New method for triton calculations in momentum space*

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A new method for a solution of the bound-state Faddeev equations in momentum space is presented. The method takes advantage of an expansion of the Faddeev amplitude into a suitable set of basis states and uses the potential directly. To demonstrate the practicability we have recalculated the triton binding energy for the Reid soft-core potential including all partial waves up to a maximal subsystem angular momentum $j = 2$. We found $E_B = -7.56 \pm 0.10$ MeV. The Reid potential truncated to the $^1S_0$ and $^3S_1$-$^3D_1$ states yields $E_B = -7.53 \pm 0.08$ MeV.

[NUCLEAR STRUCTURE ^{3}H; calculate binding energy; include all partial waves of Read potential up to $J \leq 2$; Faddeev approach.]

I. INTRODUCTION

Meson theoretical potentials are naturally given in momentum space and can be transformed to coordinate space only after appreciable approximations.[1] The same holds for modified versions of the Faddeev equations taking care of relativistic effects.[2,3] Therefore a simple and economic algorithm to calculate the three nucleon bound state in momentum space is desirable for allowing an easy test of nuclear interaction models and dynamical equations.

The starting point of our method, which has been briefly described in Ref. 4, is the Faddeev equations[5] in differential form

$$(T + V - E)|\Psi\rangle_{(2,3)} = -V(P_{12}P_{23} + P_{13}P_{23})|\Psi\rangle_{(2,3)} , \tag{1}$$

which directly contain the two-body nuclear interaction V_{23}. The high number of partial-wave amplitudes together with the dependence on two continous variables, namely, the subsystem momentum p and the spectator momentum q, prohibits a direct discretization at least for most of the present day computer facilities.

As proposed by Kim[6] one might gain a reduction of the dimension if the p dependence of Ψ can be incorporated using a few suitably chosen basis functions $\varphi_n(p)$ taken from a complete set:

$$\Psi(p,q) = \sum_n \varphi_n(p) b_n(q) . \tag{2}$$

Since the variable p describes the relative motion of two interacting particles, one should incorporate into the basis functions $\varphi_n(p)$ the specific features of the two-body interaction, especially the short range repulsion. Furthermore, since we are interested in a bound-state problem and because of technical simplifications we would like to deal with a discrete basis of square integrable functions. Therefore we suggested[4] the use of eigenfunctions of the two-nucleon Hamiltonian supplemented by a harmonic-oscillator potential:

$$h = T + V + V_{\text{ho}} . \tag{3}$$

This choice should be much more convenient than pure harmonic-oscillator functions,[7] although the basis functions are no longer given analytically. A similar technique has been used to calculate the Brueckner G matrix.[8]

Using the expansion (2) in Eq. (1) one ends up with a set of coupled integral equations in one variable for the unknown functions $b_n(q)$. This set is tractable since it turns out that the convergence with respect to the number of basis functions is rather good.

We wish to emphasize that the decomposition (2) does not correspond to a finite-rank expansion of the interaction. This is obvious from the representation

$$\Psi = G_0 t (P_{12}P_{23} + P_{13}P_{23})\Psi \tag{4}$$

of Eq. (1). Here t is the two-body t matrix and $G_0 = (z - p^2 - q^2)^{-1}$. Obviously, the presence of G_0 precludes simultaneous finite-rank expansions for Ψ and t. Nevertheless, one can handle separable interactions by our method as well, though in that case there exist more economic methods. In fact we recalculated the three-body binding energy for some separable interactions for the sole purpose of testing our computer code (see Sec. IV).

In Sec. II we define our notations and describe

our choice of basis functions. The representation of the Faddeev equation (1) in that basis is given in Sec. III. Section IV is devoted to numerical details, including test calculations with separable interactions and three spinless bosons interacting by the Reid-$^1S_0$ potential only.

In Sec. V we present our results for the full Reid soft-core potential[9] together with a detailed analysis of the p- and d-wave contributions. There is a striking difference from results found by other groups,[10-13] who quoted a value of 7.0 MeV for the truncated Reid potential. We want to stress that we also get a value close to that, namely 7.1, if we erroneously change the sign of the $l=0$, $l=2$ potential matrix elements. Perhaps this is the key to the still unbalanced results of triton calculations with the Reid potential.[10-16]

II. NOTATIONS AND CHOICE OF BASIS FUNCTIONS

For the convenience of the reader and for the sake of clearness we briefly describe the notations used. In terms of the states $|\vec{k}_1,\vec{k}_2,\vec{k}_3\rangle$ describing three free particles with momenta $\vec{k}_1$, $\vec{k}_2$, and $\vec{k}_3$ we define momentum states of relative and center of mass motion by

$$\langle\vec{k}_1\vec{k}_2\vec{k}_3|\vec{p}\vec{q}\vec{k}\rangle_{(23)} = \delta(\vec{p}-\tfrac{1}{2}(\vec{k}_2-\vec{k}_3)) \times\delta(\vec{q}-(1/\sqrt{3})[\vec{k}_1-\tfrac{1}{2}(\vec{k}_2+\vec{k}_3)]) \times\delta(\vec{k}-\vec{k}_1-\vec{k}_2-\vec{k}_3) \quad (5)$$

and the corresponding ones for the subsystems (12) and (31) in a cyclic way. Assuming the states $|\vec{k}_1\vec{k}_2\vec{k}_3\rangle$ to be normalized to δ functions and fulfilling the completeness relation without additional factors one finds

$$_{(23)}\langle\vec{p}\vec{q}\vec{k}|\vec{p}'\vec{q}'\vec{k}'\rangle_{(31)} = \left(\frac{2}{\sqrt{3}}\right)^3\delta(\vec{p}'+\tfrac{1}{2}\vec{p}+\tfrac{1}{2}\sqrt{3}\vec{q}) \times\delta(\vec{q}'-\tfrac{1}{2}\sqrt{3}\vec{p}+\tfrac{1}{2}\vec{q})\delta(\vec{k}-\vec{k}') . \quad (6)$$

Furthermore we define partial-wave states by

$$_{(23)}\langle\vec{p}\vec{q}\vec{k}|p'lmq'\lambda\mu\vec{k}'\rangle_{(23)} = \frac{\delta(p-p')}{p^2}\,\frac{\delta(q-q')}{q^2}\,Y_{lm}(\hat{p})Y_{\lambda\mu}(\hat{q})\delta(\vec{k}-\vec{k}') \quad (7)$$

which fixes the normalization and completeness relation as

$$_{(23)}\langle plmq\lambda\mu\vec{k}|p'l'm'q'\lambda'\mu'\vec{k}'\rangle_{(23)} = \left(\frac{\sqrt{3}}{2}\right)^3\frac{\delta(p-p')}{p^2}\,\delta_{ll'}\delta_{mm'} \times\frac{\delta(q-q')}{q^2}\,\delta_{\lambda\lambda'}\delta_{\mu\mu'}\delta(\vec{k}-\vec{k}') \quad (8)$$

and

$$\left(\frac{2}{\sqrt{3}}\right)^3\sum_{lm\lambda\mu}\int_0^\infty p^2dp\int_0^\infty q^2dq\int d^3k \times|plmq\lambda\mu\vec{k}\rangle_{(23)}\langle plmq\lambda\mu\vec{k}| = \underline{1} . \quad (9)$$

In complete analogy we introduce states $|\vec{r}\vec{\rho}\vec{R}\rangle_{(23)}$ by

$$\langle\vec{x}_1\vec{x}_2\vec{x}_3|\vec{r}\vec{\rho}\vec{R}\rangle_{(23)} = \delta(\vec{r}-(\vec{x}_2-\vec{x}_3)) \times\delta(\vec{\rho}-(2/\sqrt{3})[\vec{x}_1-\tfrac{1}{2}(\vec{x}_2+\vec{x}_3)]) \times\delta(\vec{R}-\tfrac{1}{3}(\vec{x}_1+\vec{x}_2+\vec{x}_3)) \quad (10)$$

and the corresponding partial-wave states $|rlm\rho\lambda\mu\vec{R}\rangle_{(23)}$ by

$$_{(23)}\langle\vec{r}\vec{\rho}\vec{R}|r'lm\rho'\lambda\mu\vec{R}'\rangle = \frac{\delta(r-r')}{r^2}\,\frac{\delta(\rho-\rho')}{\rho^2}\,Y_{lm}(\hat{r}) \times Y_{\lambda\mu}(\hat{\rho})\delta(\vec{R}-\vec{R}') , \quad (11)$$

where $\vec{x}_1$, $\vec{x}_2$, $\vec{x}_3$ denote the positions of the three particles. As a consequence of (5), (7), (10), (11), and of

TABLE I. List of partial waves in ^{3}H (total angular momentum $\mathfrak{J}=\frac{1}{2}$, parity +) which come into play, if the interaction $V^{st}_{jj'}=0$ for $j>2$.

No.	l	s	j	t	λ	J
1	0	0	0	1	0	$\frac{1}{2}$
2	0	1	1	0	0	$\frac{1}{2}$
3	2	1	1	0	0	$\frac{1}{2}$
4	0	1	1	0	2	$\frac{3}{2}$
5	2	1	1	0	2	$\frac{3}{2}$
6	1	1	0	1	1	$\frac{1}{2}$
7	1	0	1	0	1	$\frac{1}{2}$
8	1	0	1	0	1	$\frac{3}{2}$
9	1	1	1	1	1	$\frac{1}{2}$
10	1	1	1	1	1	$\frac{3}{2}$
11	1	1	2	1	1	$\frac{3}{2}$
12	3	1	2	1	1	$\frac{3}{2}$
13	1	1	2	1	3	$\frac{5}{2}$
14	3	1	2	1	3	$\frac{5}{2}$
15	2	0	2	1	2	$\frac{3}{2}$
16	2	0	2	1	2	$\frac{5}{2}$
17	2	1	2	0	2	$\frac{3}{2}$
18	2	1	2	0	2	$\frac{5}{2}$

$$\langle \vec{x}_1\vec{x}_2\vec{x}_3|\vec{k}_1\vec{k}_2\vec{k}_3\rangle = \left(\frac{1}{(2\pi)^{3/2}}\right)^3 \times \exp[i(\vec{k}_1\cdot\vec{x}_1+\vec{k}_2\cdot\vec{x}_2+\vec{k}_3\cdot\vec{x}_3)] \quad (12)$$

one gets

$$_{(23)}\langle p\,lmq\lambda\mu\vec{k}|rl'm'\rho\lambda'\mu'\vec{R}\rangle_{(23)} = \delta_{ll'}\delta_{mm'}\delta_{\lambda\lambda'}\delta_{\mu\mu'} \times \frac{2}{\pi} i^{l+\lambda} j_l(pr) j_\lambda(q\rho) \frac{1}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\vec{R}}. \quad (13)$$

We are now able to state unambigously the potential to be used in the three-particle momentum space. In the following we use the familiar j-J coupling scheme where l and the subsystem spin s are coupled to j and J is the total angular momentum of the spectator particle; t denotes the isospin of the subsystem. The first 18 partial-wave states, characterized by these quantum numbers, are shown in Table I.

The Reid potential is given as a local potential of the type

$$\langle \vec{x}_1\vec{x}_2\vec{x}_3|V_{23}|\vec{x}_1'\vec{x}_2'\vec{x}_3'\rangle = \prod_i \delta(\vec{x}_i-\vec{x}_i') \sum_\alpha V_\alpha(|\vec{x}_2-\vec{x}_3|) O_\alpha \frac{\vec{x}_2-\vec{x}_3}{|\vec{x}_2-\vec{x}_3|}. \quad (14)$$

Here O_α are the unit, the spin-orbit, and the tensor operators, respectively. Suppressing the fixed total momentum dependence we have in the three-particle momentum space

$$_{(23)}\langle p(ls)j;q(\lambda\tfrac{1}{2})J\mathfrak{T};(t\tfrac{1}{2})T|V|p'(l's')j';q'(\lambda'\tfrac{1}{2})J'\mathfrak{T};(t'\tfrac{1}{2})T\rangle_{(23)} = \left(\frac{\sqrt{3}}{2}\right)^3 \frac{\delta(q-q')}{q^2}\delta_{\lambda\lambda'}\delta_{JJ'}\delta_{SS'}\delta_{jj'}V_{ll'}^{sjt}(p,p') \quad (15a)$$

with

$$V_{ll'}^{sjt}(p,p') = \frac{2}{\pi} i^{l-l'} \sum_\alpha \int_0^\infty r^2 dr j_l(pr) V_\alpha(r) j_{l'}(p'r) O_{ll'}^{\alpha;sjt} \quad (15b)$$

and where the geometrical factors are given by

$$O_{ll'}^{\alpha;sjt} = \langle (ls)j;t|\begin{pmatrix}1\\ \vec{l}\cdot\vec{s}\\ S_{12}\end{pmatrix}|(l's)j;t\rangle = \begin{cases}\delta_{ll'},\\ \delta_{ll'}-\frac{1}{2}[j(j+1)-l(l+1)-s(s+1)]\\ \sqrt{24}(-1)^{1+j}[(2l+1)(2l'+1)]^{1/2}\langle l'0l0|20\rangle W(ll'11;2j)\end{cases}$$

Since we use a different manner of integration and discretization in the Faddeev equations than the authors of Ref. 11 we need another form of the recoupling coefficient:

$$_{(23)}\langle p(ls)j;q(\lambda\tfrac{1}{2})J\mathfrak{T};(t\tfrac{1}{2})T|P_{12}P_{23}+P_{13}P_{23}|p'(l's')j';q'(\lambda'\tfrac{1}{2})J'\mathfrak{T};(t'\tfrac{1}{2})T\rangle_{(23)}.$$

Inserting the completeness relation for the states $|\vec{p}\vec{q}\vec{k}\rangle$ and using (6) and (7) it is straightforward but tedious to arrive at the following result[17]:

$$_{(23)}\langle p(ls)j;q(\lambda\tfrac{1}{2})J\mathfrak{T};(t\tfrac{1}{2})T|P_{12}P_{23}+P_{13}P_{23}|p'(l's')j';q'(\lambda'\tfrac{1}{2})J'\mathfrak{T};(t'\tfrac{1}{2})T\rangle_{(23)} = \int_{-1}^{1} du \frac{\delta(p-|(1/\sqrt{3})\vec{q}+(2/\sqrt{3})\vec{q}'|)}{p^{2+l}} \frac{\delta(p'-|(2/\sqrt{3})\vec{q}+(1/\sqrt{3})\vec{q}'|)}{p'^{2+l'}} \mathcal{G}_{lsj\lambda Jt}^{l's'j'\lambda'J't'}(q,q',u), \quad (16a)$$

where $u=\hat{q}\cdot\hat{q}'$ and the geometrical factor $\mathcal{G}$ is given by

$$\mathcal{G}_{lsj\lambda Jt}^{l's'j'\lambda'J't'}(q,q',u) = \sum_{\substack{l_1+l_2=l\\ l_1'+l_2'=l'}} \sum_k q^{l_1+l_1'} q'^{l_2+l_2'} P_k(u) \mathcal{G}_{lsj\lambda Jt}^{l's'j'\lambda'J't'}(l_1,l_2,l_1',l_2',k)$$

and

$$g^{l's'j'\lambda'J't'}_{lsj\lambda Jt}(l_1,l_2,l_1',l_2',k) = -\hat{l}\hat{l}'\hat{s}\hat{s}'\hat{j}\hat{j}'\hat{\lambda}\hat{\lambda}'\hat{t}\hat{t}'\hat{J}\hat{J}' \frac{2^{l_2+l_1'}}{(\sqrt{3})^{l+l'}} \left(\frac{(2l+1)!(2l'+1)!}{(2l_1)!(2l_2)!(2l_1')!(2l_2')!}\right)^{1/2} (2k+1) \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & t \\ \frac{1}{2} & T & t' \end{Bmatrix}$$

$$\times \sum_{L,S} (2L+1)(2S+1) \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & s \\ \frac{1}{2} & S & s' \end{Bmatrix} \begin{Bmatrix} l & s & j \\ \lambda & \frac{1}{2} & J \\ L & S & \mathcal{T} \end{Bmatrix} \begin{Bmatrix} l' & s' & j' \\ \lambda' & \frac{1}{2} & J' \\ L & S & \mathcal{T} \end{Bmatrix}$$

$$\times \sum_{fd} (2f+1)(2d+1) \begin{pmatrix} l_1 & \lambda & f \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_1' & f & k \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_2' & \lambda' & d \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_2 & d & k \\ 0 & 0 & 0 \end{pmatrix} \begin{Bmatrix} l_2 & l_1 & l \\ \lambda & L & f \end{Bmatrix} \begin{Bmatrix} l_1' & l_2' & l' \\ \lambda' & L & d \end{Bmatrix} \begin{Bmatrix} L & l_1' & d \\ k & l_2 & f \end{Bmatrix} . \tag{16c}$$

Having clarified our notation we now want to introduce the basis functions

$$\varphi^n_{lsj}(p) = \frac{1}{p} a^n_{lsj}(p)$$

for the two-particle subsystem.

They are chosen as solutions to the subsystem Hamiltonian supplemented by a harmonic-oscillator potential with the strength κ. The corresponding eigenvalue problem reads

$$\left(\frac{\hbar^2}{M}p^2 - \kappa \frac{d^2}{dp^2} + \kappa \frac{l(l+1)}{p^2} - E_n\right) a^n_{lsj}(p) + \sum_{l'} \int_0^\infty dp' v^{sjt}_{ll'}(p,p') a^n_{l'sj}(p') = 0 . \tag{17}$$

Here we used

$$v^{sjt}_{ll'}(p,p') = pp' V^{sjt}_{ll'}(p,p') . \tag{17a}$$

The coupling occurs only in partial waves Nos. 2, 3 and 11, 12 of Table I, where the tensor force is present.

Due to the hermiticity of the problem the solution to a fixed set $\{sjt\}$ can be chosen to fulfill

$$\sum_l \int_0^\infty dp\, p^2 \varphi^{n*}_{lsj}(p) \varphi^m_{lsj}(p) = \delta_{nm} , \tag{18a}$$

$$\sum_n \varphi^n_{lsj}(p) \varphi^{n*}_{l'sj}(p') = \frac{\delta(p-p')}{p^2} \delta_{ll'} . \tag{18b}$$

Since the potential is real we may choose real solutions. Assuming the same functions $a^n_{lsj}(p)$ for partial waves differing only in quantum numbers of the spectator particle, we define our basis states in the complete three-particle space by

$$|nsj;q(\lambda\tfrac{1}{2})J\mathcal{T};(t\tfrac{1}{2})T\rangle_{(2,3)} = \sum_l \int_0^\infty p^2 dp\, \varphi^n_{lsj}(p) |p(ls)j;q(\lambda\tfrac{1}{2})J\mathcal{T};(t\tfrac{1}{2})T\rangle_{(23)} \tag{19}$$

which by construction fulfill

$${}_{(23)}\langle nsj;q(\lambda\tfrac{1}{2})J\mathcal{T};(t\tfrac{1}{2})T|n's'j';q'(\lambda'\tfrac{1}{2})J'\mathcal{T};(t'\tfrac{1}{2})T\rangle_{(23)} = \left(\frac{\sqrt{3}}{2}\right)^3 \delta_{nn'}\delta_{ss'}\delta_{jj'} \frac{\delta(q-q')}{q^2} \delta_{\lambda\lambda'}\delta_{JJ'}\delta_{tt'} \tag{20a}$$

and

$$\left(\frac{2}{\sqrt{3}}\right)^3 \sum_{\substack{nsj \\ \lambda Jt}} \int q^2 dq |nsj;q(\lambda\tfrac{1}{2})J\mathcal{T};(t\tfrac{1}{2})T\rangle_{(23)} \langle nsj;q(\lambda\tfrac{1}{2})J\mathcal{T};(t\tfrac{1}{2})T| = \underline{1} \tag{20b}$$

In any application we can take advantage of a free choice of κ for different partial waves to improve the convergence with respect to n. Choosing $l+s+t$ odd the states (19) are antisymmetric under exchange of particles 2 and 3.

III. EQUATIONS FOR THE THREE-BODY BOUND STATE

We now represent the Faddeev equation (1) in the basis states (19). Using (20b) the Faddeev amplitude $|\Psi\rangle_{(23)}$ can be written as

$$|\Psi\rangle_{(23)} = \left(\frac{2}{\sqrt{3}}\right)^3 \sum_{\substack{nsj \\ \lambda Jt}} \int q^2 dq |nsj;q(\lambda\tfrac{1}{2})J\mathcal{T};(t\tfrac{1}{2})T\rangle \times b^n_{sj\lambda Jt}(q) . \tag{21}$$

We insert this expansion into (1), project onto the states (19) and are left with the following set of coupled integral equations:

$$\left(\frac{\hbar^2}{M}q^2 - E\right)b^n_{sj\lambda Jt}(q) + \sum_{n'} c^{sjt}_{nn'} b^{n'}_{sj\lambda Jt}(q) + \sum_{n'} \sum_{\substack{s'j't' \\ \lambda'J'}} \int q'^2 dq' H^{n's'j'\lambda'J't'}_{nsj\lambda Jt}(q,q') \times b^{n'}_{s'j'\lambda'J't}(q') = 0. \quad (22)$$

The coefficients c, which are off diagonal only in the quantum number n, are given as

$$c^{sjt}_{nn'} = \frac{\hbar^2}{M} \sum_l \int dp\, a^{n*}_{lsj}(p) p^2 a^{n'}_{lsj}(p) + \sum_{ll'} \int dp\, dp'\, a^{n*}_{lsj}(p) v^{sjt}_{ll'}(p,p') a^{n'}_{l'sj}(p') . \quad (23)$$

The three-body dynamics is contained in the integral kernel $H(q,q')$ which couples all nonconserved quantum numbers:

$$H^{n's'j'\lambda'J't'}_{nsj\lambda Jt}(q,q') = \left(\frac{2}{\sqrt{3}}\right)^3 \sum_{ll'} \int_{-1}^{1} du \frac{I_{nlsj}(p_1)}{p_1^{l+1}} \mathcal{G}^{l's'j'\lambda'J't'}_{lsj\lambda Jt}(q,q',u) \times \frac{a^{n'}_{l's'j'}(p_2)}{p_2^{l'+1}} \quad (24)$$

with

$$p_1 = (\tfrac{1}{3}q^2 + \tfrac{4}{3}q'^2 + \tfrac{4}{3}qq'u)^{1/2} , \qquad p_2 = (\tfrac{4}{3}q^2 + \tfrac{1}{3}q'^2 + \tfrac{4}{3}qq'u)^{1/2} , \quad (25)$$

and

$$I_{nlsj}(p_1) = \sum_{l''} \int dp\, a^n_{l''sj}(p) v^{sjt}_{l''l}(p,p_1) . \quad (26)$$

It is the choice of physically adapted basis functions $\varphi^n_{lsj}(p)$ which allows us to truncate the infinite sums over n' in (22) to a small and easy manageable size.

IV. NUMERICAL DETAILS

The solution of Eq. (17) to determine the basis functions $a^n_{lsj}(p)$ is not trivial, because the nucleon-nucleon interaction demands a large numerical cutoff for the p' integration ($p_{max} \lesssim 35\,\mathrm{fm}^{-1}$), whereas the second derivative requires a small step length at small p values, where the functions $a^n_{lsj}(p)$ oscillate (compare Fig. 1). Therefore we used a variable step length typically of 0.1 fm^{-1} for small p values ($\lesssim 3.6\ \mathrm{fm}^{-1}$) and of 1.6 fm^{-1} for p values beyond 15 fm^{-1} in the realistic domain of the oscillator strength κ between 2 and 5 MeV fm^{-2}. Using five point formulas for the derivative and the Simpson rule for the integral, equation (17) is turned into an algebraic eigenvalue problem which is solved by inverse iteration as described in detail in Ref. 4. In all our calculations the orthogonality of the eigenvectors is less than 10^{-4}, which is not self-evident, because the matrix to be diagonalized is not symmetric.

TABLE II. Comparison of the eigenvalues E_n of Eq. (17) calculated for the Reid-soft $^1S_0$ potential in configuration and momentum space. The oscillator strength $\kappa = 0.4$ MeV fm^{-2} demanded a very fine initial step size of 0.05 fm^{-1} to reach this accuracy in the p-space calculation.

n	E_n (MeV) (r space)	E_n (MeV) (p space)
1	7.038	7.036
2	24.972	24.969
3	42.230	42.218
4	59.252	59.223
5	76.142	76.082

In addition, we did a configuration space calculation for the $^1S_0$-state Reid soft potential. The corresponding Bessel transform of Eq. (17) was solved using the Numerov method which guarantees high accuracy. Table II demonstrates the agreement between both calculations.

The Faddeev equation (22) is turned into an algebraic eigenvalue problem by representing the functions $b^n_{sj\lambda Jt}(q)$ at Gauss points. We use an integration cutoff q_{max} and map the interval $[0, q_{max}]$ onto $[-1,1]$ by using

$$q = \frac{1+x}{1/q_0 - (1/q_0 - 2/q_{max})x} , \quad (27)$$

where q_0 is another free parameter. Calculating the kernel $H(q,q')$ at the Gauss points we can take advantage of using a q_{max} cutoff by first integrating for $I_{nlsj}(p_1)$ at equidistant points p_1 and then interpolating at the actual positions. The numerical parameters are as follows:

(1) cutoff momentum q_{max};
(2) number of Gauss points N_q for q' integration;
(3) mapping parameter q_0;
(4) number of Gauss points N_u for u integration; and
(5) degree of interpolation polynomial (Aitken-Neville) N_{p_1} for integrals $I_{nlsj}(p_1)/p_1^{l+1}$ and N_{p_2} for functions $a^n_{lsj}(p_2)/p_2^{l+1}$.

The dependence on these parameters is given in Table III where triton binding energies have been calculated for the Reid potential acting only in the $^1S_0$ and $^3S_1 - {}^3D_1$ channels (Nos. 1-5 of Table I). The energy E_{B_1} refers to a calculation with only one basis function per partial wave, E_{B_2} refers to taking three basis functions in the $^1S_0$, five in the $^3S_1$-$^3D_1$ ($\lambda = 0$), and three in the $^3S_1$-$^3D_1$ ($\lambda = 2$) channels, respectively. Though q_{max} is the most critical numerical parameter, the cutoff dependence actually is negligible for $q_{max} \gtrsim 4\ \mathrm{fm}^{-1}$. Neverthe-

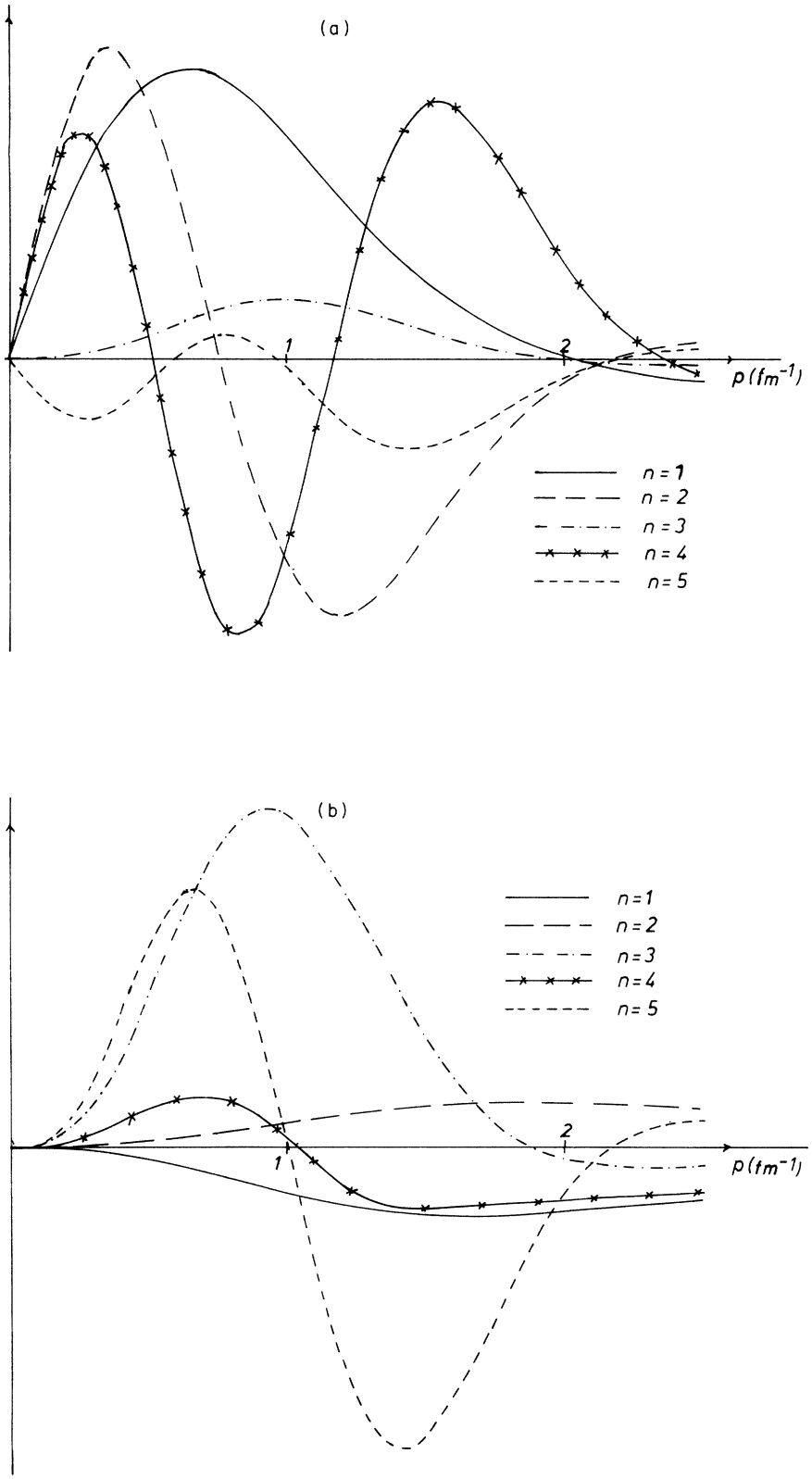

FIG. 1(a) $l = 0$ components of basis functions of $a^n_{ls\,j}(p)$ in the $^3S_1$-$^3D_1$ channel calculated with the Reid-soft potential and $\kappa = 4.0$ MeV fm^{-2}. (b) $l = 2$ components of basis functions $a^n_{ls\,j}(p)$ corresponding to (a).

TABLE III. Dependence of triton binding energy E_B on the numerical parameters as defined in the text. E_{B_1} and E_{B_2} refer to different numbers of basis functions taken into account, as indicated in parentheses.

q_{max} (fm^{-1})	N_q	q_0 (fm^{-1})	N_u	N_{p_1}	N_{p_2}	E_{B_1} (MeV) (1,1,1,1,1)	E_{B_2} (MeV) (3,5,5,3,3)
2	8	1	8	6	8	−7.032	−7.373
3	8	1	8	6	8	−7.217	−7.546
3	10	1	8	6	8	−7.215	−7.547
4	8	1	8	6	8	−7.248	−7.576
4	10	1	8	6	8	−7.245	−7.577
5	10	1	8	6	8	−7.247	−7.581
4	8	1.5	8	6	8	−7.248	−7.576
4	8	1	10	6	8	−7.248	−7.576
4	8	1	8	8	8	−7.248	−7.576
4	8	1	8	8	10	−7.248	−7.576

less, we used the whole q interval (no cutoff) in the case of three identical bosons interacting by the Reid $^1S_0$ potential. There was no change within 10 keV with respect to a calculation with cutoff. For that simple system we also expanded the Faddeev amplitude in configuration space

$$\Psi(r,\rho) = \sum_n \varphi_n(r) b_n(\rho)$$

determined $\varphi_n(r)$ as mentioned above and solved the coordinate space equation corresponding to (22). Here one is faced with a set of coupled integrodifferential equations for the functions $b_n(\rho)$. This set is turned into integral equations and solved by the method of Malfliet and Tjon.[18] Due to the fine discretization necessary we could include only two basis functions $\varphi_n(r)$. However, the three-body eigenvalue problem could be solved with high accuracy and is compared with the momentum space calculation in Table IV. The agreement is satisfying.

Purely for the sake of testing our computer code in problems with the full complexity and to examine the convergence with respect to the number of basis functions we did some more test runs with several separable potentials. In that case the three-body binding energies are very well known from calculations making use of the separability. The agreement is very good as shown in Table V and could be reached with a reasonable number of basis functions.

In order to demonstrate the convergence of E_B in the number of basis functions and the independence on the oscillator strength κ, we show in Fig. 2(a) the case of the Graz potential. Here we have taken along only the first three partial waves of

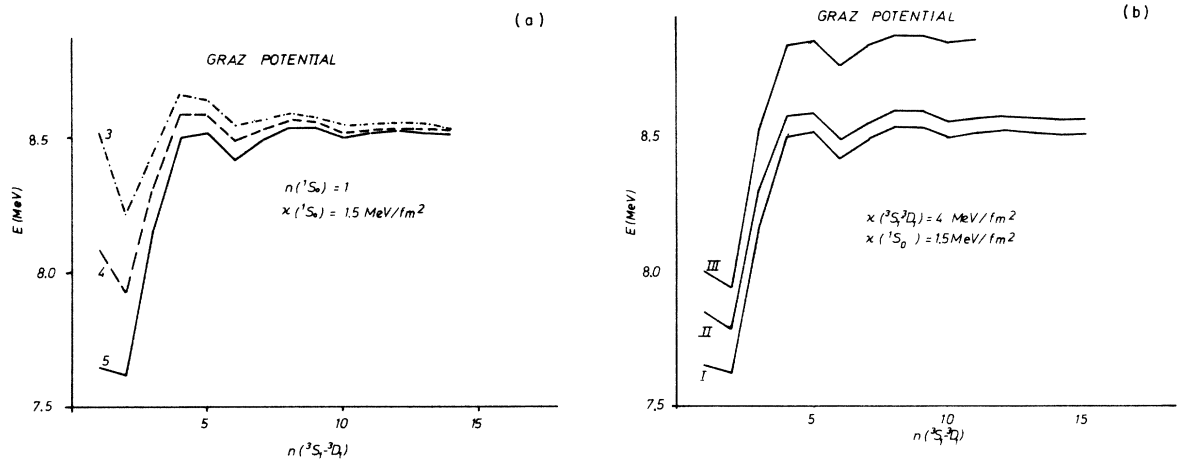

FIG. 2. (a) Dependence of E_B on the number of basis functions n and the oscillator strength κ used in the $^3S_1$-$^3D_1$ channel for the Graz potential. The angular momentum of the spectator particle is allowed to be $\lambda = 0$ only. In the $^1S_0$ channel only one basis function with a fixed $\kappa_1 = 1.5$ MeV fm^{-2} is used. (b) For the Graz potential the oscillator of the $^3S_1$-$^3D_1$ channel is fixed to $\kappa = 4$ MeV fm^{-2} and the numbers of basis states in the other partial waves are varied.

TABLE IV. Comparison of coordinate- and momentum-space calculation of the three boson bound state interacting by the $^1S_0$ Reid potential.

	E_B (r space)	E_B (p space)
1	−0.873	−0.877
2	−0.763	−0.765

Table I, and in the $^1S_0$ wave we fixed the oscillator strength $\kappa_1 = 1.5$ MeV fm^{-2} as well as the number of basis functions $n_1 = 1$. Thus Fig. 2(a) shows the poor dependence on n_2 and κ_2 in the $^3S_1$-$^3D_1$ channel. The optimal curve for $\kappa_2 = 4$ MeV fm^{-2} is sandwiched by curves with decreasing and increasing tendency. A picture like this easily allows to estimate the numerical error.

In Fig. 2(b) we fixed $\kappa_2 = 4$ MeV fm^{-2} and now varied the number of basis functions in the other partial waves. Curve I is the same as the middle one of Fig. 2(a). If we use $n_1 = 5$ we get curve II. Including the $\lambda = 2$ waves (Nos. 4, 5 of Table I) with four basis functions we arrive at curve III. Note that the curves are almost parallel. This allows for extrapolations, for instance of curve III.

V. TRITON CALCULATIONS WITH THE REID SOFT-CORE POTENTIAL

In the j-J coupling scheme the Reid potential for $j \leq 2$ acts in 18 partial waves of the three nucleon bound state (see Table I). Let us first discuss calculations for this potential truncated to the $^1S_0$ and $^3S_1$-$^3D_1$ waves. In Fig. 3 we show the convergence in the $^1S_0$ wave (No. 1 of Table I). Obviously $\kappa_1 = 2$ MeV fm^{-2} with $n_1 = 5$ (or 3) appear to be sufficient. The dependence on κ_2 and n_2 of partial waves Nos. 2 and 3 is pictured in Fig. 4. For all curves belonging to different κ_2 values, κ_1 and n_1 have been fixed to the above numbers, and in partial waves Nos. 4 and 5, $n_4 = 3$ basis functions of the same κ_2 are taken into account.

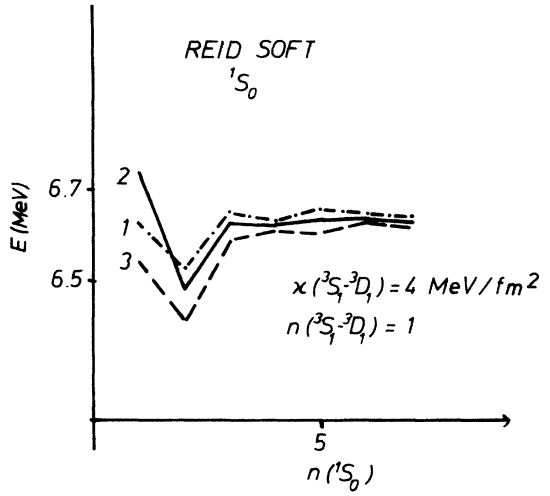

FIG. 3. For the Reid potential the dependence of E_B on κ_1 and n_1 in the $^1S_0$ channel is pictured. $\kappa_2 = 4$ MeV fm^{-2} and $n_2 = 1$ are fixed for partial waves Nos. 2 and 3 of Table I.

Because we used the same set of basis functions in partial waves Nos. 2, 3 and Nos. 4, 5, one might suspect that the κ dependence is more complicated than in Fig. 2(a), for instance. However, there is rapid convergence in Nos. 4, 5 regardless of what κ has been used (see Fig. 5). Thus we end up with a binding energy of -7.53 ± 0.06 MeV for the truncated Reid potential.

Let us next turn to the p and d waves of the Reid potential. To include them we were forced to reduce the number of basis functions in the $^3S_1$-$^3D_1$ channel because of the very limited capacity of our computer. We have chosen $n_1 = 3$ ($\kappa_1 = 2$ MeV fm^{-2}), $n_2 = 5$ and $n_4 = 2$ ($\kappa_2 = 4$ MeV fm^{-2}). For this choice

TABLE V. Results for test runs with various separable interactions.

Potential		E_B/MeV (this method)	Number of basis functions $^1S_0$	$^3S_1$-$^3D_1$ $\lambda=0$	$^3S_1$-$^3D_1$ $\lambda=2$	E_B/MeV (other methods)	Reference
Reid (UPA)	A	−6.87 ± 0.02	5	10	...	−6.8	15
	B	−7.44 ± 0.02	5	10	5	−7.45(−7.42)	15(19)
Graz	A	−8.57 ± 0.03	5	15	...		20, 21
	B	−8.86 ± 0.03	5	11	4	−8.86	
Hammann-Mongan II (no tensor)		−10.01 ± 0.01	7	7	...	−10.02	22, 23, 21
Hammann-Mongan II (with tensor)	A	−9.63 ± 0.01	5	10	...		22, 23, 21
	B	−9.69 ± 0.01	5	10	3	9.70	

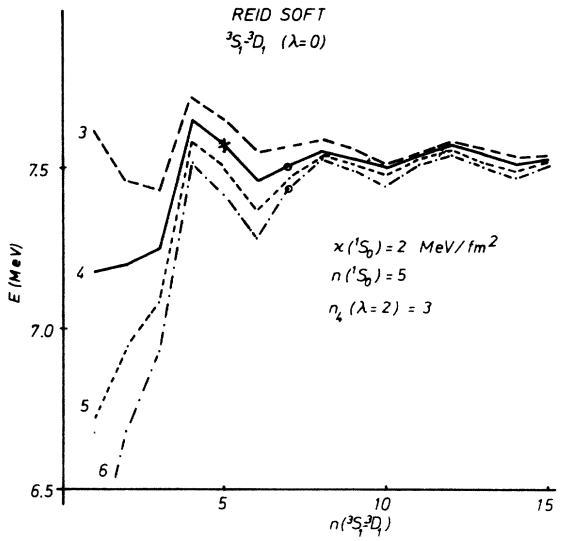

FIG. 4. Dependence of E_B on κ_2 and n_2 (partial waves Nos. 2, 3) for the Reid potential. The values of $\kappa_1 = 2$ MeV fm^{-2} and $n_1 = 5$ are fixed. In partial waves Nos. 4 and 5 we used $n_4 = 3$ basis functions with the same κ_2.

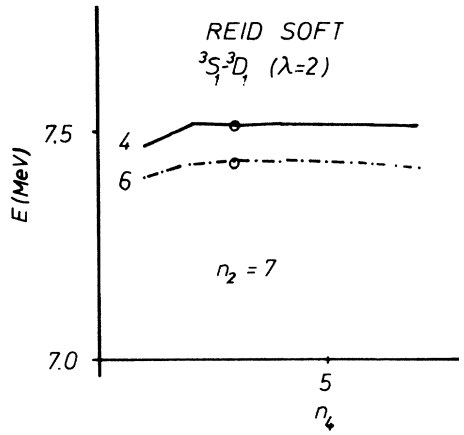

FIG. 5. Convergence with respect to n_4 for κ_2 = 4 MeV fm^{-2} and 6 MeV fm^{-2}. $n_2 = 7$ is fixed. The difference between both curves is due to the κ_2 dependence in partial waves Nos. 2, 3 as indicated by the circles in Fig. 4 and 5.

one gets −7.58 MeV for the truncated Reid potential which is not too far away from the fully converged result. We then investigated the particular influence of the $^3P_0$, $^1P_1$, $^3P_1$, $^3P_2$-$^3F_2$, $^1D_2$, and $^3D_2$ waves of the Reid potential. Their particular contributions can be taken from Table VI. In Fig. 5 the convergence and κ dependence is shown for these waves.

Including all p waves (i.e., Nos. 1–14) shows that there is almost no interference between these waves. Extrapolating from the small basis used in this calculation, especially from 5 to 15 functions in partial waves Nos. 2, 3, we get a binding energy of −7.46 ± 0.09 MeV. Thus the contribution of all p waves including the $^3P_2$-$^3F_2$ tensor force is slightly repulsive which mainly is due to the $^3P_1$ waves. The d waves contribute 100 keV attraction and we get −7.56 ± 0.10 MeV for the Reid potential including all partial waves up to $J \leq 2$.

VI. SUMMARY

We presented an economic and transparent method to solve the Faddeev equation in momentum space using directly the potential. It takes ad-

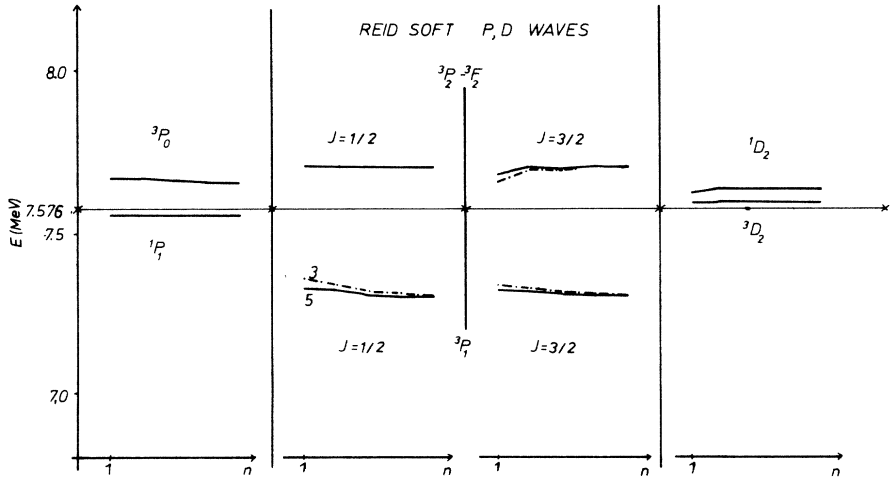

FIG. 6. Convergence and κ-dependence for partial waves Nos. 6–18 of Table I. When calculating the $^3P_1$ curves for instance, all other p and d waves have been omitted, but Nos. 9 and 10 are both included. The horizontal line corresponds to the truncated Reid calculation as marked by the cross in Fig. 4. This is basically taken in partial waves Nos. 1–5.

TABLE VI. Triton binding energy dependence on the number of partial waves calculated with the Reid-soft potential.

Partial wave No.	Potential acting in	Number of basis functions											
1	$^1S_0$	5	5	3	3	3	3	3	3	3	3	3	3
2		15	15	5	5	5	5	5	5	5	5	5	5
3	$^3S_1$-$^3D_1$	15	15	5	5	5	5	5	5	5	5	5	5
4		...	3	2	2	2	2	2	2	2	2	2	2
5		...	3	2	2	2	2	2	2	2	2	2	2
6	$^3P_0$	...	...	...	5	...	...	...	...	...	1	...	1
7	$^1P_1$	...	...	...	...	5	...	...	...	...	1	...	1
8		...	...	...	...	5	...	...	...	...	1	...	1
9	$^3P_1$	...	...	...	...	...	5	...	...	...	2	2	2
10		...	...	...	...	...	5	...	...	...	2	2	2
11	$^3P_2$-$^3F_2$	...	...	...	...	...	...	5	...	...	2	1	1
12		...	...	...	...	...	...	5	...	...	2	1	1
13		...	...	...	...	...	...	5	...	...	2	1	1
14		...	...	...	...	...	...	5	...	...	2	1	1
15	$^1D_2$	...	...	...	...	...	...	...	5	...	...	1	1
16		...	...	...	...	...	...	...	5	...	...	1	1
17	$^3D_2$	...	...	...	...	...	...	...	...	5	...	1	1
18		...	...	...	...	...	...	...	...	5	...	1	1
Binding energy (MeV)		−6.80	−7.53	−7.58							−7.55	−7.49	−7.57
Binding energy −7.58 MeV					−0.08	+0.02	+0.27	−0.14	−0.07	−0.03			
Extrapolated											−7.46	−7.48	−7.56

vantage of an expansion of the Faddeev amplitude into a physically adapted set of basis states for the subsystem. The convergence is satisfactory and thus gives an easy manageable tool to test meson theoretical potentials and relativistic generalized equations. Investigations in this direction are under way. A complete calculation including the usual five partial waves (Nos. 1–5 of Table I) takes about 45 minutes on a Telefunken TR 440, which should be reduced by a factor 8 or 10 on an IBM 370.

Our result for the Reid potential differs from results of other groups[10–12] by an intolerable amount. It is interesting to note, however, that if we remove (erroneously) the factor $i^{l-l'}$ from the potential $V^{sjt}_{ll'}(p,p')$ we found −7.1 MeV (five partial waves) which is fairly close to −7.0 MeV given in Refs. 10–12. The separable (UPA) expansion[15] of the Reid potential also gives −7.1 MeV after that modification.

The presence of that tedious factor $i^{l-l'}$ is just a matter of the convention used when defining the momentum states $|p(ls)j\rangle$. Due to our convention presented in Sec. II it must be present. Its presence or absence does neither effect the deuteron binding energy nor the two-body phase shifts and not even the three-body binding energy, when only partial waves Nos. 1–3 are taken into account, but it shows up in the coupling parameters ϵ and in the full three-body system.

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[1]K. Erkelenz, Phys. Rep. 13C, 191 (1974).
[2]R. Blankenbecler and R. Sugar, Phys. Rev. 142, 1051 (1966).
[3]V. A. Allessandrini and R. L. Omnes, Phys. Rev. B139, 167 (1965).
[4]W. Glöckle and R. Offermann, Nuovo Cimento Lett. 15, 134 (1976).
[5]L. D. Faddeev, Zh. Eksp. Teor. Fiz. 39, 1459 (1960) [Sov. Phys. JETP 12, 1014 (1961)].
[6]Y. E. Kim, J. Math. Phys. 10, 1491 (1969); Y. E. Kim and A. Tubis, Phys. Rev. C 1, 1627 (1970).
[7]A. D. Jackson, A. Lande, and P. U. Sauer, Nucl. Phys. A156, 1 (1970).
[8]B. R. Barrett, R. G. L. Hewitt, and R. J. McCarthy, Phys. Rev. C 3, 1137 (1971).

[9]R. Reid, Ann. Phys. (N.Y.) 50, 411 (1968).
[10]A. Laverne and C. Gignoux, Nucl. Phys. A203, 597 (1973).
[11]R. A. Brandenburg, Y. E. Kim, and A. Tubis, Phys. Rev. C 12, 1368 (1975); E. P. Harper, Y. E. Kim, and A. Tubis, *ibid.* 2, 877 (1970); 6, 126 (1972).
[12]I. R. Afnan and J. M. Read, Phys. Rev. C 8, 1294 (1973).
[13]M. R. Strayer and P. U. Sauer, Nucl. Phys. A231, 1 (1974).
[14]M. A. Hennell and L. M. Delves, Phys. Lett. 40B, 20 (1972).
[15]S. C. Bhatt, J.S. Levinger, and E. Harms, Phys. Lett. 40B, 23 (1972).
[16]L. M. Delves, in *Few Body Problems in Nuclear and Particle Physics*, proceedings of the Quebec Conference, 1974, edited by by R. J. Slobodrian, B. Cujec, and K. Ramavataran (Les Presses de l'Université Laval, Quebec, 1975), p. 446.
[17]The formulas have been checked by recalculating the Moshinsky transformation coefficients.
[18]R. A. Malfliet and J. A. Tjon, Ann. Phys. (N.Y.) 61, 425 (1970); Nucl. Phys. A127, 161 (1969); Phys. Lett. B29, 391 (1969).
[19]H. de Groot and J. S. Boersma, Nuovo Cimento Lett. 11, 238 (1974).
[20]L. Crepinsek, C. B. Lang, H. Oberhummer, W. Plessas, and H. F. K. Zingl, Acta Phys. Austriaca 42, 139 (1975).
[21]H. Zankel (private communication).
[22]T. F. Hammann and Q. Ho-Kim, Nuovo Cimento 64B, 356 (1969).
[23]T. R. Mongan, Phys. Rev. 178, 1597 (1969).