (1.4)], then it follows that $a_\alpha(z)=\lambda_\alpha$ and Eq. (1.5) is simplified

$$t_\alpha(z)=\frac{\lambda_\alpha}{z}u_\alpha\left[1+\frac{z^2}{h_0^{\,2}}\frac{1}{z-h_0+i\epsilon}t_\alpha(z)\right]. \tag{1.7}$$

In Sec. II we state our assumptions about the Hamiltonian, and use the techniques of Wick[8] to obtain the π-nucleus Low equation. An assumed complete set of zero and one meson-nuclear eigenstates is used to simplify the equation.

The crossing-symmetric driving term arises from the sum over zero meson-nuclear intermediate states, and depends on matrix elements of the current operator, J_k, between nuclear eigenstates. Many-body perturbation techniques are applied to obtain the necessary matrix elements of the current operator as an equivalent matrix element of an *effective* current $\hat{J}_k$ between *unperturbed* eigenstates of a chosen H_0. In general, $\hat{J}_k$ is a multinucleon operator, but as a first approximation we keep only its single nucleon term $\hat{J}_k^{(1)}$. As a further approximation, in performing the sum over intermediate states, we assume that low excitation energies dominate, and neglect the excitation energies of the nuclear states. As a result of these approximations the driving term is simply the sum over nucleons of the driving term of the π-nucleon Low equation. Such approximations are implicit in conventional multiple-scattering theories in which the fundamental interaction (driving term of the linear wave equation) is taken as a π-nucleon potential which is the same in the nuclear medium as in free space. Hence, we call these approximations the *usual* approximation. Under the usual approximation the crossing symmetric driving term is given by V/z.

The full driving term is the sum of V/z and a correction term, ΔV (i.e., ΔV is the full driving term minus V/z). Now ΔV is also given as a sum over intermediate zero-meson eigenstates. Some of these states have excitation energy equal to the pion energy of the pion. Excitations of such states in ΔV, give rise to a complex potential. Because this term is not included in the usual approximation, it is said to arise from *true* meson absorption. (The usual optical absorption arises from various iterations of the π-nucleus driving term V/z.)

In Sec. III, the Low equation under the additional approximation of neglecting the π-nucleus crossing term $C(T^\dagger D_z T)$ along with the usual approximation is solved. The π-nucleus equivalent linear equation is used in conjunction with Eq. (1.7) to obtain the multiple-scattering theory. A significant difference between this theory and conventional versions is the appearance of an additional factor of $(z/H_0)^2$ in the propagator. This factor, which arises from the $1/z$ dependence of the driving term, considerably reduces off-shell scattering.

The importance of π-nucleus crossing has been emphasized by Cammarata and Banerjee.[9] Such effects are included in the crossing term which is handled, in Sec. IV, by first including its single-nucleon contributions. This enables us to develop the multiple-scattering expansion in terms of the crossing symmetric π-nucleon T matrix of Eq. (1.5). Only the first term (involving one nucleon) is crossing symmetric. By crossing the series of many-body (two-nucleon, three-nucleon, etc.) terms and adding them to the original series, one obtains a crossing symmetric result. For the case of $N=Z$, closed shell nuclei and in the fixed scatterer approximation the multiple-scattering series has approximately the form of the conventional theory except that the propagators are of the Klein-Gordon form of Ref. 9 multiplied by the $(z/H_0)^2$ modification.

In Sec. V the one-meson truncation is relaxed. First, a π-nucleon equation similar to Eq. (1.5) but including the effects of π-nucleon inelasticities is obtained. The techniques of Sec. IV may be applied (for pion energies less than π-production threshold) to obtain a multiple-scattering series in terms of a crossing-symmetric, π-nucleon T matrix which includes inelasticities.

At this point the only effect of the field theoretic approach (other than the explicit manifestation of crossing symmetry) is the $(z/H_0)^2$ propagator modification. The implications of this factor, with respect to the Kisslinger singularity, Lorentz-Lorenz effect, and local field corrections, are discussed in Sec. VI. First-order optical model calculations are performed for elastic π-^{16}O scattering. The relationship between our theory and those of others[10,11] is also discussed.

In Sec. VII the effect of true meson absorption is estimated. It is found that at low energies the effect of the (π, NN) reaction on elastic scattering is substantial.

The nonstatic crossing corrections are defined and estimated in Sec. VIII. These are small.

A discussion of the strengths and weaknesses of this approach is presented in Sec. IX.

Various technical details are presented in Appendixes A–E.

II. π-NUCLEUS LOW EQUATION

The starting point is a field-theoretic Hamiltonian consisting of a pion interaction term $H_{\pi N}$, a noninteracting pion Hamiltonian H_π, and a nonrelativistic noninteracting nucleon Hamiltonian. We have

$$H_{\pi N}=g_0 \sum_{p,q,p'} (\langle p' | j_q | p\rangle a_q^\dagger b_{p'}^\dagger b_p + \langle p' | j_q^\dagger | p\rangle a_q b_{p'}^\dagger b_p), \tag{2.1}$$

$$H_\pi=\sum_k \omega_k a_k^\dagger a_k, \tag{2.2}$$

$$H_N=\sum_p b_p^\dagger b_p\left(\frac{p^2}{2m_0}+m_0\right), \tag{2.3}$$

where g_0 is the unrenormalized coupling constant. The operator $b_p(b_p^\dagger)$ destroys (creates) a bare nucleon of momentum $\vec{p}$, spin magnetic quantum number s_p, and isospin magnetic quantum number t_p. The operator $a_q(a_q^\dagger)$ destroys (creates) a meson of momentum $\vec{q}$ and isospin index $\lambda_q(\lambda_q=1,2,3)$. The quantities μ and m_0 are the mass (139.6 MeV) of the physical pion and the mass of the bare nucleon. The operator j_q is the π-nucleon interaction vertex function.

The S matrix element for the scattering

$$\pi(k)+M \to \pi(q)+N, \tag{2.4}$$

where M and N are any two nuclear eigenstates, is

$$S_{fi}={}^{(-)}\langle Nq | Mk\rangle^{(+)} . \tag{2.5}$$

The states $|Mk\rangle^{(\pm)}$ refer to eigenstates which consist of a nucleus plus a free pion of quantum numbers k at an infinite time before or after the scattering event.

The straightforward application of the techniques of Wick yields an equation for the scattering states

$$|Mk\rangle^{(\pm)}=a_k^\dagger|M\rangle+\frac{1}{E-H\pm i\eta}g_0 J_k|M\rangle, \tag{2.6}$$

where

$$g_0 J_k=[H, a_k^\dagger]-\omega_k a_k^\dagger \tag{2.7}$$

and $E=E_M+\omega_k$. The first term of Eq. (2.6) represents a free pion and a nucleus at $t=\pm\infty$ and the second term represents the scattering.

By using Eq. (2.6) in Eq. (2.5) one has

$$S_{fi}=\delta_{fi}-2\pi i\,\delta(E_i-E_f)T_{fi}, \tag{2.8}$$

$$\begin{aligned} T_{fi}&={}^{(-)}\langle Nq| J_k|M\rangle \\ &=g_0{}^2\left\langle N \middle| J_k\frac{1}{E_M-\omega_q-H-i\eta} J_q^\dagger \middle| M\right\rangle \\ &\quad+g_0{}^2\left\langle N \middle| J_q^\dagger\frac{1}{E_M+\omega_k-H+i\eta} J_k \middle| M\right\rangle , \end{aligned} \tag{2.9}$$

where the second line of Eq. (2.9) is obtained by the equation for $a_q|M\rangle$,

$$a_q|M\rangle=\frac{1}{E_M-\omega_q-H-i\eta} J_q^\dagger|M\rangle,$$

which is derived and discussed in Appendix A. In using our simple Hamiltonian the seagull terms have been neglected. The importance of such terms has been recently discussed by Banerjee.[12] Equation (2.9) has been derived by Cammarata.[13] The nucleon-nucleus version of (2.9) has been obtained by Ernst.[14]

The next step in solving Eq. (2.9) is to insert a complete set of intermediate eigenstates of H. In this section only the no-meson and one-meson states are included so that inelastic states involving production of an additional pion are ignored. Thus we insert the truncated set

$$1=\sum |M\rangle\langle M| + \sum |Np\rangle^{(-)(-)}\langle Nq| \tag{2.10}$$

into Eq. (2.9).

It is necessary to inquire if the states of Eq. (2.10) have the necessary orthogonality properties. In particular, states (M) with excitation energy greater than a pion mass are degenerate with some of the one-meson states. It is shown in Appendix A that

$$\langle M|Np\rangle^{(\pm)}=0\,,$$
$$^{(\pm)}\langle Mk|Np\rangle^{(\pm)}=\delta(k,p)\delta(M,N) \tag{2.11}$$

so that Eq. (2.10) is meaningful.

Using (2.10) in (2.9) we have

$$\begin{aligned}T_{fi}={}&g_0^2\sum_L \frac{\langle N|J_q^\dagger|L\rangle\langle L|J_k|M\rangle}{E_M+\omega_k-E_L+i\eta}\\ &+g_0^2\sum_L \frac{\langle N|J_k|L\rangle\langle L|J_q^\dagger|M\rangle}{E_M-\omega_q-E_L+i\eta}\\ &+g_0^2\sum_{Lp}\left(\frac{\langle N|J_q^\dagger|Lp\rangle^{(-)(-)}\langle Lp|J_k|M\rangle}{E_M+\omega_k-E_L-\omega_p+i\eta}\right.\\ &\left.+\frac{\langle N|J_k|Lp\rangle^{(-)(-)}\langle Lp|J_q^\dagger|M\rangle}{E_M-\omega_q-E_L-\omega_p+i\eta}\right).\end{aligned} \tag{2.12}$$

It is interesting to note that for states $|L\rangle$ such that $\omega_k+E_M=E_L$, $\langle L|J_k|M\rangle$ is the transition operator for the meson absorption reaction.

The π-nucleus Low equation gives T_{fi} in terms of a nonlinear equation involving other T matrices T_{ni} (where n is any one-meson state) and a driving term. The first two terms of Eq. (2.12), which come from the no-meson intermediate states are defined as the driving term $T^{(B)}$. Then in a more concise notation

$$\begin{aligned}T_{fi}={}&T_{fi}^{(B)}+\sum_n \frac{T_{nf}^\dagger T_{ni}}{E_M+\omega_k-E_n+i\eta}\\ &+C\sum \frac{T_{nf}^\dagger T_{ni}}{E_M+\omega_k-E_n+i\eta},\end{aligned} \tag{2.13}$$

where the crossing operator C designates the replacement (k,p) by $(-p,-k)$ where the minus sign does not apply to the isospin indices. An even more schematic notation which stresses the matrix character of Eq. (2.13) is

$$T=T^{(B)}+T^\dagger D_z T+C(T^\dagger D_z T), \tag{2.14}$$

where $z=E_M+\omega_k$.

In order to solve Eq. (2.14) it is necessary to obtain expressions for the matrix elements $\langle L|J_k|M\rangle$. We proceed by developing a perturbation procedure for this quantity. First, add and subtract a Hermitian nucleon-nucleon potential V_{NN} and a mass counterterm CT to the Hamiltonian, with

$$\mathrm{CT}=\sum b_p^\dagger b_p\left[\frac{p^2}{2}\left(\frac{1}{m}-\frac{1}{m_0}\right)+m-m_0\right],$$

$$V_{NN}=\frac{1}{2}\sum (p_1p_2|v|p_3p_4)b_{p_1}^\dagger b_{p_2}^\dagger b_{p_4}b_{p_3}. \tag{2.15}$$

Then one has an unperturbed Hamiltonian H_0 and a perturbation Hamiltonian H_1

$$H=H_0+H_1, \tag{2.16}$$

$$H_0=H_\pi+\sum \frac{p^2}{2m}b_p^\dagger b_p+V_{NN}, \tag{2.17}$$

$$H_1=H_{\pi N}-V_{NN}-\mathrm{CT}. \tag{2.18}$$

The unperturbed states $|N\rangle_0$ are defined by

$$H_0|N\rangle_0=E_N|N\rangle_0, \tag{2.19}$$

where we assume that V_{NN} may be chosen so that the spectrum of H_0 is a good approximation to the physical spectrum. Of course, $H_\pi|N\rangle_0=0$. Furthermore, nuclear eigenstates of H_0 form a complete set.

By using the expressions for physical states in terms of unperturbed states one may write (see Appendix B) $\langle N|J_k|L\rangle$ as a matrix element of an effective operator $\hat{J}_k$ acting between unperturbed states, and a linked cluster expansion may be obtained for $\hat{J}_k$, i.e.,

$$g_0\langle N|J_k|L\rangle={}_0\langle N|\hat{J}_k|L\rangle_0.$$

In general $\hat{J}_k$ is a many-nucleon operator and

$$\hat{J}_k=\hat{J}_k^{(1)}+\hat{J}_k^{(2)}+\hat{J}_k^{(3)}+\cdots,$$

where the superscript denotes the number of nucleons involved in the matrix element. Here we consider only the single-nucleon terms. Examination of the perturbation series (Appendix B) shows that such terms depend only on CT and $H_{\pi N}$, and furthermore, that CT cancels all terms involving $H_{\pi N}$ except for a series which renormalizes the coupling constant. Thus under the single-nucleon approximation

$$\langle N|\hat{J}_q^{(1)}|L\rangle={}_0\langle N|gJ_q|L\rangle_0 \tag{2.20}$$

and the only influence of the meson degrees of freedom of the nucleus is to renormalize the coupling constant.

Even with the single-nucleon approximation to the effective current, it is necessary to make a further approximation. It is useful to define the quantity $z=E_M+\omega_k+i\eta=E_N+\omega_q+i\eta$ and rewrite the energy denominator, $1/(z\pm\epsilon)$, of the Born term in the form

$$\frac{1}{z \pm x} = \frac{1}{z} + \left(\frac{1}{z \pm x} - \frac{1}{z} \right).$$

We are then able to write the Born term as one which has an explicit $1/z$ dependence plus a specific correction term. We have

$$T_{fi}^{(B)} = \frac{V_{fi}}{z} + \Delta V_{fi}(z), \tag{2.21}$$

where

$$\frac{V_{fi}}{z} = g^2 \sum_L \frac{{}_0\langle N | J_q^\dagger | L \rangle_0 \, {}_0\langle L | J_k | M \rangle_0 - {}_0\langle N | J_k | L \rangle_0 \, {}_0\langle L | J_q^\dagger | M \rangle_0}{z}, \tag{2.22}$$

$$\Delta V_{fi}(z) = g^2 \sum_L {}_0\langle N | J_q^\dagger | L \rangle_0 \, {}_0\langle L | J_k | M \rangle_0 \left(\frac{1}{z - E_L} - \frac{1}{z} \right) + g^2 \sum_L {}_0\langle N | J_k | L \rangle_0 \, {}_0\langle L | J_q^\dagger | M \rangle_0 \frac{1}{z} \frac{E_L - (E_M + E_N)}{z + E_L - (E_M + E_N)}$$
$$+ \sum_{\substack{L, n, m \\ n \geq 1 \\ m \geq 1}} \left\{ \frac{{}_0\langle N | \hat{J}_q^{(n)\dagger} | L \rangle_0 \, {}_0\langle L | \hat{J}_k^{(m)} | M \rangle_0}{z - E_L} - \frac{{}_0\langle N | \hat{J}_k^{(m)} | L \rangle_0 \, {}_0\langle L | \hat{J}_q^{(n)\dagger} | M \rangle_0}{z + E_L - (E_M + E_N)} \right\}. \tag{2.23}$$

Our strategy is to first solve the Low equation with the driving term V/z and then to treat $\Delta V(z)$ as a perturbation, if possible. Note that $\Delta V(z)$ includes the effects of the multinucleon current operator and is complex because there are energies E_L greater than μ. Furthermore, $\Delta V(z)$ is crossing symmetric. Because conventional theories neglect three-body π-nucleon potentials and true meson absorption, we call the approximation of neglecting $\Delta V(z)$ the *usual* approximation.

The use of the completeness property of the eigenstates of H_0 gives

$$\frac{V_{fi}}{z} = \frac{g^2}{z} \, {}_0\langle N | [J_q^\dagger, J_k] | M \rangle_0 . \tag{2.24}$$

Because the commutation of two single-nucleon operators is itself a single-nucleon operator, one may write the operator V/z as

$$\frac{V}{z} = \frac{1}{z} \sum_i v_i , \tag{2.25}$$

where the matrix element is to be taken between nuclear states. In terms of second quantized notation

$$\sum_i \frac{v_i}{z} = \frac{g^2}{z} \sum_{p_1 p_2 p_3 p_4} [b_{p_1}^\dagger b_{p_2}, b_{p_3}^\dagger b_{p_4}] (p_1 | j_q^\dagger | p_2)(p_3 | j_k | p_4)$$
$$= \frac{g^2}{z} \sum_{p_1 p_2 p_4} b_{p_1}^\dagger b_{p_4} (p_1 | j_q^\dagger | p_2)(p_2 | j_k | p_4)$$
$$- \frac{g^2}{z} \sum_{p_2 p_3 p_4} b_{p_3}^\dagger b_{p_2} (p_3 | j_k | p_1)(p_1 | j_q^\dagger | p_2) , \tag{2.26}$$

$$\sum \frac{v_i}{z} = \frac{g^2}{z} \sum_{p, p'} b_{p'}^\dagger (p' | [j_q^\dagger, j_k] | p) b_p , \tag{2.27}$$

where use of the completeness of the momentum representation is made in obtaining (2.27). This expression for the approximate driving term has been obtained by Cammarata and Banerjee.[4]

At this point it is worthwhile to remind the reader that Eq. (2.27) represents an operator, which is to be taken between the one-meson eigenstates of H_0 in order to obtain V_{fi}/z. In order for this to be meaningful, it is necessary that the wave functions of H_0 be reasonable approximations to the wave functions of the physical states.

It is often noted that within the static model for J_q, the ground state expectation value of V vanishes for closed shell nuclei with zero isospin. However, the off-diagonal matrix elements of V/z do not vanish so that V/z may be used as a driving term even in the static model.

The next step is to rewrite the equation

$$T_{fi} = V_{fi}/z + T^\dagger D_z T + C(T^\dagger D_z T) \tag{2.28}$$

as an equivalent linear equation. Before proceeding with the solution of Eq. (2.28) it is worthwhile to consider the effects of introducing other meson-nucleon couplings into the Hamiltonian. This enables one to calculate better nuclear eigenstates. If these other mesons do not interact with the pion, the current operator J_k is unchanged. The existence of other mesons requires us to modify the completeness relation, Eq. (2.10), so as to include a sum over the appropriate meson-nuclear states. However, if the coupling is weak, or if the mass of the meson is very large, such terms may be ignored and the form of the equations for the T matrix are unchanged. (See, however, Ref. 15.)

III. SOLUTION OF THE LOW EQUATION IN THE USUAL APPROXIMATION—NO CROSSING TERM

The Low equation, under the usual approximation along with neglect of the crossing term, is solved in this section. In this case the Low equation is

$$T_{fi}(z)=\frac{V_{fi}}{z}+\sum_n \frac{T_{nf}^\dagger T_{ni}}{z-E_n+i\eta}. \tag{3.1}$$

Because the driving term of (3.1) is obtained by taking matrix elements of operators between unperturbed eigenstates, we look for solutions which are also expressible as operators taken between such eigenstates.

The solution of

$$T_z=\frac{V}{z}\left(1+\left(\frac{z}{H_0}\right)^2\frac{1}{z-H_0+i\eta}T_z\right). \tag{3.2}$$

where matrix elements of T_z are to be taken between unperturbed eigenstates, is a solution of Eq. (3.1). This is proved in Appendix C.

The appearance of the $(z/H_0)^2$ factor in the propagator may also be understood from a consideration of the analytic structure of Eqs. (3.1) and (3.2). From Eq. (3.1) we see that in the limit of z approaching zero, $T_z \propto 1/z$. The same limit obtains from Eq. (3.2). Furthermore, the elimination of the $(z/H_0)^2$ factor would completely change the z equals zero limit of (3.2). Similarly, it is apparent that as z approaches infinity, the T matrices of Eqs. (3.1) and (3.2) are both proportional to $1/z$.

The linear equation (3.2) is much more tractable than the nonlinear Low equation. Indeed, except for the presence of a factor $(z/H_0)^2$, Eq. (3.2) is the starting point for the usual multiple-scattering theory. One may obtain equations for multiple-scattering expansions by simply modifying the Green's functions of the standard theories.

In the remainder of this section the optical potential of Kerman, McManus, and Thaler (KMT)[16] is obtained. A discussion of several current multiple-scattering theories has been presented by Ernst, Londergan, Miller, and Thaler[17] and we closely follow their techniques.

In order to simplify the notation define

$$u_i=v_i/z \tag{3.3}$$

and

$$G(z)=\left(\frac{z}{H_0}\right)^2\frac{1}{z-H_0+i\eta} \tag{3.4}$$

so that

$$T_z=\sum u_i[1+G(z)T_z]. \tag{3.5}$$

The procedure of Watson[18] is followed by defining $T=\sum_i T_i$ with T_i given by

$$T_i=u_i+u_iG(z)\sum_j T_j. \tag{3.6}$$

The linear equation[6]

$$t=u+u\left(\frac{z}{h_0}\right)^2\frac{1}{z-h_0+i\eta}t \tag{3.7}$$

is a solution of the π-nucleon Low equation for p-wave π-nucleon scattering [neglecting the $C(t^\dagger Dt)$ term]. Here, as in Sec. I, h_0 is a propagator which includes the recoil kinetic energy of the nucleon to order $1/m$. Then for scattering from the ith nucleon

$$t_i=u_i+u_ig(z)t_i, \tag{3.8}$$

where

$$g(z)=\left(\frac{z}{h_0}\right)^2\frac{1}{z-h_0+i\eta}. \tag{3.9}$$

By solving Eq. (3.8) for u_i in terms of t_i and inserting the result in Eq. (3.6) we have

$$T_i=t_i+t_ig(z)\sum_{j\neq i}T_j+t_i[G(z)-g(z)]\sum T_j. \tag{3.10}$$

At this point we are faced with one of the more difficult problems of multiple-scattering theory. In order to proceed we must evaluate the terms involving $(G-g)$. As this problem has not been solved and because our present interest is in specific changes caused by the field-theoretic approach the $G-g$ term of Eq. (3.10) is simply ignored.

By iterating Eq. (3.10) and summing over i we have

$$T=\sum_i t_i+\sum_{i\neq j}t_igt_j+\sum_{\substack{i\neq j\\ j\neq k}}t_igt_jgt_k+\cdots. \tag{3.11}$$

The optical potential of KMT, U', when inserted into a Lippmann-Schwinger equation, yields the pseudo T matrix $T'=[(A-1)/A]T$. It is defined by

$$T'=U'+U'gPT', \tag{3.12}$$

where P is the ground state projection operator.

In Ref. 17 it is shown that U' is given by the series

$$U'=(A-1)\left\{\sum t_i+\sum_{i\neq j}t_igQt_j+\sum_{\substack{i\neq j\\ j\neq k}}t_igQgt_jgQt_k+\cdots\right\}, \tag{3.13}$$

where $Q=1-P$ and where it is understood that we may only take matrix elements of U' between antisymmetrized states. The use of (3.13) in (3.12), along with the definition of T', gives the series (3.11).

This means that within the usual approximation the field-theoretic multiple-scattering theory has the same form as conventional theories except for the use of a modified Green's function.

IV. CROSSING

In Sec. III the term $C[T^\dagger D_z T]$ is ignored. This leads to a multiple-scattering expansion in terms of a π-nucleon T matrix which is not crossing symmetric. Furthermore, crossed π-nucleus processes, which have been emphasized by Cammarata and Banerjee,[9] are also neglected. We seek to remedy these deficiencies by the following procedure. First, the $C[T^\dagger D_z t]$ term is approximated by its one-body part. This term, when added to V/z, defines a new driving term W, which is the sum over nucleons of the driving term of Eq. (1.5). A linear equation which is a solution of the π-nucleus Low equation with W as the driving term is then obtained. This enables us to develop the multiple-scattering expansion in terms of the crossing symmetric π-nucleon T matrix of Eq. (1.5). Only the first term, $\sum_i t_i$, of this new series is crossing symmetric, while terms involving two or more nucleons are not. By crossing the series of such terms and adding them to the original series one obtains a crossing symmetric result.

In doing the algebraic manipulations a problem occurs in that for each of the spin-isospin channels the driving term has a different energy dependence. An approximation method based on the fact that the most rapid energy variation, the $1/z$ dependence, is treated exactly is developed.

Consider the equation

$$T_z = V/z + T^\dagger D_z T + C(T^\dagger D_z T)\,. \tag{4.1}$$

As a first approximation we extract the one-body piece of the crossing term,

$$C(T^\dagger D_z T) \approx \sum_i C(t^\dagger D_z^{(0)} t)_i\,, \tag{4.2}$$

where $D_z^{(0)}$ neglects effects of nucleons other than i. This approximation is suggested by the form of the π-nucleon Low equation

$$t_z = v + t^\dagger D_z^{(0)} t + C(t^\dagger D_z^{(0)} t) \tag{4.3}$$

$$\equiv w + t^\dagger D_z^{(0)} t\,, \tag{4.4}$$

where Eq. (4.4) defines w. As shown in Ref. 2 and summarized in the Introduction

$$w(q,p) = \sum_\alpha \frac{a_\alpha(z)}{z}\, u_\alpha(q,p) = \sum_\alpha w_\alpha(z) \tag{4.5}$$

and

$$t_\alpha(z) = \frac{a_\alpha(z)}{z}\left[1 + \left(\frac{z}{h_0}\right)^2 \frac{1}{z-h_0}\left(\frac{a_\alpha(h_0)}{a_\alpha(z)}\right)^2 t_\alpha(z)\right]. \tag{4.6}$$

Using Eq. (4.2) in Eq. (4.1)

$$T = \sum_i w_i(z) + T^\dagger D_z T \equiv W + T^\dagger D_z T\,. \tag{4.7}$$

We wish to obtain an equivalent linear equation. The operator $\tilde{W}$ is defined by replacing $a_\alpha(z)$ by $a_\alpha(H_0)$ in (4.5)

$$\frac{\tilde{W}}{z} = \sum_i \{[a_\alpha(H_0)]^{1/2}(u_\alpha/z)[a_\alpha(H_0)]^{1/2}\}_i\,. \tag{4.8}$$

The idea of having a single-nucleon operator depend on a many-body Hamiltonian seems unusual. However, the matrix elements of W between eigenstates of H_0 are well defined:

$$\begin{aligned} {}_0\langle Np\,|\,\tilde{W}\,|\,Mq\rangle_0 = \sum_\alpha & \frac{u(q)\,u(p)}{(2\omega_q)^{1/2}(2\omega_p)^{1/2}} \\ & \times [a_\alpha(E_M+\omega_q)a_\alpha(E_N+\omega_p)]^{1/2} \\ & \times \left\langle N\left|\sum_i e^{i(\vec{q}-\vec{p})\cdot\vec{r}_i} P_\alpha(p,q)_i\right| M\right\rangle_0\,, \end{aligned} \tag{4.9}$$

where $|Mq\rangle_0$ is a one meson eigenstate of H_0.

Consider $\tilde{T}$ where,

$$\tilde{T}_z = \tilde{W}/z + \tilde{T}^\dagger D_z \tilde{T} \tag{4.10}$$

by the proof of Sec. III and Appendix C

$$\tilde{T}_z = \frac{\tilde{W}}{z} + \frac{\tilde{W}}{z}\left(\frac{z}{H_0}\right)^2 \frac{1}{z-H_0+i\eta}\,\tilde{T}_z\,, \tag{4.11}$$

so that Eq. (4.10) is in a solvable form. However, we want a solution of Eq. (4.7). Consider the matrix elements of $\tilde{T}$. We have

$$\begin{aligned} {}_0\langle Mk\,|\,\tilde{T}_z\,|Np\rangle_0 = & \sum_i {}_0\langle M\,|\,[a_\alpha(z)]^{1/2}(u_\alpha^{(i)}/z)[a_\alpha(z)]^{1/2}\,|N\rangle_0 \\ & + \sum_{Lq} \frac{\tilde{T}^\dagger_{Lq,Mk}\tilde{T}_{Lq,Np}}{z-E_L-\omega_q+i\eta} \end{aligned} \tag{4.12}$$

but $z = E_M + \omega_k = E_N + \omega_p$.

Thus

$$\tilde{T}_z = W + \tilde{T}^\dagger D_z \tilde{T} \tag{4.13}$$

and for *on-shell* values of z, $\tilde{T}$ is equivalent to T so that it is sufficient to solve (4.11). Thus the linear equation is obtained, but at the expense of having a driving term which is a many-body operator.

Because Eq. (4.11) involves $\tilde{W}$ it is useful to define $\tilde{t}_\alpha$ and $\tilde{w}_\alpha$

$$\tilde{t}_\alpha = \frac{[a_\alpha(h_0)]^{1/2} t_\alpha [a_\alpha(h_0)]^{1/2}}{a_\alpha(z)},$$
$$\tilde{w}_\alpha = \frac{[a_\alpha(h_0)]^{1/2} w_\alpha [a_\alpha(h_0)]^{1/2}}{a_\alpha(z)}. \tag{4.14}$$

By multiplying Eq. (4.6) by $[a_\alpha(h_0)/a_\alpha(z)]^{1/2}$ from the left and right we obtain

$$\tilde{t}_\alpha = \tilde{w}_\alpha \left[1 + \left(\frac{z}{h_0}\right)^2 \frac{1}{z - h_0 + i\eta} \frac{a_\alpha(h_0)}{a_\alpha(z)} \tilde{t}_\alpha\right]. \tag{4.15}$$

Another equation is

$$\tilde{t}_\alpha = \hat{w}_\alpha \left[1 + \left(\frac{z}{h_0}\right)^2 \frac{1}{z - h_0 + i\eta} \tilde{t}_\alpha\right], \tag{4.16}$$

where the equivalence of Eqs. (4.15) and (4.16) is assured by eliminating $\tilde{t}_\alpha$ between the two equations to obtain $\hat{w}_\alpha$ in terms of $\tilde{w}_\alpha$. In Appendix D, it is shown that $\tilde{w}_\alpha$ and $\hat{w}_\alpha$ are equivalent to better than five percent at resonance and to higher accuracy at lower energies. This obtains from the slow variation of $a_\alpha(z)$ and the presence of the $(z/H_0)^2$ factor which inhibits off-shell scattering. In the following we set $\hat{w}_\alpha$ equal to $\tilde{w}_\alpha$. The corrections may be systematically inserted, if necessary.

Consider the multiple-scattering series obtained from

$$T = (\tilde{W}/z)(1 + GT),$$
$$G = \left(\frac{z}{H_0}\right)^2 \frac{1}{z - H_0 + i\eta}. \tag{4.17}$$

In the usual manner take

$$T_i = \frac{\tilde{W}_i}{z} + \frac{\tilde{W}_i}{z} G \sum_j T_j. \tag{4.18}$$

We wish to rewrite $\tilde{W}_i$ in terms of some effective scattering operator. However, $\tilde{W}_i$ depends upon H_0, but $\tilde{w}_i$ depends upon h_0. Define the operator Θ such that

$$\frac{\tilde{W}_i}{z} = \Theta w_i \Theta. \tag{4.19}$$

The required operator is

$$\langle p|\Theta|q\rangle = \sum_\alpha \frac{P_\alpha}{4\pi}(p,q)\delta(p-q)\left[\frac{a_\alpha(H_0)}{a_\alpha(h_0)}\right]^{1/2} \tag{4.20}$$

and its inverse is

$$\langle p|\Theta^{-1}|q\rangle = \sum_\alpha \frac{P_\alpha(p,q)}{4\pi}\delta(p-q)\left[\frac{a_\alpha(h_0)}{a_\alpha(H_0)}\right]^{1/2}. \tag{4.21}$$

Define

$$K_i = \Theta \tilde{t}_i \Theta$$

so that

$$K_i = (\tilde{W}_i/z)(1 + \Theta^{-1} g \Theta^{-1} K_i). \tag{4.22}$$

For a given spin-isospin channel

$$K_i^{(\alpha)} = \tilde{W}_i^{(\alpha)} + \tilde{W}_i^{(\alpha)} \left[\frac{a_\alpha(h_0)}{a_\alpha(H_0)}\right]^{1/2} g \left[\frac{a_\alpha(h_0)}{a_\alpha(H_0)}\right]^{1/2} K_i^{(\alpha)}. \tag{4.23}$$

We proceed to obtain the multiple-scattering series by solving (4.23) for $\tilde{W}$ in terms of K. One obtains

$$\begin{aligned} T_i &= K_i\Big(1 + G\sum T_j\Big) + K_i\Theta^{-1}g\Theta^{-1}T_i \\ &= K_i\Big(1 + g\sum_{j\neq i} T_j\Big) + K_i(G-g)\sum_{j\neq i} T_j \\ &\quad + K_i(\Theta^{-1}g\Theta^{-1} - g)T_i \end{aligned} \tag{4.24}$$

which defines the multiple-scattering series. Even if one makes the impulse approximation,

$$K_i(G-g)\sum_{j\neq i} T_j = 0, \tag{4.25}$$

an additional term remains. However, if the important matrix elements of $(G-g)$ are reasonably small then $\Theta^{-1}g\Theta^{-1} - g$ must also be small. This is because the replacement of G by g involves the replacement of $(z/H_0)^2/(z-H_0)$ by $(z/h_0)^2/(z-h_0)$ and these functions are more rapidly varying than $a_\alpha(h_0)/a_\alpha(H_0)$. Thus a generalized impulse approximation is defined by

$$T_i = K_i + K_i g \sum_{j\neq i} T_j \tag{4.26}$$

or

$$T = \sum_i K_i + \sum_{i\neq j} K_i g K_j + \sum_{\substack{i\neq j\\ j\neq K}} K_i g K_j g K_K + \cdots. \tag{4.27}$$

and one now has a multiple-scattering series in terms of an operator K which is equivalent to the solution of the Low equation for on-shell kinematics, $H_0 = z$. In the event that $G - g$ terms are to be included, the expression (4.24) may be used.

We now turn to a discussion of the crossing properties of the amplitude, T. Consider the first term of (4.27), T_1, for elastic scattering

$$\begin{aligned} T_1 &= \Big\langle 0\vec{k}\lambda \Big| \sum_i \tilde{K}_i \Big| 0\vec{k}'\lambda' \Big\rangle \\ &= \sum_{i,\alpha} \Big\langle 0\vec{k}\lambda \Big| t_i^{(\alpha)}(z)\left[\frac{a_\alpha(\omega_k)a_\alpha(\omega_{k'})}{a_\alpha^{\,2}(z)}\right]^{1/2} \Big| 0\vec{k}'\lambda' \Big\rangle, \end{aligned} \tag{4.28}$$

where $|0\vec{k}\lambda\rangle$ represents the one-meson, nuclear ground state eigenstate of H_0. For on-shell scattering $|\vec{k}| = |\vec{k}'|$ and $z = \omega_k$ and $[a_\alpha(\omega_k)a_\alpha(\omega_{k'})/a_\alpha^{\,2}(z)]^{1/2} = 1$. For closed-shell, $N = Z$ nuclei, the expectation value has the simple form[9]

$$T_1 = -\frac{16\pi^2}{3}\delta_{\lambda\lambda'}u(k^2)u(k'^2)\vec{k}\cdot\vec{k}'\rho(q)H^{(+)}(z), \quad (4.29)$$

where

$$H^{(+)}(z) = h_1(z) + 2h_2(z) + 2h_3(z) + 4h_4(z).$$

For solutions of the Low equation

$$H^{(+)}(z) = H^{(+)}(-z). \quad (4.30)$$

In order to ascertain crossing symmetry define the crossing operator C by

$$C(F(\vec{k}\lambda, \vec{k}'\lambda', z)) = F(-\vec{k}'\lambda', -\vec{k}\lambda, -z). \quad (4.31)$$

If $CF=F$ the operator is crossing symmetric. From (4.29) and (4.30) we see that $CT_1 = T_1$.

Next consider the higher-order terms. These are of the form

$$\sum_{\substack{i\neq j\\ j\neq k\\ k\neq l}} K_i g(z)K_j g(z)K_k\, g(z)K_l\, g(z)\cdots K_n. \quad (4.32)$$

In the term

$$K_i = \sum_\alpha \left[\frac{a_\alpha(H_0)}{a_\alpha(z)}\right]^{1/2} t_\alpha \left[\frac{a_\alpha(H_0)}{a_\alpha(z)}\right]^{1/2} \quad (4.33)$$

the factor on the left-hand side may be replaced by 1 and in the term K_n the similar term on the right-hand side may be replaced by 1. If we then express each K_m in terms of t_m one has the usual multiple-scattering series, except that each Green's function is of the form

$$\left[\frac{a_\alpha(H_0)}{a_\alpha(z)}\right]^{1/2}\frac{z^2}{h_0^{\,2}}\,\frac{1}{z-h_0}\left[\frac{a_\beta(H_0)}{a_\beta(z)}\right]^{1/2}. \quad (4.34)$$

For a given application one may use Eq. (4.34), but we wish to make a further approximation to simplify the equations. The approximation in which the factors $[a_\alpha(H_0)/a_\alpha(z)]^{1/2}$ and $[a_\beta(H_0)/a_\beta(z)]^{1/2}$ are replaced by 1 is defined to be the on-shell crossing approximation (ONCA). This approximation is suggested by the slow variation of $a_\alpha(x)$ and by the presence of the $(z/h_0)^2$ factor which dampens the off-shell scattering. Under this approximation the second- and higher-order terms are given by

$$\sum_{i\neq j} t_i g(z)t_j + \sum_{\substack{i\neq j\\ j\neq k}} t_i g(z)t_j g(z)t_k + \cdots. \quad (4.35)$$

Banerjee and Cammarata have argued that the inclusion of π-nucleus crossed graphs lead to the replacement of $g(z)$ in Eq. (4.35) by $g(z)+g(-z)$. Their considerations do not include the effects of correlations. Here we show that if pairwise correlations occur such that the two-body density $\rho(r_1, r_2) = \rho(r_2, r_1)$, and for $N=Z$, closed shell nuclei, the Cammarata-Banerjee result which we call the crossing theorem, applies to the most important of the terms of Eq. (4.35) in the fixed scatterer approximation. Of course our $g(z)$ includes the $z^2/h_0^{\,2}$ modification. The method of proof is to cross the terms of Eq. (4.35) and add the resulting terms to Eq. (4.35).

Consider the second-order terms of Eq. (4.35) T_2. Under the ONCA we have

$$T_2(\vec{k}\lambda, \vec{k}'\lambda', z) = \sum_{i\neq j}\int d^3r_i d^3r_j e^{i\vec{k}\cdot\vec{r}_i - i\vec{k}'\cdot\vec{r}_j}\rho(\vec{r}_i, \vec{r}_j)$$
$$\times\sum_{\alpha,\beta}\int\frac{d^3p}{(2\pi)^3}h_\alpha(z)P_\alpha^{(i)}(k,p)\frac{z^2}{\omega_p^{\,2}}$$
$$\times\frac{e^{-i\vec{p}\cdot(\vec{r}_i-\vec{r}_j)}}{z-\omega_p+i\eta}h_\beta(z)P_\beta^{(j)}(p,k'). \quad (4.36)$$

We wish to make (4.35) crossing symmetric by adding in the crossed version of the second- and higher-order terms. The quantity CT_2 is given by

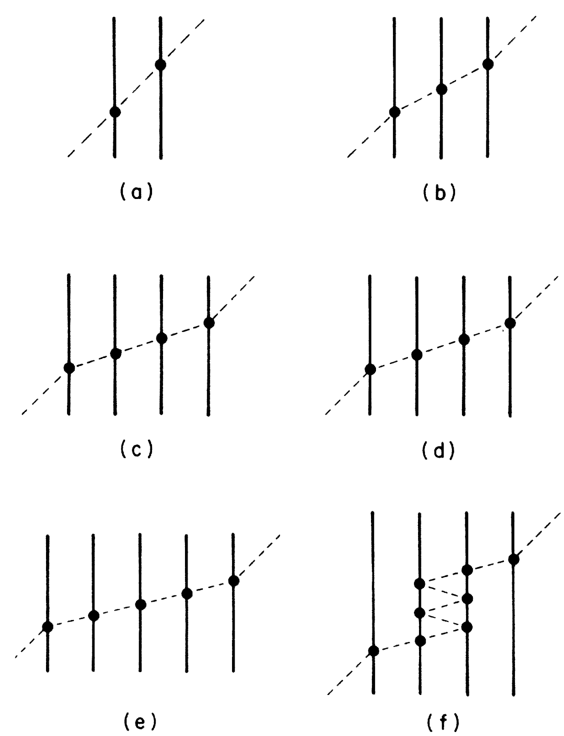

FIG. 1. Some terms of Eq. (4.35) for which the proof of Eqs. (4.36)–(4.42) applies. The dashed line represents pions, the solid line represents nucleons and the heavy dot represents t.

$$CT_2=\sum_{i\neq j}\int d^3r_i d^3r_j \rho(r_i,r_j)e^{-i\vec{k}'\cdot\vec{r}_i+i\vec{k}\cdot\vec{r}_j}$$

$$\times\int\frac{d^3p}{(2\pi)^3}e^{-ip(r_i-r_j)}\sum_{\alpha,\beta}h_\alpha(-z)P_\alpha^{(i)}(-k',p)\frac{z^2}{\omega_p^2}\,\frac{1}{-z-\omega_p}h_\beta(-z)P_\beta^{(j)}(p,-k). \tag{4.37}$$

Convert the integral over $d^3\vec{p}$ to an integral over $-d^3\vec{p}$, then use the crossing symmetry of the π-nucleon T matrix,

$$\sum_\alpha h_\alpha(-z)P_\alpha^{(i)}(-k',p)=\sum_\alpha h_\alpha(z)P_\alpha^{(i)}(-p,k') \tag{4.38}$$

to obtain

$$CT_2=\sum_{i,j}\int d^3r_i d^3r_j\rho(r_i,r_j)e^{-ik'\cdot r_i+ik\cdot r_j}$$

$$\times\int\frac{d^3p}{(2\pi)^3}e^{+ip(r_i-r_j)}\sum_{\alpha,\beta}h_\alpha(z)P_\alpha^{(j)}(k,p)\frac{z^2}{\omega_p^2}\frac{-1}{z+\omega_p}h_\beta(z)P_\beta^{(i)}(p,k'), \tag{4.39}$$

where α and β have been interchanged and where for $i\neq j$

$$[P_\alpha^{(i)},P_\beta^{(j)}]=0. \tag{4.40}$$

If

$$\rho(\vec{r}_i,\vec{r}_j)=\rho(\vec{r}_j,\vec{r}_i)$$

one may switch the indices i and j in Eq. (4.39) and find that CT_2 is obtained from T_2 merely by the replacement of the z in the Green's function by $-z$. Then the sum of the two terms is given by the replacement of

$$g(z)=\frac{z^2}{h_0^2}\frac{1}{z-h_0+i\epsilon}$$

by

$$g_c(z)=g(z)+g(-z)$$
$$=\frac{z^2}{h_0^2}\,\frac{2h_0}{z^2-h_0^2+i\eta}\ . \tag{4.41}$$

For simple models of $\rho(\vec{r}_1,\vec{r}_2)$ such as

$$\rho(\vec{r}_1,\vec{r}_2)=\rho^2[(\vec{r}_1+\vec{r}_2)/2]R(|\vec{r}_1-\vec{r}_2|) \tag{4.42}$$

the condition on $\rho(\vec{r}_1,\vec{r}_2)$ holds. However, for other models the condition is not true and the result (4.41) is not necessarily obtained. Note that the $2h_0$ factor in (4.41) serves to cancel the $(2h_0)^{-1/2}$ factor appearing in the π-nucleon T matrices.

Under the stated assumptions it is straightforward to see that the proof and result of the above paragraph holds for all terms of (4.35) in which the last nucleon struck is different than the first nucleon and in which the first and last nucleon are struck only once. Examples of such terms are given in Fig. 1.

Consider the local field correction of Foldy and Walecka[19] illustrated in Fig. 2(a).

$$T^{\rm LF}=\left\langle 0k\left|\sum_{i\neq j}t_i g t_j g t_i\right|0k'\right\rangle$$

$$=\int d^3r_i d^3r_j e^{i\vec{k}\cdot\vec{r}_i-i\vec{k}'\cdot\vec{r}_j}\int\frac{d^3p}{(2\pi)^3}\,\frac{d^3p_1}{(2\pi)^3}e^{-i\vec{p}(\vec{r}_i-\vec{r}_j)}e^{i\vec{p}_1(\vec{r}_j-\vec{r}_i)}$$

$$=\sum_{\alpha,\beta,\gamma}h_\alpha(z)P_\alpha^{(i)}(k,p)\frac{z^2}{\omega_p^2}\frac{1}{z-\omega_p+i\eta}h_\gamma(z)P_\gamma^{(j)}(p,p_1)\frac{z^2}{\omega_{p_1}}\frac{1}{z-\omega_{p_1}+i\eta}h_\beta(z)P_\beta^{(i)}(p_1,k'). \tag{4.43}$$

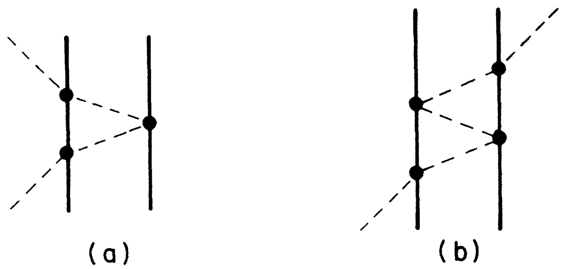

FIG. 2. (a) Local field correction. (b) Another term for which the crossing theorem can be proved.

Consider CT^{LF}, replace $\vec{p}$ and $\vec{p}_1$ by $-\vec{p}$ and $-\vec{p}_1$, use Eqs. (4.41), (4.40), and (4.39) and the fact that

$$\sum_i \langle 0|[P_\alpha^{(i)}, P_\beta^{(i)}]|0\rangle = 0 \tag{4.44}$$

for closed-shell $N=Z$ nuclei. This gives

$$T^{\mathrm{LF}} + CT^{\mathrm{LF}} = T^{\mathrm{LF}}(g_c)\,, \tag{4.45}$$

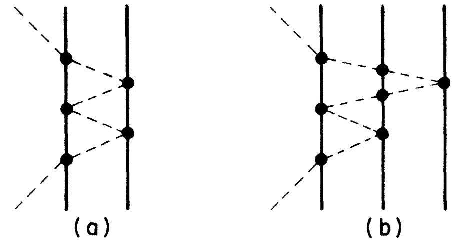

FIG. 3. The crossing theorem applies for the term (a) but not for (b).

where the notation (g_c) means g_c replaces g in Eq. (4.43).

By similar means the corresponding proof for the term of Fig. 2(b) follows immediately for $N=Z$ closed-shell nuclei.

Consider now terms in which the initial nucleon is struck more than twice such as in Fig. 3. By writing the relevant integrals one can show that the theorem obtains if the expression

$$\begin{aligned} &P_\alpha^{(i)}(k,p)P_\beta^{(i)}(p_1,p_2)P_\gamma^{(i)}(p_4,k') - P_\gamma^{(i)}(p_4,k')P_\beta^{(i)}(p_1,p_2)P_\alpha^{(i)}(k,p) \\ &\quad = [P_\alpha^{(i)}(k,p), P_\beta^{(i)}(p_1,p_2)]P_\gamma^{(i)}(p_4,k') + P_\beta^{(i)}(p_1,p_2)[P_\alpha^{(i)}(k,p), P_\gamma^{(i)}(p_4,k')] + [P_\beta^{(i)}(p_1,p_2), P_\gamma^{(i)}(p_4,k')]P_\alpha^{(i)}(k,p) \end{aligned} \tag{4.46}$$

vanishes. Consider the spin dependent terms arising from the commutator and use the fact that terms linear in $\vec{\sigma}$ vanish for a closed-shell nucleus. Straightforward algebra gives

$$\sum_{\alpha,\beta,\gamma} b_\alpha b_\beta b_\gamma (\vec{k}' \times \vec{p}_4)\cdot[(\vec{p}_2 \times \vec{p}_1) \times (\vec{p} \times \vec{k})] \tag{4.47}$$

for (4.46), where the spin-dependent part of the α projection operator is b_α. The expression (4.47) is rewritten as

$$\sum_{\alpha,\beta,\gamma} b_\alpha b_\beta b_\gamma \vec{k}' \cdot (\vec{p}_4 \times \vec{p}_1)\vec{k}\cdot(\vec{p}_2 \times \vec{p})\,, \tag{4.48}$$

where the Eq. (4.48) obtains because $\vec{p}_1 \times \vec{p}$ gives, upon integration, a term proportional to $(\vec{r}_1 - \vec{r}_2) \times (\vec{r}_1 - \vec{r}_2)$ If only two nucleons are involved [Fig. 3(a)] then $\vec{p}_1 \times \vec{p}_4$ gives a term of the form $(\vec{r}_1 - \vec{r}_2) \times (\vec{r}_1 - \vec{r}_2) = 0$ so that the theorem holds. However, for three-nucleon terms, Fig. 3(b), (4.48) does not vanish and the theorem is not true.

Similar consideration of two-nucleon terms shows that the theorem is true for all two-nucleon terms in which the last nucleon struck is also the first one (Fig. 4). It is also possible to show that all terms of the form shown in Fig. 5 obey the theorem.

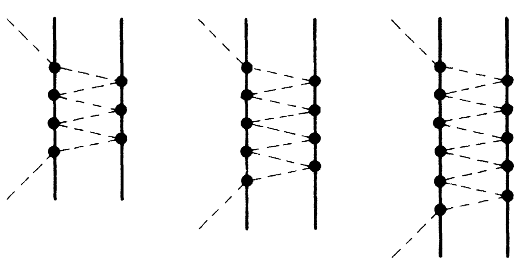

FIG. 4. More terms for which the crossing theorem can be proved.

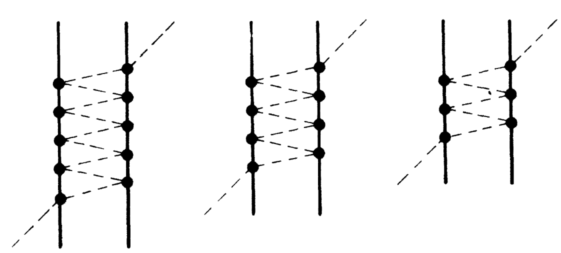

FIG. 5. More terms for which the crossing theorem can be proved.

The crossing theorem holds for the terms of Figs. 1, 2, 3(a), 4, and 5. This means that for all terms involving two struck nucleons the crossing theorem holds. It is also true that, because $T_1 = CT_1$ (and under the ONCA), all terms of the form (4.35) obey the crossing theorem under the coherent approximation (replacement of g by gP).

The correlation expansion of the optical potential has been discussed by Ernst *et al.*[17] In that work an expansion in which terms involving two struck nucleons is summed as a three-body problem is recommended. Terms involving more than two nucleons are also included but these terms do not contain successive scatterings from more than two different nucleons in which the scattering does not proceed through the ground state at some point between the scatterings.

We have showed that all terms involving two struck nucleons and all coherent terms obey the crossing theorem. By combining these results one may show that the terms of the above paragraph also obey the crossing theorem. Thus for the optical potential of Ref. 17 one has

$$T' = U' + U' g_c P T' , \tag{4.49}$$

where g_c is used in obtaining U'. This means that if the correlation expansion including two struck nucleons converges the replacement of g by g_c by in Eq. (4.35) is valid.

The crossing theorem has been proved using the simplification afforded by the fixed scatterer approximation. The result is probably true under more general considerations, but we have not attempted to obtain the necessary derivation.

V. INCLUSION OF π-NUCLEON INELASTICITIES

In this section the one-meson truncation is relaxed. We begin by examining the resulting modification of the π-nucleon scattering equation. Then the set of two-meson nuclear states is inserted into Eq. (2.9) and a solution which makes use of the modified π-nucleon equation is presented.

The π-nucleon Low equation as modified by the inclusion of two-meson states is

$$t = \frac{v}{z} + \sum_n \frac{t_{nf}^\dagger t_{ni}}{z - E_n + i\eta} + C \sum_n \frac{t_{nf}^\dagger t_{ni}}{z - E_n + i\eta} + \sum_m \frac{\Gamma_{mf}^\dagger \Gamma_{mi}}{z - E_m + i\eta} + C \sum \frac{\Gamma_{mf}^\dagger \Gamma_{mi}}{z - E_m + i\eta} , \tag{5.1}$$

where Γ_{mi} is the T matrix element for the process $\pi + N \rightarrow \pi + \pi + N$ and the sum over m is a sum over the two-meson states. Equation (5.1) is given in the π-nucleon center of mass and (for on-shell elements) $z = W_p = (p^2 + \mu^2)^{1/2} + p^2/2m$ where p is the on-shell center-of-mass momentum. Our notation for the various terms of (5.1) is given in Eqs. (1.1)–(1.5). The observed separable form of v and t_z result from the simple form of the interaction current,

$$J_q = g_0 \frac{\vec{\sigma} \cdot \vec{q}}{(2\omega_q)^{1/2}} u(q) \tau .$$

If one considers a few π-nucleon scattering graphs which have a two-meson cut one finds that a similar representation may be made for the terms involving Γ. We write

$$\sum_m \frac{\Gamma_{mf}^\dagger \Gamma_{mi}}{z - E_m + i\eta} = \sum_\alpha \gamma_\alpha(z) u_\alpha(q, p) , \tag{5.2}$$

where $\gamma_\alpha(z)$ depends on the model of Γ_{mi}. At this point we do not investigate such models but merely show that one can find a linear equation, the solution of which is a solution of the Low equation of (5.1).

The analytic structure of Eq. (5.1) tells us that $\gamma_\alpha(z)$ has a cut for $z \geq m + 2\mu$. We assume that $\gamma_\alpha(z)$ has no other singularities and write

$$\gamma_\alpha(z) = \frac{1}{\pi} \int_{m+\mu}^{\infty} \frac{dW_p}{z - W_p + i\eta} \, |U_\alpha(W_p)|^2 , \tag{5.3}$$

where $|U_\alpha(W_p)|^2 = -\mathrm{Im}\gamma_\alpha(W_p)$ and $U_\alpha(W_p) = 0$ for $W_p < m + 2\mu$.

By using Eqs. (5.2) and (5.3) in (5.1) we find

$$t_\alpha(z) = \frac{a'_\alpha(z)}{z} u_\alpha + \gamma_\alpha(z) u_\alpha + \int \frac{d^3p}{(2\pi)^3} \frac{|t_\alpha(W_p)|^2}{z - W_p + i\eta} , \tag{5.4}$$

where

$$a'_\alpha(z) = \lambda_\alpha + z b_\alpha(z) + z b'_\alpha(z) \tag{5.5}$$

and where $b'_\alpha(z)$ arises from the crossed $\Gamma^\dagger\Gamma$ term. The analytic structure of $b'_\alpha(z)$ is assumed to consist of a cut for $z < -(m + 2\mu)$. The use of (5.3) in (5.4) followed by the evaluation of the angular integrals gives

$$h_\alpha(z) = \frac{a'_\alpha(z)}{z} + \frac{1}{\pi} \int \frac{dW_p}{\omega_p} \left(\frac{dp}{dW_p} \right) \times \frac{p^4 u^2(p)}{z - W_p + i\eta} |h_\alpha(W_p)|^2 + \frac{1}{\pi} \int \frac{dW_p}{z - W_p + i\eta} \, |U_\alpha(W_p)|^2 . \tag{5.6}$$

The evaluation of Eq. (5.6) at $z = W + i\epsilon$ and $z = W - i\epsilon$ followed by the taking of the difference between the two quantities gives

$$-\mathrm{Im} h_\alpha(W) = |h_\alpha(W)|^2 \rho(W) + |U_\alpha(W)|^2 , \tag{5.7}$$

where

$$\rho(W)=\frac{p^3}{(1+\omega_p/m)}u^2(p)$$

evaluated at $W_p=W$. We use the form

$$h_\alpha(W)=-e^{i\delta_\alpha-\eta_\alpha}\frac{\sin(\delta_\alpha+i\eta_\alpha)}{\rho(W)} \tag{5.8}$$

with $\delta_\alpha, \eta_\alpha$ real and $\eta_\alpha>0$. The use of (5.8) in (5.7) gives

$$|U_\alpha(W)|^2=(1-e^{-4\eta_\alpha})/4\rho(W) \tag{5.9}$$

which shows that η_α depends only on the reaction cross section and $\rho(W)$. Furthermore, the equation (5.9) is consistent with the usual condition, based on conservation of total flux, that the reaction cross section in a given channel is less than or equal to $\frac{1}{4}$ of the maximum elastic cross section in that channel.

It is useful to combine $|h_\alpha(W_p)|^2$ and $|U_\alpha(W_p)|^2$ as follows

$$\rho(q)|h_\alpha(W_q)|^2+|U_\alpha(W_q)|^2=|h_\alpha(W_q)|^2\rho(W_q)f_\alpha{}^2(W_q)\,. \tag{5.10}$$

The use of (5.8) and (5.9) in (5.10) gives

$$f_\alpha{}^2(W_q)=\frac{2-2\cos 2\delta_\alpha}{1-2\cos 2\delta_\alpha e^{-2\eta_\alpha}+e^{-4\eta_\alpha}}\,. \tag{5.11}$$

Clearly $f_\alpha{}^2(W_q)$ is positive definite, and indeed is greater than 1.

The use of (5.10) in (5.6) gives

$$h_\alpha(z)=\frac{a_\alpha(z)}{z}+\frac{1}{\pi}\int dW_p\rho(W_p)\frac{|h_\alpha(W_p)|^2}{z-W_p+i\eta}f_\alpha{}^2(W_p)\,. \tag{5.12}$$

A relation between $f_\alpha{}^2(W)$ and $\gamma_\alpha(W)$ is obtained by comparing (5.6) and (5.12)

$$-\rho(W)[f^2(W)-1]=\mathrm{Im}\gamma_\alpha(W)/|h_\alpha(W)|^2\,. \tag{5.13}$$

We claim that a linear equation which is a solution of (5.12) is

$$t_\alpha(z)=\frac{a'_\alpha(z)}{z}u_\alpha\left[1+\left(\frac{a'_\alpha(h_0)}{h_0}\right)^2\frac{z^2}{a'^2_\alpha(z)}\frac{1}{z-h_0+i\eta}\times f^2(h_0)t_\alpha(z)\right] \tag{5.14}$$

where h_0 includes the recoil kinetic energy of the nucleon. This is established by defining $\bar{t}_\alpha$ and $\bar{u}_\alpha$

$$\bar{t}_\alpha(z)=f(h_0)t_\alpha(z)f(h_0)\,,$$
$$\bar{u}_\alpha(z)=f(h_0)u_\alpha f(h_0) \tag{5.15}$$

so as to recast Eq. (5.14)

$$\bar{t}_\alpha(z)=\frac{a'_\alpha(z)}{z}\bar{u}_\alpha\left[1+\left(\frac{a'_\alpha(h_0)}{a'_\alpha(z)}\right)^2\frac{z^2}{h_0{}^2}\times\frac{1}{z-h_0+i\eta}\bar{t}_\alpha(z)\right]. \tag{5.16}$$

From Ref. 5 we know that (5.16) is a solution of

$$\bar{t}_\alpha(z)=\frac{a'_\alpha(z)}{z}\bar{u}_\alpha+\int\frac{d^3p}{(2\pi)^3}\frac{|t_\alpha(W_p)|^2}{z-W_p+i\eta}\,. \tag{5.17}$$

The result (5.12) is reestablished by multiplying Eq. (5.17) from the left and right by $f^{-1}(h_0)$ and using (5.15). Thus (5.14) is a linear equation equivalent to the Low equation when inelasticities are included. Note that $a_\alpha(z)$ and $f_\alpha(z)$ depend on $h_\alpha(z)$ so that (5.14) is not necessarily an aid in solving the Low equation.

We now turn to the use of these ideas and Eq. (5.14) in π-nucleus scattering. Inclusion of the sum $\sum|N\pi_n\pi_m\rangle\langle N\pi_n\pi_m|$ in Eq. (2.9) gives the additional terms

$$\Delta T_{fi}=\sum_M\frac{\hat{\Gamma}^\dagger_{Mf}\hat{\Gamma}_{Mi}}{z-E_M+i\eta}+C\sum_M\frac{\hat{\Gamma}^\dagger_{Mf}\hat{\Gamma}_{Mi}}{z-E_M+i\eta}\,, \tag{5.18}$$

where $\hat{\Gamma}_{Mi}$ is the T matrix for the π-nuclear reaction $\pi+A\to\pi+\pi+A'$. The quantity ΔT includes various complicated many-body effects and our treatment of it closely parallels the treatment of the crossing term. As a first approximation, only the terms in (5.18) involving single nucleons are kept and we take

$$\Delta T_{fi}\approx\sum_i(\Gamma^\dagger\tilde{D}_z\Gamma)_i+C\sum_i(\Gamma^\dagger\tilde{D}_z\Gamma)_i\,, \tag{5.19}$$

where $\tilde{D}_z$ is the two-meson nucleon propagator. This enables us to write the modified π-nucleus Low equation as

$$T=\sum_i\left[\sum_\alpha\left(\frac{a'_\alpha(z)}{z}+\gamma_\alpha(z)\right)u_\alpha\right]_i+T^\dagger D_zT\,. \tag{5.20}$$

Equation (5.20) is very similar to Eq. (4.7) except for the significant difference that $\gamma_\alpha(z)$ is complex. However, the main interest of this investigation is on energies up to and slightly above the (3, 3) resonance. For such energies $\mathrm{Re}[\gamma_\alpha(z)]$ has a significant effect on π-nucleon elastic scattering,[20] but $\mathrm{Im}[\gamma_\alpha(z)]$ is important only at higher (>400 MeV) energies. Thus for the energies of interest here, it is reasonable to ignore $\mathrm{Im}\gamma_\alpha(z)$. This means that the techniques of Sec. IV may be used with $a_\alpha(z)$ replaced by $a'_\alpha(z)+\mathrm{Re}\gamma_\alpha(z)$, and one is therefore able to obtain a multiple-scattering expansion, equivalent to the Low equation, in terms of a π-nucleon T matrix which includes the effects of crossing and pion inelasticities.

In making the approximation (5.19) certain many-body effects are ignored. For example, terms in Fig. 6 in which two-pion system is created on one nucleon and destroyed on another nucleon are neglected. For a thorough discussion of such effects, see Londergan and Moniz.[21]

In order to be specific, we have restricted our discussion to the two-pion inelastic channels.

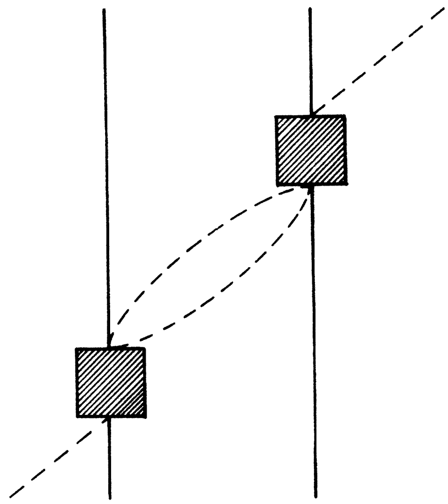

FIG. 6. Term not included in our treatment. The square box represents Γ.

However, the same considerations apply for any π-nucleon inelastic channel for which Eq. (5.2) is valid.

VI. SIGNIFICANCE OF THE $(z/H_0)^2$ FACTOR

At this point the only effect of the field theoretic approach (other than the explicit manifestation of crossing symmetry), is the modification of the propagators by the factor $(z/H_0)^2$. By considering processes in which a sum over virtual states is involved, one sees that the factor $(z/H_0)^2$ decreases the contribution of the high energy states considerably. As a result the importance of effects which depend on the high energy part of such sums is expected to be decreased. In order to illustrate these ideas we consider the consequences of the $(z/H_0)^2$ factor for three effects which are of present interest. These are the Kisslinger singularity, the Lorentz-Lorenz–Ericson-Ericson effect (LLEE), and local field corrections.

The "Kisslinger singularity" is the result of the attractive p-wave nature of π-nucleon scattering at energies below the resonance. In a uniform medium of density ρ

$$p^2 - K^2 + 4\pi\rho f K^2 = 0, \tag{6.1}$$

where $p^2 = z^2 - \mu^2$ and f is the on-shell π-nucleon forward scattering amplitude divided by p^2. The solution of (6.1) is

$$K^2 = \frac{p^2}{1 - 4\pi\rho f} \tag{6.2}$$

and the Kisslinger singularity arises at low energies because the imaginary part of f is small and because $\mathrm{Re}(4\pi\rho f) \sim 1$ for values of ρ near the nuclear density, $\rho_0 (= \frac{1}{6}\ \mathrm{fm}^{-3})$. Thus K^2 takes on a very large and purely imaginary value. For scattering by a finite nucleus the imaginary part of f presents a barrier so that the effective density is much lower than ρ and the problem is mitigated. However, even in situations where potentials of the Kisslinger form fit the elastic scattering data the π-nucleus wave functions have kinks inside the nucleus.

In our theory the equation corresponding to (6.2) is

$$p^2 - K^2 + 4\pi\rho f K^2 \frac{z^2}{K^2 + \mu^2} = 0. \tag{6.3}$$

Even if we solve (6.3) for the extreme case, $4\pi\rho f = 1$, we find that K is finite. This is because the K^2 factor in the denominator of (6.3) prevents the p-wave scattering term from being too large. For example, at a kinetic energy of 50 MeV one has $K/p = 1.55$ instead of infinity. Thus the use of the modified propagator destroys the Kisslinger singularity.

It is also interesting to consider the effects of the propagator modification at resonance energies where the neglect of meson annihilation and multinucleon currents is better justified. Ericson[22] and Hüfner parametrized the scattering amplitude by the crude (see Ref. 2) formula $f = C/(z - E_R + i\Gamma/2)$, with $C < 0$, and obtained the solution of (6.2)

$$n - 1 = -2\pi\rho C/(z - E_R + i\Gamma/2 - 3\pi\rho C), \tag{6.4}$$

where $n = K/p$. The term $-3\pi\rho C$ pushes the position where the real part of the denominator of (6.4) vanishes down in energy by about 30 MeV. This downward shift is caused by the K^2 behavior of t_c. Using the same scattering amplitude, the solution of (6.3) is

$$n - 1 = -2\pi\rho C/(z - E_R + i\Gamma/2 + \pi\rho C) \tag{6.5}$$

and the position of the zero in the denominator *rises* by about 10 MeV. The result (6.5) is also obtained from (6.3) by noticing that for resonance energies $p^2 \approx 4\mu^2$ and the factor $z^2/(K^2 + \mu^2)$ is well approximated by p^2/K^2. In this case, the wave equation (6.3) reduces to the equation of the standard theory, but with the local Laplacian off-shell extension of t_c.

If one does first-order, optical model calculations in finite nuclei one finds that the propagator modification causes significant differences. For scattering on ^{16}O at 30 MeV, the unmodified model, corresponding to Eq. (6.3), gives a reaction cross section σ_R of 210 mb and $d\sigma/d\Omega$ (180°) = 1.5 mb/sr. The same calculation done with the propagator modification gives 129 mb for σ_R and $d\sigma/d\Omega$ (180°) = 1.1 mb/sr.

If one uses various off-shell models of t_c that cut off at large (~1000 MeV) but finite momenta,

TABLE I. Effects of $(z/H_0)^2$ factor on reaction cross sections (in mb) for (π^+, ^{16}O) scattering. The column labeled $(z/H_0)^2G$ includes our modification in the Green's function, the column labeled G does not.

E_π	$(z/H_0)^2G$	G
30	118.28	118.68
66	347.8	350.2
150	587.3	587.1

the effects of our modification in first-order optical potential calculations are negligible. Table I shows that the inclusion of the $(z/H_0)^2$ factor changes the reaction cross section by less than 1%. Furthermore, negligible changes in angular distributions are also obtained. For a variety of energies and targets no change greater than 3% was obtained, and this only at the diffraction minima. The insensitivity occurs because the first order optical potential is proportional to $u(k)u(k')\rho(|\vec{k}-\vec{k}'|)$. For $\vec{k}=\vec{k}'$, the cutoff function $u(k')$ removes the Kisslinger singularity. For $k \equiv k'$ the density cuts off contributions for large momentum transfer so that $(z/H_0)^2$ takes on values of the order of 1.

Next we turn to the LLEE. As shown by several workers,[23] this effect is caused by successive iterations of the term of Fig. 1(a), i.e., Figs. 1(b), 1(c), 1(e), etc. To obtain the optical potential one removes the uncorrelated part of Fig. 1(a) and the remaining piece has a large value which is independent of the details of off-shell π-nucleon scattering and the correlation function. In nuclear matter the contribution to the optical potential from the term of Fig. 1(a) is proportional to L,

$$L = \int d^3s[R(s)-1]\int \frac{d^3q}{(2\pi)^3}\,\frac{(\vec{K}\cdot\vec{q})^2}{p^2-q^2+i\eta}\,e^{i(\vec{q}-\vec{K})\cdot\vec{s}}, \tag{6.6}$$

where $R(s)$ is the two-nucleon relative density. By doing the $\hat{q}$ integral we have

$$L = \int d^3s[R(s)-1]\frac{e^{i(\vec{q}-\vec{K})\cdot\vec{s}}}{p^2-q^2}\,\frac{K^2q^2}{3}\,dq \times [P_0(\hat{q})+2P_2(\hat{q})], \tag{6.7}$$

where we have taken the direction of K parallel to the $\hat{p}_0$ axis. The next step is to make the small $\vec{K}\cdot\vec{s}$ approximation in the medium, i.e., we set $e^{-i\vec{K}\cdot\vec{s}}$ equal to 1. If $R(s)$ is spherically symmetric, the first error in this approximation is of order $(\vec{K}\cdot\vec{s})^2$. Then

$$L = \frac{K^2}{3}\int d^3s[R(s)-1]\int \frac{d^3q}{(2\pi)^3}\,\frac{q^2}{p^2-q^2+i\eta}\,e^{i\vec{q}\cdot\vec{s}}. \tag{6.8}$$

If we use

$$\frac{q^2}{p^2-q^2} = -1+\frac{p^2}{p^2-q^2} \tag{6.9}$$

and neglect terms of order p^2, the integral over d^3q is $-\delta^{(3)}(s)$ and one has

$$L = \tfrac{1}{3}K^2 + \mathcal{O}(z^2, K^2). \tag{6.10}$$

The $\frac{1}{3}K^2$ term is the origin of the LLEE.

In the present theory, under the same approximations, we have

$$L = \frac{K^2}{3}\int d^3s[R(s)-1]\int \frac{d^3q}{(2\pi)^3}\,e^{i\vec{q}\cdot\vec{s}}\,\frac{q^2z^2}{(p^2-q^2)(q^2+\mu^2)} \tag{6.11}$$

which becomes, after doing the d^3q integration,

$$L = \frac{K^2}{3}\left(-\frac{1}{4\pi}\right)\int d^3s\,\frac{R(s)-1}{s}\,(p^2e^{i\vec{p}\cdot\vec{s}}+\mu^2e^{-\mu s}). \tag{6.12}$$

The integral over d^3q gives no δ function and the lowest order contribution to the proper self-energy is of order p^2, a^2, or μ^2a^2, where a is a typical correlation length. These terms are much smaller than $\frac{1}{3}$ and the conventional LLEE is suppressed. A simple evaluation of (6.12) using $R(s)=0$ for $s<a$ and $R(s)=1$ for $s\geq a$ gives $L=(K^2/3)(0.23+0.05i)$ for 50 MeV pions and $a=0.7$ fm.

Another effect is the local field correction shown in Fig. 2(a). Keister[24] has shown the correction to the optical potential is proportional to M,

$$M = \int x^2dxR(x)[h_0{}^2(x)+2h_2{}^2(x)], \tag{6.13}$$

where $x=p_0s$. In order to estimate M we use the simple correlation function of Fig. 7 and find a huge correction which results mainly from the large values of $h_2(x)$ for small x near pa. The modification of the propagator by the factor $(z/H_0)^2$ causes M to be replaced by M':

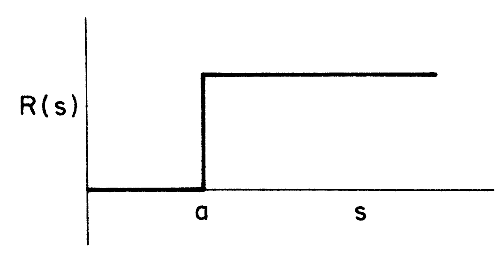

FIG. 7. Correlation function $R(s)$.

$$M' = \int x^2 dx R(x)\left\{\left[h_0(x) - \frac{\mu^3}{ip^3}h_0\left(\frac{i\mu x}{p}\right)\right]^2 + 2\left[h_2(x) - \frac{\mu^3}{ip^3}h_2\left(\frac{i\mu x}{p}\right)\right]^2\right\}. \tag{6.14}$$

Now $\lim_{y\to 0} h_2(y)$ is $3i/y^3$ so we see that

$$\lim_{x\to 0}\left[h_2(x) - \frac{\mu^3}{ip^3}h_2\left(\frac{i\mu x}{p}\right)\right] = \mathcal{O}(1/x) \tag{6.15}$$

and not $\propto x^{-3}$ so that the large value of M is destroyed. A straightforward computation gives $M = -59 + 4.9i$ and $M' = 7.57 - 2.5i$ for 58 MeV pion ($p = \mu$), so that the effect of local field corrections is reduced by a factor of about 7.

The results of this section may be summarized by the statement that the $(z/H_0)^2$ factor introduces a cutoff at a low momentum (strong cutoff) into the multiple-scattering series even though the fundamental form factor contains either no or a very weak cutoff.

It is interesting to obtain a direct relationship with theories which use the conventional Green's function. Our field-theoretic multiple-scattering series, under the impulse and ONCA approximations, is given by

$$T = \sum t_i + \sum_{i\neq j} t_i g_c t_j + \sum_{\substack{i\neq j\\ j\neq k}} t_i g_c t_j g_c t_K + \cdots . \tag{6.16}$$

By defining

$$\overline{T} = \frac{z}{h_0} T \frac{z}{h_0}, \qquad \overline{t} = \frac{z}{h_0} t \frac{z}{h_0}, \tag{6.17}$$

one obtains

$$\overline{T} = \sum \overline{t}_i + \sum_{i\neq j} \overline{t}_i g_c \overline{t}_j + \sum_{\substack{i\neq j\\ j\neq K}} \overline{t}_i g_c \overline{t}_j g_c \overline{t}_K + \cdots , \tag{6.18}$$

where

$$g_c = \frac{2h_0}{z^2 - h_0^2 + i\eta} \tag{6.19}$$

and we again note that the $2h_0$ cancels the $(2\omega)^{-1/2}$ factors of the π-nucleon T matrix. By redefining $\overline{t}$ which has a different off-shell behavior than t, we have converted our theory into one which uses the conventional Klein-Gordon Green's function. However, the operators $\overline{t}$ fall off rapidly for momenta for off shell. Thus the series (6.18) bears a striking resemblance to theories such as that of Moniz and collaborators[10] and Gibbs *et al.*[11] which employ π-nucleon T matrices which are strongly damped for off-shell momenta.

VII. EFFECTS OF TRUE MESON ABSORPTION

So far nuclear excitation energies and multinucleon contributions to the driving term (2.21) have been neglected. These terms contain specific field theoretic features not found in conventional theories. However, I am not yet able to include many of these terms within the framework of obtaining linear equations equivalent to the Low equation. The problem is that these terms are complex and energy dependent. The driving terms considered previously are energy dependent but of a form such that an equivalent linear equation could be found. The terms of the present section are not.

We may, however, calculate the leading contributions to $\Delta V(z)$ [cf. Eq. (2.23)] and determine the conditions under which meson annihilation is significant. If $\Delta V(z)$ is important it is used as an additional term to be combined with V/z or $W(z)$. If V/z and $W(z)$ were energy independent and $\Delta V(z)$ were Hermitian as well as energy independent then such a procedure would be equivalent to solving the Low equation. These conditions do not obtain, our procedure is not a solution of the Low equation, and hence our results should be taken as an order-of-magnitude estimate. (If one is using a perturbative or other *linear* scheme this procedure would be valid.)

In this section effects of true meson absorption are considered. Of course many kinds of states are excited when a meson deposits its entire energy into the nucleus. Here we examine only the effects of the (π, N) and (π, NN) reactions. Because these reactions occur at high momentum transfer it is expected that the (π, N) effects are small and that the (π, NN) effects will be more important.

We first focus on effects of the (π, N) reaction, discussed in Ref. 25, where generally modest corrections to standard theories are obtained. We provide, here, the promised derivation of Eq. (1) of that reference. Start with that part of $\Delta V(z)$, $V_1(z)$, obtained from the one-body effective currents [the first term of Eq. (2.23)]

$$V_2(x) \approx \sum_L g^2 {}_0\langle 0|J_q^\dagger|L\rangle_0 \langle L|J_k|0\rangle_0 \times \left(\frac{1}{z - E_L + i\eta} - \frac{1}{z}\right), \tag{7.1}$$

where $|0\rangle_0$ is the unperturbed ground state. The crossed version of (7.1) has no cut and is discussed in Sec. VIII. To examine the effects of the (π, N) vertex consider only those states $|L\rangle_0$ that asymptotically consist of one fast nucleon and a residual $(A-1)$-nucleon state $|L'\rangle_0$.

The term ${}_0\langle L|\hat{J}_k^{(1)}|0\rangle_0$ which is the matrix element of the single-nucleon current is not the complete matrix element for pion absorption which appears in (2.12), as it is well known that pion rescattering processes play a dominant role. However, we consider $V_1(z)$ as a correction to be evaluated between distorted waves of the optical potential of the previous sections. Thus a reasonable representation of the (π, N) and (N, π) matrix element is used.

Next consider the sum over intermediate nuclear states as a Green's function $G^{(1)}(z)$,

$$G^{(1)}(z) = \sum_L |L\rangle_{00}\langle L| \left(\frac{1}{z - E_L + i\eta} - \frac{1}{z}\right), \qquad (7.2)$$

where the state $|L\rangle_0$ is approximated by $b_\lambda^\dagger |L'\rangle_0$ with λ representing a single-nucleon continuum wave function. We consider only single-nucleon transfer in which the core is inert. Then the Green's function may be written

$$G_z^{(1)}(r, r') = \sum_\lambda \phi_\lambda^*(r)\hat{\phi}_\lambda(r')\left(\frac{1}{z - \epsilon_\lambda + i\eta} - \frac{1}{z}\right), \qquad (7.3)$$

where ϵ_λ is the excitation energy of the continuum nucleon. The quantity $G_z^{(1)}(r, r')$ is just the coordinate representation of the operator $G_z^{(1)}$,

$$G_z^{(1)} = \frac{1}{z - H_{sp} + i\eta} - \frac{1}{z}, \qquad (7.4)$$

where H_{sp} is an effective single-nucleon Hamiltonian, consisting of the kinetic energy plus nucleon optical potential. The biorthogonal basis is used in Eq. (7.3) because H_{sp} is complex. The use of (7.4) in (7.1) allows us to write the correction due to the (π, N) process V_1 as

$$V_1 = g^2 \sum_l (J_q^\dagger G_z^{(1)} J_k)_l = \sum_l V_1(l), \qquad (7.5)$$

A picture which represents V_1 is given in Fig. 8.

In developing the multiple-scattering expansion replace the π-nucleon scattering operator t_l by $\hat{t}_l$ where

$$\hat{t}_l = t_l + V_1(l). \qquad (7.6)$$

The optical potential is then given by

$$U' = \frac{A-1}{A}\left(\sum \hat{t}_i + \sum_{i \neq j} \hat{t}_i G Q \hat{t}_j + \cdots\right) \qquad (7.7)$$

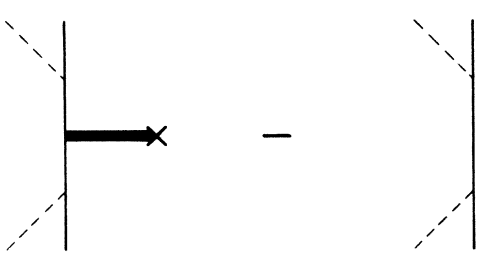

FIG. 8. Effect of (π,N) reaction. The heavy bar represents H_{sp}.

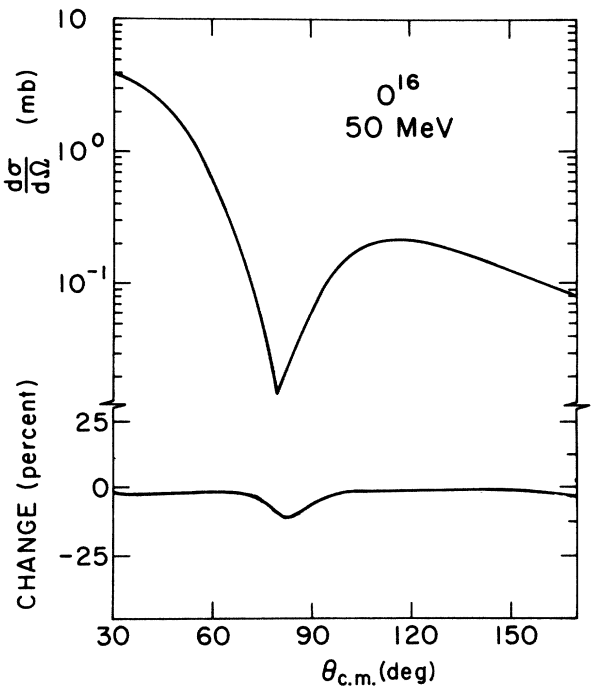

FIG. 9. Effect of (π, N) reaction at 50 MeV.

which is Eq. (1) of Ref. 25 (except that binding effects are neglected here).

Technical details for the calculation of V_1 and the corresponding change in the optical potential are provided in Ref. 25. For completeness we include the angular distributions of Figs. 9 and 10

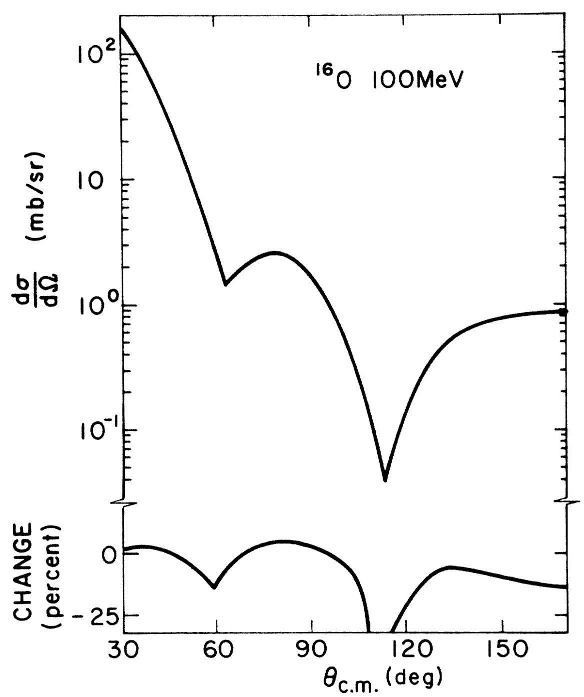

FIG. 10. Effect of (π, N) reaction at 100 MeV.

TABLE II. Effects of the (π,N) reaction channels on total reaction cross sections (given in mb) on $(\pi^+, {}^{16}O)$ scattering. The notation IA designates the standard impulse approximation. The notation (π,N) designates the inclusion of that channel.

E	IA	(π,N)
30	86	99.7
50	153	175
75	288	312
100	417	429
125	528	526
150	567	562
175	547	543
200	496	489

and the reaction cross sections of Table II. The effects of the (π, N) reaction are very small. Furthermore, the results of Figs. 9 and 10 represent an overestimate of the effect. Recent data[26] show that the procedure of Ref. 25 (using distorted waves from the local Laplacian potential) leads to ${}^{16}O(\pi, p){}^{15}N$ cross sections that are too large. Thus the present results should be taken as an upper limit, and it is safe to ignore the effects of the (π, N) reaction.

Next we examine the effects of the (π, NN) reaction which are expected to be more important. We sum over states $|L\rangle_0$ that asymptotically consist of two nucleons plus an inert, $(A-2)$-nucleon core. The reaction $p+p\to d+\pi^+$ is dominated by terms [Fig. 11(a)] in which the pion scatters from one nucleon and is absorbed on another (see, however, Ref. 27). The two-nucleon emission process in nuclei is expected to be dominated by a similar term. Our procedure is to determine the two-nucleon pion absorption (or production) operator from the Wick reduction technique, and to perform the sum over two-nucleon states by Green's function methods. The two-nucleon current is obtained in Appendix E and is described by

$$ {}_0\langle L|\hat{J}_k^{(2)}|0\rangle_0 = {}_0\langle L|\Gamma(k)|0\rangle_0 . \tag{7.8}$$

$\Gamma(k)$ is a two nucleon operator of the form

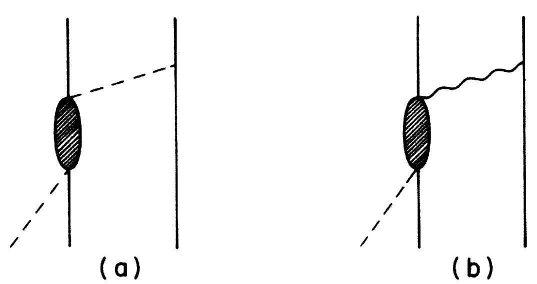

FIG. 11. Dominant mechanisms for the $d\pi\to pp$ reaction. The wiggly line represents the ρ meson.

$$\Gamma(k)=\frac{1}{2}\sum \langle p_1p_2|\gamma(k)|p_3p_4\rangle b^\dagger_{p_1}b^\dagger_{p_2}b_{p_4}b_{p_3} . \tag{7.9}$$

To calculate the contribution $\Delta V^{(2)}$, due to two-nucleon intermediate states, take

$$|L\rangle_0 = b^\dagger_{p_1}b^\dagger_{p_2}|L'\rangle_0 . \tag{7.10}$$

Use (7.8) and (7.10) in the $n=m=2$ term of (2.23) and assume $|L'\rangle_0$ is inert. Then, defining the contribution of the (π, NN) reaction as $\Delta V^{(2)}$,

$$\Delta V^{(2)} = \sum_{ab} \frac{\langle ab|\gamma^\dagger(q)|p_1p_2\rangle\langle p_1p_2|\gamma(k)|ab\rangle_a}{z-\epsilon_{p_1}-\epsilon_{p_2}+i\eta} , \tag{7.11}$$

where $\epsilon_{p_1}+\epsilon_{p_2}$ is the excitation energy of the nucleon pair, a, b denote states below the Fermi sea, and the subscript a denotes the antisymmetric matrix element

$$|ab\rangle_a = |ab\rangle - |ba\rangle . \tag{7.12}$$

In Eq. (7.11) we have used the dominant part of the reaction mechanism for both $\langle 0|J_q|L\rangle$ and $\langle L|J_k|0\rangle$.

The calculation is facilitated by defining the two-nucleon Green's function G_{NN},

$$G_{NN} = \sum_{p_1p_2} \frac{|p_1p_2\rangle\langle p_1p_2|}{z-\epsilon_{p_1}-\epsilon_{p_2}+i\eta} \tag{7.13}$$

which is approximated by the use of plane waves for $|p_1\rangle$ and $|p_2\rangle$. The coordinate space representation of G_{NN} is then

$$G_{NN}(\vec{R}_1\vec{s};\vec{R}'\vec{s}\,') = \int \frac{d^3p\,d^3p'}{(2\pi)^6} e^{+i\vec{P}\cdot(\vec{R}-\vec{R}')+i\vec{p}\cdot(\vec{s}-\vec{s}\,')} \times \frac{1}{z-P^2/4M-p^2/m+i\eta} , \tag{7.14}$$

where

$$\vec{R}=\tfrac{1}{2}(\vec{r}_1+\vec{r}_2), \quad \vec{s}=\vec{r}_1-\vec{r}_2,$$
$$\vec{R}'=\tfrac{1}{2}(\vec{r}\,'_1+\vec{r}\,'_2), \quad \vec{s}\,'=\vec{r}\,'_1-\vec{r}\,'_2,$$
$$\vec{P}=\vec{p}_1+\vec{p}_2, \quad \vec{p}=\tfrac{1}{2}(\vec{p}_1-\vec{p}_2).$$

If one considers the two nucleons as an independent pair moving about the nucleus, the relative momentum may be quite high, whereas the total momentum is generally smaller and is bounded by twice the Fermi momentum. We use this as guidance, and neglect the $P^2/4M$ term in Eq. (7.14). This provides an important simplification because then

$$G_{NN}(\vec{R}\vec{s},\vec{R}'\vec{s}\,') = \delta^{(3)}(\vec{R}-\vec{R}')\int \frac{d^3p}{(2\pi)^3}\frac{e^{+i\vec{p}\cdot(\vec{s}-\vec{s}\,')}}{z-p^2/m+i\epsilon}$$
$$\equiv \delta^{(3)}(R-R')g(\vec{s},\vec{s}\,') . \tag{7.15}$$

The δ function considerably simplifies the evaluation of Eq. (7.11). Note that $\mathrm{Im}(G_{NN})$ is proportional to $\delta(p-q_0)$, where $q_0=\sqrt{mz}$. Thus $\Delta V^{(2)}$ will be complex.

TABLE III. Effects of the (π,NN) reaction channels on $(\pi^+, {}^{16}O)$ scattering. E is the pion's kinetic energy in MeV; B is given in fm^6; Im$(\Delta U(0))$ is given in fm^{-2} as is $U^{(1)}(0)$ (these are potentials evaluated at the origin). IA designates the impulse approximation result for σ_R. The column with ΔU represents the effects of including ΔU.

E	B	Im $\Delta U(0)$	Im $U^{(1)}(0)$	IA	With ΔU
50	$7.2 + 0.61i$	0.24	0.17	153	206
75	$7.4 + 0.53i$	0.25	0.37	288	339
100	$8.2 + 0.49i$	0.28	0.62	417	451
125	$9.1 + 0.48i$	0.31	1.0	528	540
150	$9.1 + 0.42i$	0.31	1.4	567	570
175	$7.0 + 0.29i$	0.22	1.6	547	548
200	$4.6 + 0.17i$	0.16	1.3	491	493
225	$2.8 + 0.092i$	0.094	0.88	433	435
250	$1.7 + 0.051i$	0.056	0.60	379	381
275	$0.83 + 0.023i$	0.028	0.43	341	343

The remainder of the calculation is outlined in Appendix E, where it is shown that the change in the optical potential is given by the formula

$$\Delta V^{(2)}(\vec{k},\vec{k}') = \frac{f^2}{\mu^2 z}\frac{imq_0}{4\pi}|h_4(z)|^2 F(q)\left(\frac{mz}{mz+\mu^2}\right)^2 \times \tfrac{8}{9}\tfrac{22}{15}\frac{1}{1+2z/m} \times \{1+i[\tfrac{5}{2}(\mu/q_0)^3 - \tfrac{3}{2}(\mu/q_0)^5]\}, \quad (7.16)$$

where $F(q)$ is the Fourier transform of the square of the nuclear density and where μ^2, z, m, and q_0 are given in MeV and all other momenta in fm^{-1}. At 50 MeV $q_0/\mu \approx 3$ and is larger for higher energies. This means that $\mathrm{Re}(\Delta V^{(2)})/\mathrm{Im}(\Delta V^{(2)}) \lesssim 0.10$ at all energies of interest. The magnitude of $\mathrm{Re}\Delta V^{(2)}$ is of interest because it simulates the Lorentz-Lorenz effects. However, the value obtained here corresponds to a contribution of less than 5% to the parameter ξ. In coordinate space

$$\Delta V^{(2)}(\vec{r}) = iC\,\vec{\nabla}\cdot\rho^2(r)\vec{\nabla}, \quad (7.17)$$

where

$$C = \frac{(2\pi)^3}{4\pi}\frac{f^2}{\mu^2 z} mq_0\hbar^2c^2|h_4(z)|^2\left(\frac{mz}{mz+\mu^2}\right)^2 \times \tfrac{8}{9}\tfrac{22}{15}\frac{1}{(1+2z/m)} \times [1+i(\tfrac{5}{2}(\mu/q_0)^3 - \tfrac{3}{2}(\mu/q_0)^5)], \quad (7.18)$$

$$\Delta V = \frac{2w}{\hbar^2c^2}V$$

so that one has

$$\Delta V = iB\nabla\rho^2\nabla \quad (7.19)$$

with $B = 2wC/(\hbar c)^2$. The values of B are given in Table III, which also includes a comparison of $\Delta U(r=0)$ to $\mathrm{Im}V^{(1)}(r=0)$. The effect of ΔU on first-order optical-potential calculations is shown in Figs. 12 and 13. At low energies the reaction cross sections are increased by as much as 30% and the angular distributions are, for the most part, unaffected. At energies greater than 100 MeV the effect of true meson annihilation is essentially negligible. This is because of the use of a strongly absorbing first-order potential.

The size of the effect caused by ΔU is, somewhat sensitive to the details of the distorting optical potential. For Figs. 12 and 13 we use the local Laplacian model. If the Kisslinger model is used, the effect of ΔV is decreased because there is greater absorption in the Kisslinger model.

Recent work[27] shows that inclusion of ρ exchange [Fig. 11(b)] plays an important role in reducing

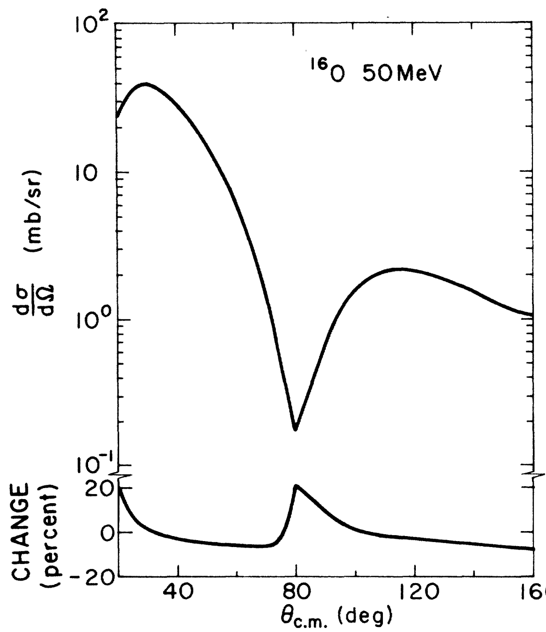

FIG. 12. Effect of the (π,NN) reaction –50 MeV.

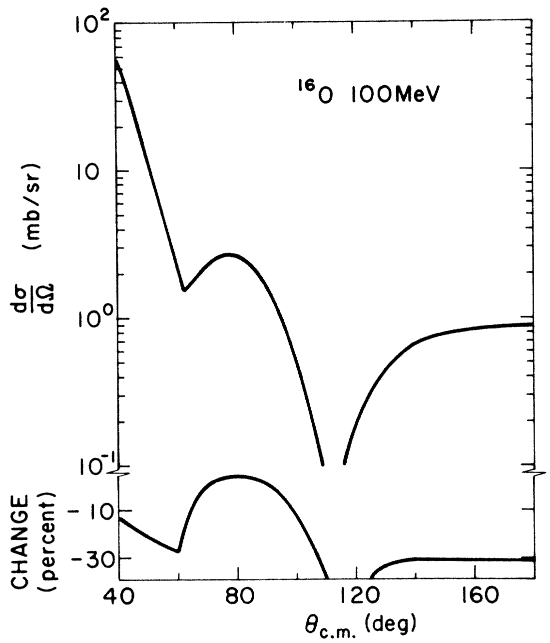

FIG. 13. Effect of the (π, NN) reaction -100 MeV.

$\pi d \to pp$ calculated cross sections and in improving agreement with experiment. We have not included ρ exchange. However, $\Gamma(k)$ includes only terms in which the intermediate meson propagates forward in time (see Appendix E). The neglect of intermediate states with mesons propagating backward in time also has the effect of reducing the annihilation cross section. It turns out that when $\Gamma(k)$ is used in calculations of the $\pi d \to pp$ cross section good agreement with experiment is obtained. These deuteron calculations will be reported elsewhere.

It is worthwhile to review the development and to assess the approximations used in obtaining (7.16). We have used dynamics that are consistent with the $pp \to d\pi^+$ reaction in obtaining the operator Γ. This is the strongest feature of the calculation. However, several approximations have been made. Nucleon plane waves have been used in obtaining the two-nucleon Green's function G. Whereas this is adequate[27] for $pp \to d\pi^+$, the optical distortions provided by the nucleus are neglected. However, for nucleons of about 100 MeV such distortions are relatively weak. A further approximation in obtaining G is the neglect of the center-of-mass momentum of the pair. Finally, the simple quasilocal form of (7.17) obtains from neglecting the variation of the pion wave function over the distance between the two nucleons and by setting the relative two-nucleon wave function to one except at the origin where it vanishes. Because we are interested in pions of small momentum and because of the short-range of the two-nucleon relative wave function, these approximations should not be too bad. It seems that the result (7.17) represents a qualitatively reasonable estimate of the effects of the (π, NN) reaction.

VIII. NONSTATIC CROSSING TERM

Although inclusion of the correct energy denominators in the second term of Eq. (2.23) introduces no imaginary piece into the driving term, it is interesting to see if there are significant nonstatic corrections. One way to obtain an estimate is to cross the terms of Sec. VII. Instead we simply employ a sum-rule technique.

To first order in excitation energy, the nonstatic crossing term ΔV_c, given by the single nucleon currents, is

$$\Delta V_c = -g^2 \sum \frac{E_L}{z^2}\, {}_0\langle 0|J_q|L\rangle_0\, {}_0\langle L|J_k^\dagger|0\rangle_0 . \qquad (8.1)$$

By performing some algebraic manipulations and using the time-reversal invariance properties of the ground state, we have

$$\Delta V_c = -\frac{g^2}{2z^2}\, {}_0\langle 0|[J_q, [H_0, J_k^\dagger]]|0\rangle_0 . \qquad (8.2)$$

Here H_0 is

$$H_0 = \sum_i \frac{p_i^2}{2m} + \sum_{i>j} v_{ij} \qquad (8.3)$$

and we further assume that v_{ij} is local, spin and isospin independent. Then the only terms contributing to the double commutator come from the kinetic energy. We take $|0\rangle$ to be an $N=Z$, closed shell nucleus so that terms arising from commutators of spin and isospin operators vanish.

Using

$$J_q = \sqrt{4\pi}\,\frac{f}{\mu}\,\frac{1}{(2\omega_q)^{1/2}}\,\vec{\sigma}\cdot\vec{q}\,\tau e^{iq\cdot r} \qquad (8.4)$$

and performing the required manipulations of Eq. (8.2) gives

$$\Delta V_c = 4\pi \frac{f^2}{\mu^2}\,\frac{1}{z^2}\,\frac{(\vec{q}\cdot\vec{k})^2}{2M}\,\frac{1}{(2\omega_q 2\omega_k)^{1/2}} F(|\vec{q}-\vec{k}|), \qquad (8.5)$$

where the form factor is defined by

$$F(q) = {}_0\left\langle 0\left| \sum_i e^{iq\cdot r_i} \right| 0\right\rangle_0 . \qquad (8.6)$$

This is to be compared with the first-order optical potential which is dominated by terms arising from the (3, 3) phase shifts

$$U = 4\pi \frac{f^2}{\mu^2}\,\frac{1}{z}\left(\frac{4}{3}\right)^2 \frac{D_4(z)}{(4\omega_q\omega_k)^{1/2}}\,\vec{q}\cdot\vec{k} F(|\vec{q}-\vec{k}|), \qquad (8.7)$$

where $D_4(z)=h_4(z)/\lambda_4$. We consider the ratio of the term of Eqs. (7.6) and (7.7). In the forward direction

$$\frac{\Delta V_c}{U}\approx\frac{q^2}{2M}\frac{1}{z}\left(\frac{3}{4}\right)^2 D_4^{-1}(z). \tag{8.8}$$

At low energies $D_4(z)\approx 1$ and at 50 MeV we have $\delta V_c/U\approx 0.02$ so that the correction is very small. At energies nearer to the (3, 3) resonance $|D_4^{-1}(z)|$ is greater than 1 so that the effect of δV_c is even smaller.

IX. DISCUSSION

This work is an attempt to learn if solutions or useful approximate solutions of the Low equation for π nucleon and π nucleus scattering could be obtained. The procedure of looking for linear equations equivalent to the Low equations is employed. Under the usual approximation, the driving term of the π-nucleus Low equation is a single-nucleon operator of the form V/z. In this case, and neglecting the crossing $C(T^\dagger DT)$ term, a linear equation equivalent to the Low equation is obtained. The use of this equation in conjunction with a corresponding equation for π-nucleon scattering enables us to obtain a field-theoretic derivation of the multiple-scattering series. The difference between this theory and conventional ones is the appearance of factors $(z/H_0)^2$ and $(z/h_0)^2$ in the various Green's functions. Such factors, which arise from the assumed $1/z$ energy dependence of the driving term, significantly reduce off-shell scattering. Three specific consequences of this are the elimination of the Kisslinger singularity, reduction of the Lorentz-Lorenz effect (correlations), and the reduction of the local field correction.

The principal problem of applying our method is that equivalent linear equations are difficult to obtain for cases in which the π-nucleus driving term has a more complicated energy dependence or is non-Hermitian.

The first place where this difficulty occurs is in the inclusion of the crossing term. The single-nucleon approximation to this term, when combined with V/z, results in a single-nucleon driving term which has a different energy dependence in each spin-isospin channel. This difficulty is handled by replacing the term $a_\alpha(z)u_\alpha$ by $[a_\alpha(H_0)]^{1/2}\times a_\alpha[a_\alpha(H_0)]^{1/2}$. A linear equation equivalent to the Low equation (for on-shell π-nucleus scattering) is obtained at the cost of having the π-nucleon potential depend on H_0. However, the multiple-scattering series can then be developed as an expansion in a crossing symmetric π-nucleon T matrix. Furthermore, for closed-shell, $N=Z$ target nuclei, in the fixed scatterer approximation, and under an additional approximation (ONCA) the analog of the Cammarata-Banerjee crossing theorem may be proved. That is, for the most important terms, the series

$$T=\sum t_i+\sum_{i\neq j} t_i g(z)t_j+\sum_{\substack{i\neq j\\ j\neq k}} t_i g(z)t_j g(z)t_k+\cdots,$$

is rendered crossing symmetric by making the replacement $g(z)\to g(z)+g(-z)$. In our case, $g(z)=(z/h_0)^2(z-h_0)^{-1}$.

The effects of π-nucleon inelasticities can also be included, to a certain extent. First with the relaxation of the one-meson truncation, a linear equation equivalent to the π-nucleon Low equation is obtained. The techniques of Sec. IV may then be applied. However, for pion energies above pion-production threshold, the π-nucleus driving term is complex. I have been unable to find the necessary equivalent linear equation in this energy region. However, the main interest of this study is on energies up to and slightly above the (3, 3) resonance. For such energies $\mathrm{Re}[\gamma_\alpha(z)]$ (Sec. V) has a significant effect on π-nucleon elastic scattering, but $\mathrm{Im}\gamma_\alpha(z)$ is important only at higher (>400 MeV) energies. Thus for the energies of interest here, it is reasonable to ignore $\mathrm{Im}\gamma_\alpha(z)$. In this case, we obtain a field-theoretic derivation of the multiple-scattering series as an expansion in a π-nucleon T matrix which includes the nucleon pole, crossing symmetry, and inelastic channels.

The effects of true meson absorption leads to important π-nucleus inelasticities. At low energies the imaginary part of the no-meson driving term is comparable to the imaginary part, ΔV, of the first order optical potential. The necessary linear equation equivalent to the π-nucleus Low equation has not been obtained. Instead we use ΔV, together with the optical potential of the previous sections, to obtain crude estimates of the size of the effect. For scattering at 50 MeV the reaction cross section is increased by about 30% but for energies greater than 100 MeV the effects of true meson absorption are negligible.

Recently, the importance of virtual ρ-meson exchange has been emphasized.[15] Suppose, for example, we include a ρ-nucleon coupling term in the Hamiltonian. This has two consequences. The first is that we are able to obtain better nuclear eigenstates. The second is that a sum of ρ-nuclear states must be inserted into Eq. (2.9) and a more complicated equation is obtained. With the couplings,[15] dominated by the (3, 3) resonance, we are able to find the equivalent linear equation. This will be reported elsewhere.

Note added. In the time since this manuscript was accepted, I have been able to solve the Low equation including the effects of true meson absorption. The procedure of using ΔV as an additional optical potential is essentially correct. I have also been able to show that $\mathrm{Re}(\Delta V)$ depends strongly on the π-nucleon cutoff.

I thank R. A. Friedenberg, E. M. Henley, and J. F. Walker for many useful discussions.

APPENDIX A

The orthogonality properties of the various wave functions of the theory are discussed in this Appendix. By following the arguments of Wick one obtains

$$|Np\rangle^{(+)} = a_p^\dagger|N\rangle + \frac{1}{E_N+\omega_p-H+i\eta} J_p|N\rangle\,. \tag{A1}$$

It is also necessary to obtain $a_k|M\rangle$. We have

$$\begin{aligned} Ha_k|M\rangle &= [H,a_k]|M\rangle + E_M a_k|M\rangle \\ &= (-J_k^\dagger - \omega_k a_k + E_M a_k)|M\rangle\,. \end{aligned} \tag{A2}$$

One then solves Eq. (A2) by keeping only the inhomogeneous term,

$$a_k|M\rangle = \frac{1}{E_M-\omega_k-H+i\epsilon} J_k^\dagger|M\rangle\,. \tag{A3}$$

The addition of the term $i\epsilon$ requires some explanation. In applications by previous workers, $|M\rangle$ was the ground state and $E_M=0$ so that no singularity could occur. However, in some cases $E_M>\mu$ so that one must choose the sign of ϵ correctly. The choice given in (A3) guarantees the orthogonality of $|M\rangle$ and $|Np\rangle^{(+)}$ and, furthermore, if one looks at quantities involving $a_k|M\rangle$ in a time dependent representation these quantities vanish as $t\to\infty$ so that asymptotically there are no mesons present in the no-meson state $|M\rangle$.

To show the orthogonality consider $\langle M|Np\rangle^{(+)}$

$$\begin{aligned} \langle M|Np\rangle^{(+)} &= \langle M|a_p^\dagger|N\rangle + \frac{1}{E_N+\omega_p-E_M+i\epsilon}\langle M|J_p|N\rangle \\ &= \langle M|J_p|N\rangle\left(\frac{1}{E_M-\omega_p-E_N-i\epsilon} + \frac{1}{E_N+\omega_p-E_M+i\epsilon}\right) \\ &= 0\,. \end{aligned} \tag{A4}$$

The choice $+i\epsilon$ in (A2) gives orthogonality with the state $|Np\rangle^{(+)}$, a choice of $-i\epsilon$ in (A2) would give orthogonality with the state $|Np\rangle^{(-)}$. In obtaining Eq. (2.9) the state $|M\rangle$ must be orthogonal to the state $|Nq\rangle^{(-)}$ and we use $-i\epsilon$.

The next task is to prove the orthogonality of the different one-meson states. Consider

$$\langle Mk^{(\pm)}|Np^{(\pm)}\rangle = \langle M|a_k|Np\rangle^{(\pm)} + \left\langle M\left|J_k^\dagger \frac{1}{E_M+\omega_k-E_N-\omega_p\mp i\eta}\right|Np\right\rangle^{(\pm)} \tag{A5}$$

$$\begin{aligned} &= \delta(k,p)\,\delta(M,N) + \langle M|a_p^\dagger a_k|N\rangle + \left\langle M\left|J_k^\dagger \frac{1}{E_M+\omega_k-E_N-\omega_p\mp i\eta}\right|Np\right\rangle^{(\pm)} \\ &\quad + \left\langle M\left|a_k \frac{1}{E_N+\omega_p-H\pm i\eta}J_p\right|N\right\rangle \end{aligned} \tag{A6}$$

$$= \delta(k,p)\,\delta(M,N) + \Delta\,.$$

Our task to show that the sum of the three terms in Δ vanishes. The use of (A3) in the first term of Δ gives

$$\langle M|a_p^\dagger a_k|N\rangle = \frac{1}{-\omega_k+E_N+\omega_p-E_M\pm i\eta}\left\langle M\left|J_p\left(\frac{1}{-\omega_p-H+E_M\mp i\eta} - \frac{1}{-\omega_k-H+E_N\pm i\eta}\right)J_k^\dagger\right|N\right\rangle. \tag{A7}$$

The second term of Δ is

$$\begin{aligned} \left\langle M\left|J_k^\dagger \frac{1}{E_M+\omega_k-E_N+\omega_p\mp i\eta}\right|Np\right\rangle^{(\pm)} &= \frac{1}{E_M+\omega_k-E_N-\omega_p\mp i\eta}\left\langle M\left|J_p \frac{1}{-\omega_p-H+E_M\mp i\eta}J_k^\dagger\right|N\right\rangle \\ &\quad + \frac{1}{E_M+\omega_k-E_N-\omega_p\mp i\eta}\left\langle M\left|J_k^\dagger \frac{1}{E_N+\omega_p-H\pm i\eta}J_p\right|N\right\rangle\,. \end{aligned} \tag{A8}$$

Consider the last term of (A6). The identity

$$\left[a_k, \frac{1}{z-H}\right] = \frac{1}{z-H}(\omega_k a_k + J_k^\dagger)\frac{1}{z-H} \tag{A9}$$

is useful. Taking $z=E_N+\omega_p\pm i\eta$ and defining the last term of (A6) as X we have

$$\begin{aligned} X &= \frac{1}{z-E_M-\omega_k}\left\langle M\left|J_p \frac{1}{-\omega_k-H+E_N\pm i\eta}J_k^\dagger\right|N\right\rangle \\ &\quad + \frac{1}{z-E_M-\omega_k}\left\langle M\left|J_k^\dagger \frac{1}{z-H}J_p\right|N\right\rangle. \end{aligned} \tag{A10}$$

The sum of X and the terms of (A7) and (A8) vanish. Thus one has

$$\langle Mk^{(\pm)}|Np^{(\pm)}\rangle = \delta(M,N)\,\delta(k,p)\,. \tag{A11}$$

APPENDIX B. CONSTRUCTION OF $\langle L \mid J_k \mid M\rangle$

The unperturbed Hamiltonian H_0 and perturbation H_1 are defined in Eqs. (2.16)–(2.18). We use the time-independent Brillouin-Wigner techniques and follow a discussion given by Brandow.[28] First a projection operator P is defined,

$$P = \sum_M |M\rangle_0\,{}_0\langle M|\,, \tag{B1}$$

where

$$H_0|M\rangle_0 = E_M^{(0)}|M\rangle_0$$

and

$$H_\pi|M\rangle_0 = 0\,. \tag{B2}$$

The Schrödinger equation is

$$E_M|M\rangle = (E_M^{(0)} + \Delta E_M)|M\rangle = (H_0 + H_1)|M\rangle\,. \tag{B3}$$

By using standard techniques one may show

$$\Delta E_M P|M\rangle = PH_1P|M\rangle + PH_1Q\,\frac{1}{E_M - H_{QQ}}\,QH_1P|M\rangle\,, \tag{B4}$$

where $Q = 1 - P$, and $H_{QQ} = QHQ$. It is useful to use the normalization ${}_0\langle M|M\rangle_0 = 1 = {}_0\langle M|M\rangle$. Multiplying (B4) by ${}_0\langle M|$ we have

$$\Delta E_M = {}_0\!\left\langle M\left|H_1\left(P + Q\,\frac{1}{E_M - H_{QQ}}\,QH_1P\right)\right|M\right\rangle_0 \tag{B5}$$

which may be used to define the operator Ω_m

$$\begin{aligned}\Delta E_m &= {}_0\langle M|H_1\Omega_m|M\rangle_0\,,\\ \Omega_m &= P + Q\,\frac{1}{E_M - H_{QQ}}\,QH_1P\,.\end{aligned} \tag{B6}$$

However, multiplication of (B3) by ${}_0\langle M|$ gives

$$\Delta E_m = {}_0\langle M|H_1|M\rangle\,. \tag{B7}$$

Hence

$$|M\rangle = \Omega_m|M\rangle_0\,. \tag{B8}$$

We have ignored difficulties arising from possible degeneracies of the unperturbed states. A renormalized wave function $|M\rangle = \Omega_m|M\rangle_0/({}_0\langle M|\Omega_m^\dagger\Omega_m|M\rangle_0)^{1/2}$ is defined so that

$$\langle L|J_k|M\rangle = \frac{{}_0\langle L|\Omega_l^\dagger J_k\Omega_m|M\rangle_0}{{}_0\langle L|\Omega_l^\dagger\Omega_l|L\rangle_0^{1/2}\,{}_0\langle M|\Omega_m^\dagger\Omega_m|M\rangle_0^{1/2}}\,. \tag{B9}$$

A many-body effective current is defined by

$$g_0\langle L|J_k|M\rangle = {}_0\langle L|\hat{J}_k|M\rangle_0 \tag{B10}$$

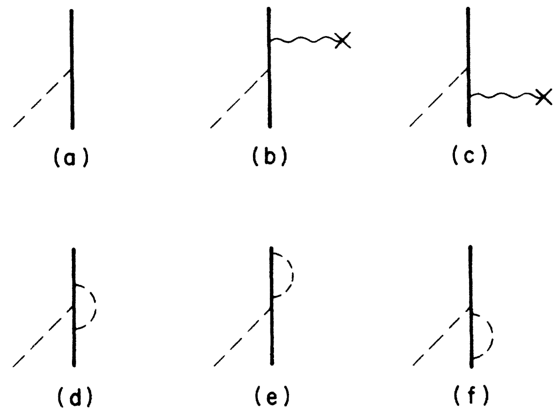

FIG. 14. First-, second-, and some third-order contributions to $\hat{J}_k^{(1)}$. The wiggly line with the $\times$ represents CT.

and

$$\hat{J}_k = \frac{g_0\Omega_l^\dagger J_k\Omega_m}{{}_0\langle L|\Omega_l^\dagger\Omega_l|L\rangle_0^{1/2}\,{}_0\langle M|\Omega_m^\dagger\Omega_m|M\rangle_0^{1/2}}\,. \tag{B11}$$

The operator $\hat{J}_k$ is a many-body operator, and Brandow[29] has shown how to derive the necessary linked-cluster expansions. Our main concern is with the single-nucleon part $\hat{J}_k^{(1)}$ of $\hat{J}_k$. The terms up to third order in CT and fourth order in $H_{\pi N}$ are shown in Figs. 14–16. For $\hat{J}_k^{(1)}$, terms involv-

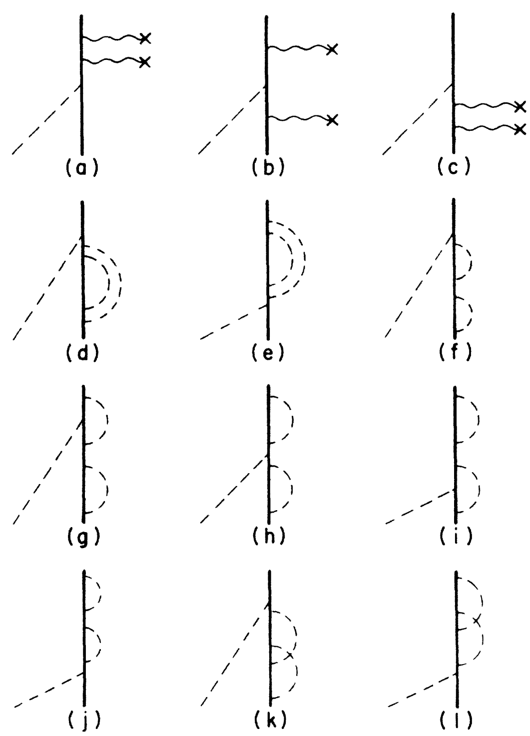

FIG. 15. Some second- and fourth-order contributions to $\hat{J}_k^{(1)}$.

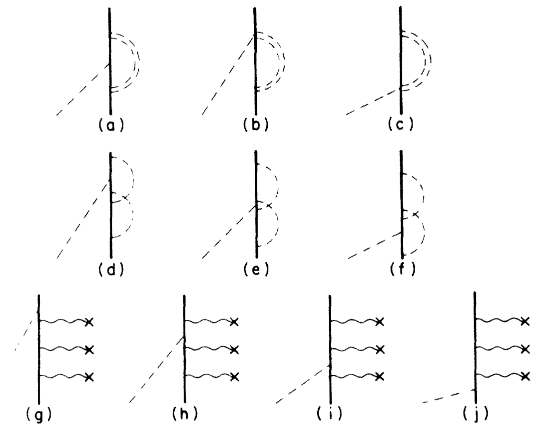

FIG. 16. Some fourth- and third-order contributions to $\hat{J}_k^{(1)}$.

ing V_{NN} do not appear explicitly. The terms of Figs. 14(b) and 14(c) are chosen to cancel the terms of Figs. 14(e) and 14(f). The terms of Figs. 15(a), 15(b), and 15(c) are chosen to cancel the terms of Figs. 15(d)–15(l). The terms of Figs. 16(g)–16(j) are used to cancel the sixth-order graphs. The remaining series, Figs. 14(a), 14(d) and Figs. 16(a)–16(f), are almost identical to the series found by Wick which simply replaces the unrenormalized coupling constant by the renormalized coupling constant. The only difference is the appearance of the energy E_M in the operator Ω_m. The important contributions to the graphs of Figs. 14(a), 14(d) and Figs. 16(a)–16(f) come from the virtual states with excitation energy on the order of the cutoff energy which is about a nucleon mass. As all of the states of interest have excitation energies much lower than 1000 MeV we neglect these energies in evaluating the remaining diagrams. Hence our renormalized coupling constant is the (free) physical coupling constant.

APPENDIX C. EQUIVALENCE OF EQS. (3.1) AND (3.2)

To prove that Eqs. (3.1) and (3.2) are equivalent, define $T_z' = (z/H_0)T_z$. Then

$$T_z' = \frac{1}{H_0} V + \frac{1}{H_0} V \frac{z}{H_0} \frac{1}{z - H_0 + i\eta} T_z' \tag{C1}$$

which may be written in the equivalent Lippmann-Schwinger form

$$T_z' = \tilde{V} + \tilde{V} \frac{1}{z - H_0 + i\eta} T_z' . \tag{C2}$$

The equivalence of (C1) and (C2) are assured by equating them and obtaining

$$\tilde{V} = \frac{1}{H_0} V + \frac{1}{H_0} V \frac{1}{H_0} \tilde{V} . \tag{C3}$$

Although $\tilde{V}$ is non-Hermitian it is *energy independent*, so that a complete bi-orthogonal set of basis states may be set up. By solving Eq. (C3) for $(1/H_0)V$ and using the resulting expression in the second term of (C3) one finds

$$\tilde{V} = \frac{1}{H_0} V + \tilde{V} \frac{1}{H_0 + \tilde{V}} \tilde{V} . \tag{C4}$$

The non-Hermiticity of $\tilde{V}$ requires us to examine the scattering caused by $\tilde{V}^\dagger$. This defines $\overline{T}_z$,

$$\overline{T}_z = \tilde{V}^\dagger + \tilde{V}^\dagger \frac{1}{z - H_0 + i\eta} \overline{T}_z . \tag{C5}$$

The next step is to define wave functions $|\psi_{n,z}\rangle^{(-)}$ and $|\overline{\psi}_{n,z}^{(-)}\rangle$ which correspond to solutions of the Schrödinger equations, with the potentials $\tilde{V}$ and $\tilde{V}^\dagger$. Solution of Eq. (C2) by eliminating T_z' from the right-hand side gives

$$T_z' = \tilde{V} + \tilde{V} \frac{1}{z - \tilde{V} - H_0 + i\eta} \tilde{V} , \tag{C6}$$

which together with the assumed completeness of the bi-orthogonal basis and Eq. (C4) gives

$$T_z' = \frac{1}{H_0} V + \sum_n \frac{\overline{T}_{nf}^{\dagger} T_{ni}'}{z - E_n + i\eta} \frac{z}{E_n} . \tag{C7}$$

In general $\overline{T}_z$ and T_z' have no simple relationship (see Ref. 30, for example). In the present case one may show that $\tilde{V}^\dagger H_0 = H_0 \tilde{V}$ and then with (C2) and (C5) that $\overline{T}_z H_0 = H_0 T_z'$. This relation gives

$$T_z' = \frac{1}{H_0} V + \frac{z}{H_0} \sum_n \frac{T_{nf}'^{\dagger} T_{ni}'}{z - E_n + i\eta} , \tag{C8}$$

where $z/H_0 = z/E_f$ when acting on an eigenstate $|f\rangle$ of H_0. For the half-shell matrix elements of T' appearing in (C8), z/H_0 corresponds to E_n/E_n and in that case $T = T'$. The proof is completed by multiplying by H_0/z from the left.

Note that we use the term equivalent to mean that the solution of (3.2) is a solution of (3.1).

APPENDIX D. EQUIVALENCE OF $\tilde{w}_\alpha$ AND $\hat{w}_\alpha$

The potential $\tilde{w}_\alpha$ has the form

$$\tilde{w}_\alpha(q,k) = \frac{[a_\alpha(W_q)]^{1/2}}{z} u_\alpha(q,k)[a_\alpha(W_k)]^{1/2} . \tag{D1}$$

The elimination of $\tilde{t}_\alpha$ between Eqs. (4.15) and (4.16) gives

$$\hat{w}_\alpha = \tilde{w}_\alpha + \tilde{w}_\alpha \left(\frac{a_\alpha(h_0)}{a_\alpha(z)} - 1 \right) g(z) \hat{w}_\alpha . \tag{D2}$$

The separability of u_α implies that the solution of (D2) is of the form

$$\hat{w}_\alpha = \frac{\gamma_\alpha(z)}{z} \tilde{w}_\alpha(q,k) . \tag{D3}$$

Insertion of (D3) into (D2) and performing the necessary angular integrations gives

$$\gamma_\alpha(z)=1+\frac{z}{\pi}\frac{\gamma_\alpha(z)}{a_\alpha(z)}\int\frac{q^4dq}{\omega_qW_q^{\,2}}\frac{u^2(q)a_\alpha(W_q)}{z-W_q}\times[a_\alpha(W_q)-a_\alpha(z)]\,. \qquad \text{(D4)}$$

In Eq. (1.6) it is convenient to define $\sigma_\alpha(W_k)$

$$\sigma_\alpha(W_k)=\sum A_{\alpha\beta}\frac{k^3}{(1+\omega_k/m)}|h_\beta(W_k)|^2 \qquad \text{(D5)}$$

so that

$$a_\alpha(x)=\lambda_\alpha-\frac{x}{\pi}\int\frac{dW_k}{x+W_k}\sigma_\alpha(W_k)\,. \qquad \text{(D6)}$$

The use of (D6) allows us to write

$$\gamma_\alpha(z)=1+\frac{z\gamma_\alpha(z)}{\pi^2}\int\frac{q^4dq}{\omega_qW_q^{\,2}}u^2(q)\frac{a_2(W_q)}{a_2(z)}\times\int\frac{dW_kW_k}{(W_q+W_k)}\frac{\sigma_\alpha(W_k)}{(z+W_k)}\,. \qquad \text{(D7)}$$

It is our intention to make a crude estimate of the integrals in (D7). To this end we assume that the (3, 3) channel dominates the dW_k integral and furthermore approximate $\sin^2\delta_\alpha(W_k)$ by a δ function $\delta(W_k-W_R)$ times a width $\Gamma=120$ MeV which simulates the resonance peaking. We take

$$\sigma_\alpha(W_k)=\frac{A_{\alpha 4}}{(1+\omega_R/m)^3}\frac{\Gamma}{k_R^{\,3}}\delta(W_k-W_R)\,, \qquad \text{(D8)}$$

where the subscript R implies that the quantity is to be evaluated at the resonance energy. The use of (D8) in (D7) gives

$$\gamma_\alpha(z)=1+\frac{z\gamma_\alpha(z)}{\pi}\frac{W_R\Gamma}{k_R^{\,3}}\left(\frac{m}{m+\omega_R}\right)^3\times\int\frac{q^4dqu^2(q)a_\alpha(W_q)}{\omega_qW_q^{\,2}(z+W_R)a_\alpha(z)(W_R+W_q)}\,. \qquad \text{(D9)}$$

We may also use (D8) in (D6) to obtain

$$\frac{a_\alpha(W_q)}{\lambda_\alpha}=1-\frac{y}{(2.42+y)}\frac{x_\alpha}{\pi}\,, \qquad \text{(D10)}$$

where $x_\alpha=(-\frac{4}{9},-\frac{4}{9},-\frac{4}{9},\frac{1}{18})$ and $W_q=y\mu$. It is therefore reasonable to replace $a_\alpha(W_q)/a_\alpha(z)$ by 1 in Eq. (D9) because the maximum value of y is of the order of a nucleon mass $=6.7$. This gives

$$\gamma_\alpha(z)=1+\frac{z\gamma_\alpha(z)}{\pi^2}W_R\Gamma\left(\frac{m}{m+\omega_R}\right)^2\times\int\frac{q^4dq}{\omega_qW_q^{\,2}}\frac{u^2(q)}{(z+W_R)(W_R+W_q)}\,. \qquad \text{(D11)}$$

The integral over dq is evaluated by effective range techniques, and we find

$$\gamma_\alpha(z)=1+\frac{z\gamma_\alpha(z)}{\pi}\frac{W_R\Gamma}{k_R^{\,3}}\left(\frac{m}{m+\omega_R}\right)^2\frac{m-\mu}{(z+W_R)}A_{\alpha 4}\,, \qquad \text{(D12)}$$

where a cutoff of the order of a nucleon mass is assumed. Evaluation of (D12) gives

$$\begin{aligned}&0.944\le\gamma_1(z)\le 0.966\,,\\&0.986\le\gamma_{2,3}(z)\le 0.992\,,\\&0.998\le\gamma_4(z)\le 0.999\end{aligned} \qquad \text{(D13)}$$

for energies $\mu\le z\le 3\mu$ so that setting $\hat{w}_\alpha$ equal to $\bar{w}_\alpha$ is a good approximation for all of the energies of interest.

These estimates are substantiated reasonably well by numerical calculation.

APPENDIX E. TWO-NUCLEON ABSORPTION VERTEX AND TRUE MESON ABSORPTION

In this Appendix the two-nucleon absorption vertex is obtained and used to calculate $\Delta V^{(2)}$.

We start with the pion current J_k which is defined by $g_0J_k=[H,a_k^\dagger]-\omega_ka_k^\dagger=[H_{\pi N},a_k]$. It is useful to define a nucleon current by $g_0L_p=[H,b_p^\dagger]-(M+p^2/2m)b_p^\dagger$. Consider the T matrix Γ for the absorption of a pion on a pair of bound nucleons leading to a two-nucleon final state,

$$\Gamma(k)=g_0\,{}^{(-)}\langle p_1p_2|J_k|i\rangle\,, \qquad \text{(E1)}$$

where $|i\rangle$ is the initial bound state and (E1) is obtained by reducing the pion in the expression for the S matrix. The techniques of Wick as applied to the nucleon-nucleon wave function give

$$|p_1p_2\rangle^{(-)}=b_{p_1}^\dagger|p_2\rangle+\frac{1}{E_{p_1p_2}-H-i\eta}g_0L_{p_1}|p_2\rangle\,. \qquad \text{(E2)}$$

The use of (E2) in (E1) gives

$$\begin{aligned}\Gamma(k)=&\,g_0\langle p_2|b_{p_1}J_k|i\rangle\\&+g_0^{\,2}\sum_{p_3p_4}\frac{\langle p_2|L_{p_1}^\dagger|p_3p_4\rangle^{(-)}\,{}^{(-)}\langle p_3p_4|J_k|i\rangle}{E_{p_1p_2}-E_{p_3p_4}+i\eta}\\&+g_0^{\,2}\sum_{n\pi}\frac{\langle p_2|L_{p_1}^\dagger|n\pi\rangle^{(-)}\,{}^{(-)}\langle n\pi|J_k|i\rangle}{E_{p_1p_2}-E_{n\pi}+i\eta}\,,\end{aligned} \qquad \text{(E3)}$$

where a set of two-nucleon and two-nucleon one-meson states has been used in the intermediate state sum. It is necessary to identify the various matrix elements of (E3). By reducing the S matrix element ${}^{(-)}\langle p_1p_2|p_3p_4\rangle^{(+)}$ it is easy to show that ${}^{(-)}\langle p_1p_2|L_{p_3}|p_4\rangle$ represents the nucleon-nucleon T matrix. It is also possible to show that $\langle p_2|L_{p_1}|n\pi\rangle^{(-)}$ is ${}^{(-)}\langle p_1p_2|J_k|n\rangle$ for $E_{p_1p_2}=E_{n\pi}$. It is well known that nucleon-nucleon rescattering

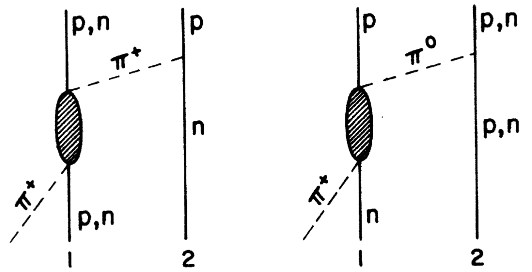

FIG. 17. Charge states contributing to $\Gamma(k)$.

plays a minor role in the $d\pi \to pp$ reaction, and we neglect the second term of (E3). Furthermore, terms involving both nucleons dominate. The matrix element ${}^{(-)}\langle p_1 p_2 | J_k | n\rangle$ is approximated by its Born term Γ_B and the Born term of (E3) is neglected. Then

$$\Gamma(k) = g_0 \sum_{n\pi} \Gamma_B^\dagger \frac{\langle n\pi^{(-)} | J_k | i\rangle}{z - E_{n\pi} + i\eta} . \tag{E4}$$

The matrix element $\langle n\pi^{(-)} | J_k | i\rangle$ includes π-two-nucleon scattering. The quantity $\Gamma(k)$ is the T matrix element for the full π-nucleus absorption process. However, we are interested in the absorption process involving only two nucleons and neglect terms involving more than two nucleons. We approximate (E4) by replacing Γ_B by a renormalized single-nucleon absorption operator and $\langle n\pi^{(-)} | J_k | i\rangle$ by a pion scattering on one other nucleon. Thus

$$\Gamma(k) = \sum_{n\pi} \left(\frac{\Gamma_{B(1)}^\dagger T_{\pi\pi}^{(2)}(E_{n\pi})}{z - E_{n\pi} + i\eta} + 1 \rightleftarrows 2 \right) . \tag{E5}$$

A picture representing this term is given in Fig. 17.

The isospin and space spin parts of $\Gamma(k)$ can be treated separately. To use (E5) in evaluating (7.11) we include π scattering only in the dominant (3, 3) channel. The isospin factor appropriate to Figs. 17(a) and 17(b) is defined as $T(1,2)$ and is given by

$$T(1,2) = \tau_f(2) - \tau_f(1) \frac{\vec{\tau}_1 \cdot \vec{\tau}_2}{3} . \tag{E6}$$

The space spin part of Fig. 17(a) or 17(b) is defined as $F(1,2)$ and

$$F(1,2) = \frac{h_4(z) f}{\sqrt{\omega}\,\mu} \int \frac{d^3p}{(2\pi)^3} \frac{[2\vec{k}\cdot\vec{p} - i\sigma_1 \cdot (\vec{k}\times\vec{p})]}{\omega_p{}^2} \vec{\sigma}_2 \cdot \vec{p} \times e^{i\vec{p}\cdot(\vec{r}_2 - \vec{r}_1)} e^{ik\cdot\vec{R}} . \tag{E7}$$

It is useful to decompose $T(1,2)$ and $F(1,2)$ into parts even and odd under interchange of particle labels. The decompositions

$$T(\pm) = \frac{T(1,2) \pm T(2,1)}{2} ,$$
$$T(1,2) = T(+) + T(-) \tag{E8}$$

and

$$F(\pm) = \frac{F(1,2) \pm F(2,1)}{2} ,$$
$$F(1,2) = F(+) + F(-) \tag{E9}$$

are useful because we take a ground state expectation value and only terms even in the spin, isospin, and space coordinates survive.

Using (E7) and (E9), one may show that the quantities $F(\pm)$ are given by

$$F(+) = \frac{f}{\sqrt{\omega}\,\mu} \frac{h_4(z)}{(2\pi)^3} e^{i\vec{k}\cdot\vec{R}} \int \frac{d^3p}{\omega_p{}^2} e^{i\vec{p}\cdot\vec{s}} \left[2\vec{k}\cdot\vec{p}\vec{S}\cdot\vec{p} - \frac{i}{2}(\vec{\sigma}_1\times\vec{k})\cdot\vec{p}\sigma_2\cdot\vec{p} - \frac{i}{2}(\vec{\sigma}_2\times\vec{k})\vec{\sigma}_1\cdot\vec{p} \right] ,$$

$$F(-) = \frac{f}{\sqrt{\omega}\,\mu} \frac{h_4(z)}{(2\pi)^3} e^{i\vec{k}\cdot\vec{R}} \int \frac{d^3p}{\omega_p{}^2} e^{-i\vec{p}\cdot\vec{s}} \left[-k\cdot p\vec{\Sigma}\cdot\vec{p} - \frac{i}{2}(\vec{\sigma}_1\times\vec{k})\cdot p\vec{\sigma}_2\cdot\vec{p} + \frac{i}{2}(\vec{\sigma}_2\times\vec{k})\cdot\vec{p}\vec{\sigma}_1\cdot\vec{p} \right] , \tag{E10}$$

where

$$\vec{S} = \frac{\vec{\sigma}_1 + \vec{\sigma}_2}{2} , \quad \vec{\Sigma} = \vec{\sigma}_1 - \vec{\sigma}_2 ,$$

$$R = \frac{r_1 + r_2}{2} , \quad \vec{s} = \vec{r}_1 - \vec{r}_2 .$$

In obtaining (E10) a low momentum approximation is used to set $e^{ik\cdot s/2} = 1$.

The calculation of Eq. (7.11) requires the quantity

$$[T(1,2)F(1,2) + T(2,1)F(2,1)] \times [T(1,2)F'(1,2) + T(2,1)F'(2,1)]^\dagger ,$$

where the prime designates the replacement of s and k in Eq. (E10) by s' and k'. This term equals [via (E8) and (E9)]

$$4T(+)T^\dagger(+)F(+)F'^\dagger(+) + T(-)T^\dagger(-)F(-)F'^\dagger(-) \tag{E11}$$

because the ground state expectation values of $T_+ T_-^\dagger = 0$.

The use of (E11) and (7.15) in (7.11) gives

$$\Delta V^{(2)} = \frac{4f^2}{\mu^2 z}|h_4(z)|^2 \sum_{\substack{ab\\ SM_S\\ TM_T}} \langle SM_S|\otimes\langle TM_T| \int d^3R\, d^3s\, d^3s'\, \phi_a^*(\vec{R}+\tfrac{1}{2}\vec{s})\phi_b^*(\vec{R}-\tfrac{1}{2}\vec{s}')g(\vec{s},\vec{s}')$$
$$\times[T(+)T^\dagger(+)F(+)F^\dagger(+)+T(-)T^\dagger(-)F(-)F^\dagger(-)]$$
$$\times[\phi_a(\vec{R}+\tfrac{1}{2}\vec{s})\phi_b(\vec{R}-\tfrac{1}{2}\vec{s})-(-1)^{S+T}\phi_a(\vec{R}-\tfrac{1}{2}\vec{s})\phi_b(\vec{R}+\tfrac{1}{2}\vec{s})]\,|SM_S\rangle\otimes|TM_T\rangle \qquad \text{(E12)}$$

when the sum over a,b in (7.11) is decomposed into its space, spin, and isospin parts.

In evaluating (E12) we neglect the $\vec{s}$ and $\vec{s}'$ dependence of the single particle wave functions except to assume that the relative wave function is zero at $s=0$. This means that $S+T=1$. From (E6) and (E8) it may be shown that only terms with $T=1$ contribute, so that $S=0$. Then after evaluating the spin matrix elements

$$\Delta V^{(2)} = \frac{8}{9}\int d^3R\, e^{i\vec{Q}\cdot\vec{R}}\rho^2(R)\int \frac{d^3p\,d^3p'}{(2\pi)^6\omega_p^{\,2}\omega_{p'}^{\,2}}e^{i\vec{p}\cdot\vec{s}-i\vec{p}'\cdot\vec{s}'}g(\vec{s},\vec{s}')$$
$$\times[-2\vec{k}\cdot\vec{p}\,\vec{k}'\cdot\vec{p}'\,\vec{p}\cdot\vec{p}'+4\vec{k}\cdot\vec{p}\,\vec{k}'\cdot\vec{p}p'^2+4\vec{k}\cdot\vec{p}'\vec{k}'\cdot\vec{p}'p^2+\vec{k}\cdot\vec{k}'(\vec{p}\cdot\vec{p}')^2]\,\frac{f^2}{z\mu^2}|h_4(z)|^2. \qquad \text{(E13)}$$

To simplify (E13) use the partial wave decomposition of $g(\vec{s},\vec{s}')$

$$g(\vec{s},\vec{s}') = -\frac{imq_0}{\hbar^2c^2}\sum_l j_l(q_0s_<)h_l^{(1)}(q_0s_>)\sum_m Y_{lm}^*(\hat{s})\,Y_{lm}(\hat{s}'), \qquad \text{(E14)}$$

where $q_0=\sqrt{mz}$ and $s_{<(>)}$ is the lesser (greater) of s and s'. Then defining $F(Q)$ as the Fourier transform of $\rho^2(R)$ and performing the four angular integrals of (E13) obtain

$$\Delta V^{(2)} = \frac{8}{9}\frac{f^2}{z\mu^2}|h_4(z)|^2\vec{k}\cdot\vec{k}'F(Q)\,\frac{1}{\pi^3}$$
$$\times\sum_{l=0,2}\int_\epsilon^\infty s^2ds\int_\epsilon^\infty s'^2ds'\int\frac{p^4dp}{p^2+\mu^2}\int\frac{p'^4dp'}{p'^2+\mu^2}j_l(ps)j_l(p's')j_l(q_0s_<)h_l^{(1)}(q_0s_>)$$
$$\times\left[-\tfrac{2}{3}(2l+1)\begin{pmatrix}l&1&1\\0&0&0\end{pmatrix}^2+\tfrac{8}{3}\delta(l,0)+\begin{pmatrix}l&1&1\\0&0&0\end{pmatrix}^2\right], \qquad \text{(E15)}$$

where ϵ is an infinitesimal quantity to be set equal to zero after performing the integrals.

The integrals in (E15) are most easily handled by doing the coordinate integrals before the momentum integrals. Careful attention to the limits is necessary. The result is simply

$$\Delta V^{(2)} = -i\tfrac{22}{15}\,\tfrac{8}{9}\,\frac{f^2}{\mu^2 z}\,\frac{|h_4(z)|^2}{1+2z/m}\,mq_0\vec{k}\cdot\vec{k}'\,\frac{F(Q)}{4\pi}\{1+i[\tfrac{5}{2}(\mu/q_0)^3-\tfrac{3}{2}(\mu/q_0)^5]\}, \qquad \text{(E16)}$$

where the $(1+2z/m)$ results from the kinematic transformation of the momentum.

A similar calculation has recently been performed with similar results by Rockmore *et al.*[31]

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