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Three-body approach to the single-scattering optical potential*

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We investigate the single-scattering optical potential in the multiple scattering approaches of Watson and of Kerman, McManus, and Thaler. Since the kinematics of single scattering is three body in nature, we build a three-body model of this term. This approach can include the proper kinematics for the struck nucleon, the identity of the target nucleons, and the binding interaction of the struck nucleon. Integral equations of the Faddeev type are derived for both the Watson and Kerman-McManus-Thaler single-scattering optical potentials. Unitarity relations are investigated and we observe that these relations can be expanded in order to identify the intermediate states responsible for the absorptive parts. The transition amplitudes to the inelastic states implicit in the model are extracted and evaluated. This permits one to understand the physical meaning of the imaginary part in precise terms. The same procedure is applied to the closure and impulse approximations for the single-scattering term and their implicit inelastic states and reaction amplitudes are identified. These approximations are evaluated by analyzing the inelastic data. We conclude that the impulse approximation to the Watson single-scattering term should provide the best two-body approximation to a single-scattering optical potential.

[NUCLEAR REACTIONS Study optical potential in Watson and KMT multiple-scattering theory, three-body model introduced, Faddeev equations derived, unitarity studied; applied to intermediate energy nucleon-nucleus scattering.]

I. INTRODUCTION

Most calculations of the optical potential for hadrons scattering from nuclei at intermediate energies are based on the multiple-scattering series of Watson[1] in the rearranged form of Kerman, McManus, and Thaler (KMT).[2] The lowest order term of this series, the single-scattering term, gives the optical potential arising from scattering with uncorrelated single nucleons. Single scattering can be expected to give the dominant contribtuion to the optical potential at energies above about 100 MeV for nucleon projectiles and from even lower energies for pions and kaons.[1]

This term yields an optical potential which is the folding of an effective hadron-nucleon interaction over the nuclear wave function. The exact effective interaction prescribed by the theory can not be calculated exactly, so approximations must be made. In practice, hadron-nucleon scattering amplitudes are usually used as effective interactions. These amplitudes are complex and therefore lead to optical potentials with absorptive (imaginary) parts. These absorptive parts play an essential role in determining the elastic scattering.

The physical meaning of the imaginary part constructed from folding a scattering amplitude is somewhat obscure. Crudely, one can say that any scattering leads to an absorption so the imaginary part corresponds to something like the total projectile-nucleon cross section times the nuclear density. Since different approximations for the single-scattering term lead to different imaginary parts for the optical potential, it would be useful to have a detailed relation between the effective interaction used and the reactions which are implicitly causing the absorption. It is the goal of this paper to provide such a relation via the study of unitarity.

A sharper understanding of the single-scattering optical potential is interesting for two reasons. First, microscopic calculations of nuclear reactions usually employ a distorted wave approximation. These calculations require optical wave functions in the neighborhood of the nuclear surface. The optical potentials giving these wave functions are obtained by fitting the elastic scattering data phenomenologically. Unfortunately, the optical potential is not uniquely determined by the asymptotic wave function (experimental elastic cross section). The resulting ambiguity leaves an uncertainty in the optical wave function inside the

nucleus. The theoretical structure of the potential needs to be better understood in order to remove the uncertainties from distorted wave calculations.

A second reason for a careful study of the single-scattering term is that one hopes to extract information about nuclear correlations from elastic scattering. The multiple-scattering series can be arranged in such a way[3] that the second order term depends on the correlations of pairs of nucleons in the nucleus, the third on correlations of triples, and so on. These correlations are of considerable interest in nuclear physics, and the small corrections they make to elastic scattering have been proposed as a way to measure them. Since these corrections are small compared to the first order term, the first order term must be known with high accuracy before correlations can be extracted from the data.

The optical potential in the single-scattering approximation is given by a sum of terms corresponding to the projectile scattering from one nucleon at a time. This is physically reasonable since most of the time the individual nucleons are moving in uncorrelated shell-model orbits. A dimensionless size parameter also suggests the importance of scattering from one nucleon at a time. The range of the strong part of the hadron-nucleon force, r_0, is about 1.5 fm for nucleon projectiles and less than 0.5 fm for pion and kaon projectiles. The average separation of nucleons in the nucleus, d, is about 1.8 fm. The ratio of the force range to the average spacing r_0/d, is therefore between 0.3 and 0.8 for the various projectiles. The nucleus appears dilute to the probe so it should most often see one nucleon at a time. This is consistent with the experimental observation that most of the reaction cross section at intermediate energies goes into nucelon knockout.[4]

The kinematics of the single-scattering approximation is effectively that of a three-body problem, the three particles being the projectile, the struck nucleon, and the rest of the nucleus or "core." We construct a three-body model of the single-scattering optical potential and write Faddeev equations for both its Watson and KMT forms. This permits us to include the correct collision kinematics and the effect of the nucleon-core binding potential in intermediate states.

The difficult part of constructing a three-body model of a projectile-nucleus optical potential is that for a target having A particles there are A distinct three-body problems involved. These must be superposed in a manner which does not do violence to the true many-particle structure of the amplitude. To be certain that our model yields a reasonable weaving together of three- and many-body aspects, and to obtain more physical insight into the reactive content of our model, we study the unitarity relation.

Standard manipulations of the Lippmann-Schwinger (LS) equation with an optical potential show that the imaginary part of the forward elastic amplitude is equal to the total elastic cross section plus a unitarity defect.[5] This defect is given by a distorted wave matrix element of the imaginary part of the optical potential. Since the optical potential in our model is obtained from an operator which satisfies a unitarity theorem of its own, an expanded unitarity relation can be obtained. Care must be used in constructing this relation since Lippmann-Schwinger expressions do not specify the asymptotic boundary conditions uniquely in a three-body problem and "hidden" singularities corresponding to rearrangement states can result. Here, the Faddeev structure of our representation of the optical potential allows us to use the standard methods of three-body unitarity and to demonstrate the absence of hidden states.

Our unitarity relation shows that the imaginary part of the forward elastic amplitude is the sum of the total elastic cross section and total reaction cross sections to pickup and knockout states.[6] It has a correct many-body structure and permits the identification of the pickup and knockout amplitudes implicit in the three-body construction of the optical model. A brief description of these results has been given in a recent letter.[7]

This expanded form of unitarity is most convenient for assessing the validity and applicability of approximations to the optical model. The standard impulse[2] and closure[5] approximations to the single-scattering optical potential can be obtained as approximations to our three-body model. We can therefore determine the reactions responsible for absorption in these approximations and even the reaction amplitudes which are implicit in the model. In both methods, the coordinates of only a single nucleon may be modified so knockout is the only possible inelastic intermediate state. No pick-up states arise in these approximations since neither retains the off-energy-shell pole of the scattering amplitude. The two approximations differ from each other and from our three-body model in the kinematics of the allowed inelastic states and in the reaction amplitudes to these states.

The structure of the paper is as follows. In Sec. II we review the derivation of the Watson and KMT versions of the multiple-scattering series for the optical potential, specifying the first order term carefully. We present the three-body model in Sec. III. The nature of the intermediate states involved is discussed, and the Faddeev equations for the optical potential operators derived. Techniques for deriving explicit unitarity relations are devel-

oped in Sec. IV and the absorptive reaction amplitudes in the three-body model are extracted and interpreted. In Sec. V the closure and impulse approximations are studied and their reactive content obtained. The resulting reaction amplitudes are considered for the two approximations made both for the Watson and KMT single scattering operators. The paper is summarized in Sec. VI and a discussion and evaluation of the different methods is given. The derivation of the three-body unitarity relations used in Secs. IV and V are given in the Appendix. This Appendix also contains discussions of the various limits taken in the paper to convert discontinuity to unitarity relations.

II. MULTIPLE-SCATTERING THEORY

We consider the scattering of a projectile of mass m_0 from a target consisting of A nucleons and described by the Hamiltonian H_T. The nucleon mass we write as m.

We choose to describe the scattering by Watson's multiple-scattering expansion of the optical potential.[1] We begin by reviewing the derivation of this expansion briefly. The methods of Watson and Kerman, McManus and Thaler[2] for obtaining the elastic scattering due to the first term of the Watson expansion are reviewed and the relationship between the two methods is explained. The consequences of the identity of the target nucleons is considered at the end of this section.

The transition operator for elastic and inelastic scattering to bound target states is given by the Lippmann-Schwinger (LS) equation

$$T_e(E) = V + VG(E)T_e(E)\,, \tag{1}$$

where

$$G(E) = (E + i\epsilon - K - H_T)^{-1}\,, \tag{2}$$

K is the kinetic energy operator for the projectile, and $V = \sum_{i=1}^{A} v_i$ is the sum of the interactions of the projectile with each target nucleon. The many-body scattering problem described by Eq. (1) can be formally reduced to a two-body elastic scattering problem by introducing the projection operators

$$\begin{aligned} P_e &= \int d^3k\, |\phi_e, \vec{k}\rangle\langle\vec{k}, \phi_e|\,, \\ Q_e &= 1 - P_e \end{aligned} \tag{3}$$

where ϕ_e is the target ground state and $\vec{k}$ is the projectile momentum. The subscript e is used to label the elastic channel quantities. Splitting the Green function in Eq. (1) into the components $G = P_eG + Q_eG$, the operator $T_e(E)$ can be separated into the pair of equations

$$T_e(E) = U_e(E) + U_e(E)\Gamma_e(E)T_e(E)\,, \tag{4}$$

$$U_e(E) = V + VQ_eG(E)U_e(E)\,, \tag{5}$$

where we have defined, for convenience,

$$\Gamma_e(E) = P_eG(E)\,. \tag{6}$$

The on-shell elastic scattering amplitude is $\langle\vec{k}_e'\phi_e|T_e(E)|\phi_e\vec{k}_e\rangle$ with $|\vec{k}_e'| = |\vec{k}_e|$ and $E = E_e + k_e^2/2\mu_0$. We use E_e to denote the internal energy of the target ground state and μ_0 is the projectile-target reduced mass. When target ground state matrix elements of Eq. (4) are taken, the standard two-body Lippmann-Schwinger equation for elastic scattering is obtained. The optical potential is given by the target ground state matrix elements of U_e.

All the complexity of the many-body problem still appears in Eq. (5). Separating U_e into components in which the projectile interacts last with a particular target nucleon, we have

$$\begin{aligned} U_e &= \sum_{i=1}^{A} U_e^{(i)}\,, \\ U_e^{(i)} &= v_i(1 + Q_eGU_e)\,. \end{aligned} \tag{7}$$

From the perturbation expansion of Eq. (7) we isolate the infinite series involving v_i only, by defining the Watson single-scattering operator

$$\begin{aligned} \tau_e^{(i)}(E) &= v_i + v_iQ_eGv_i + v_iQ_eGv_iQ_eGv_i + \cdots \\ &= v_i + v_iQ_eG(E)\tau_e^{(i)}(E) \\ &= v_i + \tau_e^{(i)}(E)Q_eG(E)v_i\,. \end{aligned} \tag{8}$$

This operator describes the scattering of the projectile from one of the bound nucleons. In between interactions with the projectile, all the target nucleons—including the struck one—propagate by the full target Hamiltonian. As a result $\tau_e^{(i)}$ is a many-body operator. Upon eliminating v_i in Eq. (7) in favor of $\tau_e^{(i)}$ by multiplying on the left with $(1 + \tau_e^{(i)} \times Q_eG)$, we obtain

$$U_e^{(i)} = \tau_e^{(i)} + \tau_e^{(i)}Q_eG\sum_{j\neq i}^{A} U_e^{(j)}\,. \tag{9}$$

Iterating Eq. (9) and summing over i yields the Watson multiple-scattering series

$$U_e = \sum_{i=1}^{A}\tau_e^{(i)} + \sum_{i\neq j}^{A}\tau_e^{(i)}Q_eG\tau_e^{(j)} + \sum_{i\neq j\neq k}^{A}\cdots\,. \tag{10}$$

The first term of this series is the single-scattering term to which the remainder of this paper will be devoted. Higher order terms describe processes in which the projectile scatters from at least two bound nucleons and depend on two-nucleon correlations in the target.[3]

Taking only the first term of Eq. (10), the corresponding elastic scattering operator is given by

$$T_e = \sum_{i=1}^{A} \tau_e^{(i)}(1+\Gamma_e T_e) = \sum_{i=1}^{A} T_e^{(i)} , \tag{11}$$

$$T_e^{(i)} = \tau_e^{(i)} + \tau_e^{(i)}\Gamma_e \sum_{j=1}^{A} T_e^{(j)} \quad . \tag{12}$$

In many circumstances the presence of the projection operator Q_e in Eq. (8) renders the operator $\tau_e^{(i)}$ difficult to work with. Following a technique employed in KMT calculations[2] we introduce an auxiliary operator defined by

$$\hat{\tau}_e^{(i)}(E) = v_i + v_i G(E)\hat{\tau}_e^{(i)}(E) . \tag{13}$$

It is straightforward to relate the operators $\tau_e^{(i)}$ and $\hat{\tau}_e^{(i)}$ by

$$\begin{aligned}\hat{\tau}_e^{(i)} &= \tau_e^{(i)} + \tau_e^{(i)}\Gamma_e\hat{\tau}_e^{(i)} \\ &= \tau_e^{(i)} + \hat{\tau}_e^{(i)}\Gamma_e\tau_e^{(i)} .\end{aligned} \tag{14}$$

Upon eliminating $\tau_e^{(i)}$ in favor of $\hat{\tau}_e^{(i)}$ by multiplying from the left with $(1+\hat{\tau}_e^{(i)}\Gamma_e)$, Eq. (12) becomes

$$T_e^{(i)} = \hat{\tau}_e^{(i)} + \hat{\tau}_e^{(i)}\Gamma_e \sum_{j\neq i}^{A} T_e^{(j)} . \tag{15}$$

Comparing Eqs. (14) and (11), we see that $\hat{\tau}_e^{(i)}$ is the elastic scattering operator that would result if the full optical potential were taken as $\tau_e^{(i)}$. In other words, comparing Eqs. (15) and (12), we see that the introduction of $\hat{\tau}_e^{(i)}$ to replace $\tau_e^{(i)}$ has selectively solved part of the elastic scattering equation given in Eq. (11). In KMT calculations the Green function G is approximated by a propagator for the relative motion of the projectile-nucleon system, and $\hat{\tau}_e^{(i)}$ becomes a free projectile-nucleon t matrix. The model used in this present work can also be formulated as a specific approximation to G as will be seen in the following section.

Once $\hat{\tau}_e^{(i)}$ has been constructed, the elastic scattering calculation may proceed in two ways. The Watson single-scattering operator may be constructed by solving Eq. (14) in the form

$$\begin{aligned}\tau_e^{(i)} &= \hat{\tau}_e^{(i)} - \hat{\tau}_e^{(i)}\Gamma_e\tau_e^{(i)} \\ &= \hat{\tau}_e^{(i)} - \tau_e^{(i)}\Gamma_e\hat{\tau}_e^{(i)} .\end{aligned} \tag{16}$$

The amplitude T_e is then obtained from the two-body Lippmann-Schwinger equation (11). Alternatively, Eq. (15) may be solved directly. When the identity of the target nucleons is introduced this equation may be reduced to Lippmann-Schwinger form. Using a properly antisymmetrized target ground state we may write

$$\left\langle \vec{k}, \phi_e \middle| \sum_{i=1}^{A} \hat{\tau}_e^{(i)} \middle| \phi_e, \vec{k} \right\rangle = A\langle \vec{k}', \phi_e | \hat{\tau}_e | \phi_e, \vec{k} \rangle , \tag{17}$$

where any choice may be made for the struck target nucleon described by $\hat{\tau}_e$ and the label is omitted. Since this property also holds for the target matrix elements of τ_e, we may rewrite Eqs. (11) and (15) as

$$T_e = A\tau_e(1+\Gamma_e T_e) , \tag{18}$$

$$T_e = A\hat{\tau}_e + (A-1)\hat{\tau}_e\Gamma_e T_e . \tag{19}$$

Defining an auxiliary elastic operator $\hat{T}_e$ by

$$T_e = \frac{A}{A-1}\hat{T}_e , \tag{20}$$

Eq. (19) reduces to the Lippmann-Schwinger form

$$\hat{T}_e = (A-1)\hat{\tau}_e(1+\Gamma_e\hat{T}_e) . \tag{21}$$

The consequences of identical particle symmetry for the nonelastic intermediate states involved in the optical potential will be discussed in Sec. V.

It should be noted that up to this stage no approximations have been made except for the truncation of the multiple-scattering series (10) at the first term. Therefore the resulting elastic scattering amplitude calculated through each of the routes discussed above will be the same. We will refer to Eq. (18) as the Watson method, and to Eqs. (20) and (21) as the KMT method for the calculation of elastic scattering due to the single-scattering optical potential.

III. THREE-BODY MODEL FOR THE SINGLE-SCATTERING TERM

In this section we consider in detail the operator given in Eq. (13). This operator describes the scattering of the projectile and a given bound target nucleon. By virtue of the many-body Hamiltonian H_T contained in the propagator G this scattering operator is necessarily of many-body character. Equation (13) can be written

$$\hat{\tau}_e^{(i)}(E) = v_i + v_i \frac{1}{E+i\epsilon-K-K_i-K_{R_i}-H_{R_i}-w_i}\hat{\tau}_e^{(i)}(E) , \tag{22}$$

where we have used the decomposition $H_T = K_i + K_{R_i} + H_{R_i} + w_i$. The terms are, respectively, the kinetic energy of the ith nucleon, the center of mass energy of the residual nucleus R_i, its internal Hamiltonian, and the interaction of the ith nucleon with the remaining nucleons given by $w_i = \sum_{j=i}^{A} v_{ij}$. This latter term can give rise to many-body intermediate states.

The nonelastic intermediate states in Eq. (22) provide the absorption for the optical potential. For intermediate energy nucleon projectiles we can infer that the nucleon knockout states play a dominant role.[4] Consequently we will ignore bound excited target states and also excited states of the residual nucleus. We concentrate on the problem

of treating the continuum states of the projectile and struck nucelon including the recoil of the residual nucleus and the distortion effect of the binding potential w_i. Following the theories of the nuclear bound state we introduce a single particle potential by writing $w_i = u_i + (w_i - u_i)$. We then expand in powers of the residual interaction $(w_i - u_i)$ and ignore terms of first order and above. The model target Hamiltonian is therefore $H_T^{(i)} = K_i + K_{R_i} + H_{R_i} + u_i$. At this level of approximation, Eq. (22) becomes a three-body scattering problem. We define the free Hamiltonian of the three-body model as

$$H_0^{(i)} = K + K_i + K_{R_i} + H_{R_i} . \tag{23}$$

Since the internal Hamiltonian of the residual nucleus, H_{R_i}, does not involve the coordinates of either the projectile or the struck nucleon, it commutes with all the other operators. When the target wave function is expressed in a parentage expansion (in terms of the eigenstates of H_{R_i}) the operator H_{R_i} will produce a simple shift of the effective energy. Within this model, Eq. (22) becomes

$$\hat{\tau}_e^{(i)} = v_i + v_i G_e^{(i)} \hat{\tau}_e^{(i)} , \tag{24}$$

where

$$G_e^{(i)} = (E + i\epsilon - H_0^{(i)} - u_i)^{-1} . \tag{25}$$

The corresponding model for the Watson single-scattering operator $\tau_e^{(i)}$ can be obtained by substituting Eq. (24) into Eq. (16) to yield

$$\tau_e^{(i)} = v_i + v_i \left[G_e^{(i)} - \Gamma_e(E) \right] \tau_e^{(i)} . \tag{26}$$

To ensure that Eq. (26) has no elastic channel intermediate states (and so produce a valid optical potential), we must ensure that the target ground state contained in the propagator Γ_e is the same as that generated from the model target Hamiltonian $H_T^{(i)}$. How this is done can be seen from making a parentage expansion of the target ground state. One expands in eigenstates of the core Hamiltonian, H_{R_i}. Explicitly, we write

$$|\phi_e\rangle = \sum_\lambda |\phi_i^\lambda\rangle \times |\phi_{R_i}^\lambda\rangle , \tag{27}$$

where $|\phi_{R_i}^\lambda\rangle$ is an eigenfunction of H_{R_i} of energy E_λ. The coefficient $|\phi_i^\lambda\rangle$ is given by the overlap integral $\langle \phi_{R_i}^\lambda | \phi_e \rangle$.

If the single particle potential is chosen so that the ϕ_i^λ are eigenstates of the single particle Hamiltonian $K_i + u_i$ with energy $E_e - E_\lambda$ then the cancellation at the pole is exact.[8] Thus we will write Eq. (26) as

$$\tau_e^{(i)} = v_i + v_i \frac{Q_e}{E + i\epsilon - H_0^{(i)} - u_i} \tau_e^{(i)} , \tag{28}$$

where Q_e has the interpretation, within this model, of projecting onto those three-body states in which nucleon i is not bound to the residual nucleus to form the target ground state. A more definitive description of these states will be provided by the study of unitarity relations presented in Sec. IV.

The Green function in the conventional Lippmann-Schwinger equation (24) does not give the most convenient representation of the intermediate states involved. An exact solution of the three-body problem represented by Eq. (24) will involve intermediate state contributions from the break-up eigenfunctions of $H_0^{(i)}$ and the bound eigenfunctions of $H_0^{(i)}$ and the bound eigenfunctions of $H_0^{(i)} + u_i$ and $H_0^{(i)} + v_i$ describing the elastic and rearrangement channels. A theory of three-body scattering in which the boundary conditions for all physical processes are properly specified is provided by the Faddeev[9] integral equations. The kernel of these equations does not give rise to disconnected diagrams and allows the use of standard matrix methods for the solution. We therefore employ the Faddeev technique to restructure Eq. (24) not only as a means of clarifying the nature of the intermediate states, but also as a starting point for a practical calculation.

To introduce this standard three-body description, we eliminate the potentials u_i and v_i in favor of the two-body t matrices defined by

$$t_e^{(i)}(E) = u_i + u_i G_0^{(i)}(E) t_e^{(i)}(E) , \tag{29}$$

$$t_p^{(i)}(E) = v_i + v_i G_0^{(i)}(E) t_p^{(i)}(E) , \tag{30}$$

where

$$G_0^{(i)}(E) = (E + i\epsilon - H_0^{(i)})^{-1} . \tag{31}$$

The subscript p labels the pickup channel. We multiply Eq. (24) on the left by $(1 + t_p^{(i)} G_0^{(i)})$ to obtain

$$\hat{\tau}_e^{(i)} = t_p^{(i)} + t_p^{(i)} (G_e^{(i)} - G_0^{(i)}) \hat{\tau}_e^{(i)} . \tag{32}$$

Using the relation

$$G_e^{(i)} = G_0^{(i)} + G_0^{(i)} t_e^{(i)} G_0^{(i)} , \tag{33}$$

Eq. (32) becomes

$$\hat{\tau}_e^{(i)} = t_p^{(i)} + t_p^{(i)} G_0^{(i)} t_e^{(i)} G_0^{(i)} \hat{\tau}_e^{(i)} . \tag{34}$$

It is useful to recast Eq. (34) into the coupled channel form

$$\hat{\tau}_e^{(i)} = t_p^{(i)} G_0^{(i)} \hat{\tau}_p^{(i)} , \tag{35}$$

$$\hat{\tau}_p^{(i)} = G_0^{(i)-1} + t_e^{(i)} G_0^{(i)} \hat{\tau}_e^{(i)} , \tag{36}$$

by introducing the operator $\hat{\tau}_p$. Equations (35) and (36) are three-body integral equations of the Alt, Grassberger, and Sandhas (AGS) form.[10] In the present case only two pair-wise interactions are present. The effects of the single particle binding

potential u_i enter through $t_e^{(i)}$. If we take $u_i = t_e^{(i)} = 0$, then $\hat{\tau}_e^{(i)} \to t_p^{(i)}$, and the folding model of the KMT method is obtained apart from the three-body kinematics implicit in Eq. (30). We will discuss this special case of our model in more detail in the next section.

The physical meaning of the operator $\hat{\tau}_p^{(i)}$ can be clarified in the following way. Equation (24) can be written

$$\hat{\tau}_e^{(i)} = v_i + v_i(E + i\epsilon - H_0^{(i)} - u_i - v_i)^{-1} v_i . \tag{37}$$

Substituting Eq. (37) into Eq. (36) gives

$$\hat{\tau}_p^{(i)} = G_0^{(i)-1} + t_e^{(i)} G_0^{(i)} (v_i + v_i g^{(i)} v_i) , \tag{38}$$

where

$$g^{(i)} = (E + i\epsilon - H_0^{(i)} - u_i - v_i)^{-1} . \tag{39}$$

Applying the identity $t_e^{(i)} G_0^{(i)} = u_i G_e^{(i)}$, and the resolvent relation $G_e^{(i)} + G_e^{(i)} v_i g^{(i)} = g^{(i)}$ to Eq. (38) we have

$$\begin{aligned} \hat{\tau}_p^{(i)} &= G_0^{(i)-1} + u_i g^{(i)} v_i \\ &= u_i + u_i g^{(i)} v_i + (E + i\epsilon - H_0^{(i)} - u_i) . \end{aligned} \tag{40}$$

When $\hat{\tau}_p^{(i)}$ operates to the right upon the on-shell elastic state $|\phi_e, \vec{k}_e\rangle$, the last term of Eq. (40) does not contribute. Thus $\hat{\tau}_p^{(i)}$ can be identified as an off-shell transformation of the pickup transition operator for the three-body system.[10] We assume for convenience that the projectile-nucleon system has one bound state, which we refer to generically as a "deuteron."

The corresponding integral equations for the calculation of $\tau_e^{(i)}$ are obtained by multiplying Eqs. (35) and (36) on the right by $(1 - \Gamma_e \tau_e^{(i)})$ and using Eq. (16) to obtain

$$\tau_e^{(i)} = t_p^{(i)} G_0^{(i)} \tau_p^{(i)} , \tag{41}$$

$$\tau_p^{(i)} = G_0^{(i)-1} + \left(t_e^{(i)} G_0^{(i)} - G_0^{(i)-1} \Gamma_e \right) \tau_e^{(i)} , \tag{42}$$

where we have defined

$$\tau_p^{(i)} = \hat{\tau}_p^{(i)} (1 - \Gamma_e \tau_e^{(i)}) . \tag{43}$$

The term of Eq. (42) in parentheses has the property of prohibiting on-shell propagation of the elastic channel in intermediate states, and thus takes account of the projection operator Q_e in the defining equation (28). To clarify this we note that in the neighborhood of the elastic pole, $t_e^{(i)}$ has the form $u_i \Gamma_e u_i$, so $G_0^{(i)} t_e^{(i)} G_0^{(i)}$ has the form $G_0^{(i)} u_i \Gamma_e \times u_i G_0^{(i)}$. Since $G_0^{(i)} u_i$ is unity when applied on shell to the target ground state, $G_0^{(i)} t_e^{(i)} G_0^{(i)}$ agrees with Γ_e at the pole. We emphasize that the manner in which the elastic intermediate states are excluded from Eq. (42) is not an approximate interpretation of the projector Q_e, but is an exact representation of its role in the three-body model we have set up.

Equations (41) and (42) are completely equivalent to, but more practical than, Eq. (28). A direct way to establish this equivalence is to derive Eqs. (41) and (42) from Eqs. (26) or (28). To do this, v_i must be eliminated in favor of $t_p^{(i)}$ by multiplication from the left with $(1 + t_p^{(i)} G_0^{(i)})$. Use of Eq. (33) and separation into coupled channel form recovers Eqs. (41) and (43) immediately. The three-body AGS integral equations for the single-scattering optical potential operators constitute one of the principal results of this work. The numerical methods developed in three-body theory may be applied to these equations to include three-body kinematics and binding effects into the optical potential calculations.

A somewhat similar application of three-body equations to the multiple-scattering series is found in the work of Revai.[11]

IV. UNITARITY AND THE ABSORPTIVE REACTION AMPLITUDES

The non-Hermitian part of the optical potential arises from the nonelastic channels implicit in the model. For any microscopic model of the optical potential the reaction amplitudes for these nonelastic channels are implicitly contained in the singularities of the elastic operator and may be revealed through an examination of the unitarity relation. In this section we analyze this relation in detail for our three-body model of the single-scattering optical potential. We develop explicit operator unitarity equations which permit us to identify the types of reaction responsible for the absorption and their amplitudes. In order to do this, the use of a connected kernel (Faddeev) formulation of the three-body model is essential.

The unitarity relations are obtained by investigating the discontinuity of the scattering operators across the right-hand (unitarity) cut in the energy plane. For operators related by the equation

$$A(E) = B(E) + B(E) C(E) A(E) , \tag{44}$$

the discontinuity relation is

$$\Delta A = A^\dagger \Delta C A + (A^\dagger C^\dagger + 1) \Delta B (1 + CA) , \tag{45}$$

where

$$\Delta A(E) = A(E + i\epsilon)^\dagger - A(E + i\epsilon) . \tag{46}$$

We refer to Eq.(45) as the *operator unitarity theorem*. The discontinuity across the cut is defined by

$$2\pi i \, \mathrm{disc} A(E) = \lim_{\epsilon \to 0} \Delta A(E) . \tag{47}$$

This relation must be used with extreme care. In general, the $\epsilon \to 0$ limit may not be taken inside an expression of the form $A^\dagger \Delta C A$ if A and $A^\dagger$ have singularities. If the matrix elements of ΔC are

not sufficiently connected, the δ functions may result in more than one pole of A occurring at the same energy. Since the poles of $A^\dagger$ and A will be just below and just above the real axis, respectively, taking the limit $\epsilon \to 0$ can pinch an integration contour and lead to the production of an unsuspected singularity. An example of this and a discussion of all relevant cases are given in the Appendix.

If we apply the operator unitarity theorem (45) to the elastic scattering LS equation (4) we obtain

$$\Delta T_e = T_e^\dagger \Delta\Gamma_e T_e + (T_e^\dagger \Gamma_e^\dagger + 1)\Delta U_e (1 + \Gamma_e T_e) . \tag{48}$$

In the first term, the presence of the projection operator P_e permits us to take the limit $\epsilon \to 0$ inside (see the discussion of a similar term in the Appendix) giving

$$\text{disc}\, T_e = T_e^\dagger \Lambda_e T_e + \frac{1}{2\pi i} \lim_{\epsilon \to 0} (T_e^\dagger \Gamma_e^\dagger + 1)\Delta U_e (1 + \Gamma_e T_e) , \tag{49}$$

where

$$\Lambda_e(E) = \text{disc}\Gamma_e \tag{50}$$

$$= \int d^3k \, |\vec{k}, \phi_e\rangle \delta(E - E_e - k^2/2\mu_0) \langle \phi_e, \vec{k}| . \tag{51}$$

When forward on-shell matrix elements of Eq. (49) are taken, the first term can be identified as the total elastic cross section. The optical theorem is therefore obtained in the form $\sigma_{\text{tot}} = \sigma_{\text{el}} + \sigma_{\text{abs}}$.

If we were dealing with a complete and exact microscopic theory of the system we would expect to obtain a unitarity relation of the form

$$\text{disc}\, T_e = T_e^\dagger \Lambda_e T_e + \sum_n T_n^\dagger \Lambda_n T_n \tag{52}$$

in which n labels all possible nonelastic channels, and T_n and Λ_n are the associated transition amplitude and on-shell projector, respectively. Thus, when U_e is the exact optical potential, the absorptive term is expressible in the form

$$\frac{1}{2\pi i} \lim_{\epsilon \to 0} (T_e^\dagger \Gamma_e^\dagger + 1)\Delta U_e (1 + \Gamma_e T_e) = \sum_n T_n^\dagger \Lambda_n T_n . \tag{53}$$

The corresponding relation for an approximate optical potential will involve only selected reaction channels described by approximate reaction amplitudes.

In the case of the model of the single-scattering optical potential given in the previous section, U_e is given by

$$U_e = \sum_{i=1}^{A} \tau_e^{(i)} . \tag{54}$$

The operator $\tau_e^{(i)}$ is essentially a three-body operator so $\Delta\tau_e^{(i)}$ may be obtained by using the unitarity relations for three-body scattering. As these are normally expressed in terms of operators satisfying equations without projectors in the kernel it will be convenient to express the discontinuities in terms of the operators $\hat{\tau}_e^{(i)}$. Since τ_e and $\hat{\tau}_e$ are related by Eq. (16), which has the structure of Eq. (44), the operator unitarity theorem (45) may be applied to recover $\Delta\tau_e^{(i)}$ in the form

$$\Delta\tau_e^{(i)} = -\tau_e^{(i)\dagger} \Delta\Gamma_e \tau_e^{(i)} + (1 - \tau_e^{(i)\dagger}\Gamma_e^\dagger)\Delta\hat{\tau}_e^{(i)}(1 - \Gamma_e \tau_e^{(i)}) . \tag{55}$$

Therefore, the reactive content of the optical potential in our model may be explicated by expressing ΔU_e in Eq. (48) in terms of $\Delta\tau_e^{(i)}$ using Eq. (54), expressing $\Delta\tau_e^{(i)}$ in terms of the three-body operator $\Delta\hat{\tau}_e^{(i)}$ via Eq. (55), utilizing three-body unitarity on the operator $\hat{\tau}_e^{(i)}$, and finally taking the limit $\epsilon \to 0$. The expression implied by the substitutions takes the form

$$\begin{aligned} \Delta T_e = {} & T_e^\dagger \Delta\Gamma_e T_e \\ & + \sum_i (T_e^\dagger \Gamma_e^\dagger + 1)(1 - \tau_e^{(i)\dagger}\Gamma_e^\dagger)\Delta\hat{\tau}_e^{(i)}(1 - \Gamma_e \tau_e^{(i)}) \\ & \quad \times (1 + \Gamma_e T_e) \\ & - \sum_i (T_e^\dagger \Gamma_e^\dagger + 1)\tau_e^{(i)\dagger} \Delta\Gamma_e \tau_e^{(i)} (1 + \Gamma_e T_e) . \end{aligned} \tag{56}$$

When target ground state elements of this equation are taken, the operator $\Delta\hat{\tau}_e^{(i)}$ and all outside transition operators are averaged over the target ground state. Since each of these operators has at least one interaction of the projectile with a target nucleon, each matrix element is fully connected. The singularities of the outside operators are therefore never allowed to match and cannot produce a singularity through a pinch (see the Appendix for more details). Therefore, all possible on-shell intermediate state processes are accounted for by evaluating $\Delta\Gamma_e$ and $\Delta\hat{\tau}_e^{(i)}$ in the limit $\epsilon \to 0$.

The limit of $\Delta\Gamma_e$ is given in Eq. (51). To determine the limit of $\Delta\hat{\tau}_e^{(i)}$, we first apply the operator unitarity theorem (45) to Eq. (24). The result is

$$\begin{aligned} \Delta\hat{\tau}_e^{(i)} &= \hat{\tau}_e^{(i)\dagger} \Delta G_e^{(i)} \hat{\tau}_e^{(i)} \\ &= \hat{\tau}_p^{(i)\dagger} G_0^{(i)\dagger} t_p^{(i)\dagger} \Delta G_e^{(i)} t_p^{(i)} G_0^{(i)} \hat{\tau}_p^{(i)} , \end{aligned} \tag{57}$$

where the second equality follows from Eq. (35). The intermediate states in Eq. (57) receive contributions from the elastic channel and the nucleon knockout channel due to the poles of $G_e^{(i)}$. However, when we now apply the limit $\epsilon \to 0$ carefully, the disconnected (or δ function) part of $\Delta G_e^{(i)}$ allows a contribution from the nucleon pickup channel due to the bound state pole present in $t_p^{(i)\dagger}$ and $t_p^{(i)}$. In the Appendix we give the details of a derivation of

this typical three-body unitarity relation for $\hat{\tau}_e^{(i)}$ from this point of view. The result is[12]

$$\text{disc}\,\hat{\tau}_e^{(i)} = \hat{\tau}_e^{(i)\dagger}\Lambda_e\hat{\tau}_e^{(i)} + \hat{\tau}_p^{(i)\dagger}\Lambda_p^{(i)}\hat{\tau}_p^{(i)} + \hat{\tau}_e^{(i)\dagger}\Omega_e^{(i)\dagger}\Lambda_0^{(i)}\Omega_e^{(i)}\hat{\tau}_e^{(i)}\,, \tag{58}$$

where the wave operator $\Omega_e^{(i)}$ and the projection operators $\Lambda_p^{(i)}$ and $\Lambda_0^{(i)}$ are given by

$$\Omega_e^{(i)} = 1 + t_e^{(i)}G_0^{(i)}\,, \tag{59}$$

$$\Lambda_p^{(i)} = \int d^3k_p\,|\phi_p^{(i)},\vec{k}_p\rangle \times\delta\left(E - \frac{k_p^2}{2\mu_p} - E_p^{(i)}\right)\langle\vec{k}_p,\phi_p^{(i)}|\,, \tag{60}$$

$$\Lambda_0^{(i)} = \int d^3k_0 d^3p_0\,|\phi_R^{(i)},\vec{p}_0,\vec{k}_0\rangle \times\delta\left(E - \frac{k_0^2}{2\mu_0} - \frac{p_0^2}{2\mu} - E_R^{(i)}\right)\langle\vec{p}_0,\vec{k}_0,\phi_R^{(i)}|\,. \tag{61}$$

The operator $\Lambda_p^{(i)}$ projects onto the on-shell states of the channel corresponding to pickup of the ith target nucleon. A state in this channel is specified by the product of the internal wave functions of the deuteron and residual nucleus denoted by $\phi_p^{(i)}$, and the relative momentum of the deuteron and the residual nucleus, $\vec{k}_p$. The operator $\Lambda_0^{(i)}$ projects onto the on-shell states corresponding to knockout of the ith nucleon. This state is specified by the state of the residual nucleus, $\phi_R^{(i)}$, and two momenta $\vec{k}_0$ and $\vec{p}_0$ specifying a three-body plane-wave state. The reduced masses appearing in Eqs. (60) and (61) are as follows: μ_p, the reduced mass for the relative motion of the projectile-nucleon bound state and the core ($\mu_p^{-1} = m_d^{-1} + m_R^{-1}$); μ_0, the reduced mass for the relative motion of the projectile and the target ($\mu_0^{-1} = m_0^{-1} + m_T^{-1}$); and μ, the reduced mass for the relative motion of the struck nucleon and the core ($\mu^{-1} = m^{-1} + m_R^{-1}$).

The explicit unitarity relation is obtained by taking the limit $\epsilon \to 0$ in Eq. (56) and inserting Eq. (58) to obtain

$$\begin{aligned}\text{disc}\,T_e = {} & T_e^\dagger\Lambda_e T_e + \sum_i (T_e^\dagger\Gamma_e^\dagger + 1)(1 - \tau_e^{(i)\dagger}\Gamma_e^\dagger) \times\hat{\tau}_p^{(i)\dagger}\Lambda_p^{(i)}\hat{\tau}_p^{(i)}(1 - \Gamma_e\tau_e^{(i)})(1 + \Gamma_e T_e) \\ & + \sum_i (T_e^\dagger\Gamma_e^\dagger + 1)(1 - \tau_e^{(i)\dagger}\Gamma_e^\dagger)\hat{\tau}_e^{(i)\dagger}\Omega_e^{(i)\dagger}\Lambda_0^{(i)} \times\Omega_e^{(i)}\hat{\tau}_e^{(i)}(1 - \Gamma_e\tau_e^{(i)})(1 + \Gamma_e T_e)\,.\end{aligned} \tag{62}$$

If we introduce the operators

$$T_p^{(i)} = \tau_p^{(i)}(1 + \Gamma_e T_e)\,, \tag{63}$$

$$T_0^{(i)} = \Omega_e^{(i)}\tau_e^{(i)}(1 + \Gamma_e T_e) \tag{64}$$

then Eq. (62) becomes

$$\text{disc}\,T_e = T_e^\dagger\Lambda_e T_e + \sum_i T_p^{(i)\dagger}\Lambda_p^{(i)}T_p^{(i)} + \sum_i T_0^{(i)\dagger}\Lambda_0^{(i)}T_0^{(i)}\,. \tag{65}$$

Equation (43) has been used to replace $\hat{\tau}_p^{(i)}$ by $\tau_p^{(i)}$ and Eq. (16) to replace $\hat{\tau}_e^{(i)}$ by $\tau_e^{(i)}$. This unitarity relation has the desired structure [of Eq. (52)] so the operators $T_p^{(i)}$ and $T_0^{(i)}$ may be identified as amplitudes for the pickup and knockout of the ith particle, respectively.

We note that the limit $\epsilon \to 0$ cannot simply be taken inside the operator string if a three-body treatment of the optical potential is not employed. Thus, if the operator unitarity theorem is applied directly to the LS Eq. (1) one obtains

$$\begin{aligned}\Delta T_e^\dagger &= T_e^\dagger\Delta G_e T_e \\ &= T_e^\dagger P_e\Delta G_e T_e + T_e^\dagger Q_e\Delta G_e T_e\,.\end{aligned} \tag{66}$$

In the first term one may take the limit inside due to the connectivity of the operators as discussed above and in the Appendix. In the second term, however, the limit may *not* be taken inside since Q_e does not produce a connected operator. The discontinuity of G_e only begins at the elastic threshold, while it is clear that reactive channels may open at energies below this. These reactive contributions describe rearrangement channels and will arise due to the singularities of T_e. Our present three-body model can demonstrate this. Substituting Eq. (57) into Eq. (56), and using Eqs. (16) and (12), we have

$$\Delta T_e = T_e^\dagger\Delta\Gamma_e T_e + \sum_i T_e^{(i)\dagger}(\Delta G_e^{(i)} - \Delta\Gamma_e)T_e^{(i)}\,. \tag{67}$$

This operator relation has a structure which is as close as possible (within the confines of our three-body model) to that of Eq. (66) satisfied by the exact T_e. From the foregoing discussion it is clear that the components $T_e^{(i)}$ will introduce the same pickup channel singularities into Eq. (67) that $\hat{\tau}_e^{(i)}$ introduced into Eq. (57). In the case of the exact operator relation of Eq. (66), these considerations associated with the limit $\epsilon \to 0$ should be borne in mind when discussing the reactive content of the second term.

We now turn to the interpretation of our main result, Eq. (65). As expected from the three-body structure of our single-scattering model, the reactions possible in the model are pickup and knockout of a single nucleon. We note that the unitarity relation Eq. (65) has a proper many-body struc-

ture, i.e., distinct many-body channels enter independently in the sum. This therefore demonstrates that the proposed model weaves together the three-body structure of the assumed reaction mechanism with the many-body nature of the true problem in a consistent manner.

We can interpret from the structure of the operator strings what effective reaction mechanisms are included in the model description of the reactions which are occurring in the intermediate states. Both the pickup and knockout operators [given by Eq. (63) and (64), respectively] begin on the right with the operator $1+\Gamma_e T_e$. When applied to the initial plane wave state, $|\phi_e \vec{k}_e\rangle$, this operator produces the state

$$|\phi_e \chi^{(+)}_{k_e}\rangle \equiv (1+\Gamma_e T_e)|\phi_e \vec{k}_e\rangle \tag{68}$$

which has the structure of the target in its ground state times a distorted wave in the relative coordinate. Using Eq. (4) it can be easily shown that $\chi^{(+)}_{k_e}$ satisfies the two-body LS equation

$$|\chi^{(+)}_{k_e}\rangle = |\vec{k}_e\rangle + \frac{1}{E+i\epsilon-E_e-K}\langle\phi_e|U_e|\phi_e\rangle|\chi^{(+)}_{k_e}\rangle . \tag{69}$$

The operators $T_p^{(i)}$ and $T_0^{(i)}$ therefore can be seen to have a quasidistorted wave structure. The operators $\tau_p^{(i)}$ and $\Omega_p^{(i)}\tau_e^{(i)}$ are the three-body transition operators for pickup and knockout of the ith particle given the constraint that the target is never in its ground state. These reactions therefore take place as follows: the projectile comes in, distorting in the optical potential field as it goes. It then interacts with the ith particle extracting it from the nucleus. These particles then go out, the projectile continuing to interact with particle i as often as it wants, possibly binding it to itself. Particle i continues to interact with the binding potential of the core, but the projectile and the core no longer interact.

These amplitudes therefore do not have a full distorted wave structure. The final state absorption is restricted to that which arises from interactions with the extracted nucleon. This makes sense as, for the outgoing projectile to interact absorptively with the core, it would have to extract a second nucleon from the core to interact with. This leads to a four-body intermediate state. Models including some outgoing distortion in a three-body framework (for example, by the introduction of a third potential in the projectile-ith nucleon-core three-body problem) can be constructed. This will be discussed in a subsequent paper.

V. COMPARISON WITH CLOSURE AND ON-SHELL APPROXIMATIONS

In actual calculations of single-scattering optical potentials the three-body states are usually suppressed so that a one- or two-body treatment is obtained. In this section we consider the reactive content of two standard approximations, the closure and the on-shell impulse approximations.

We begin this section by converting our equations to the "reference particle" notation. Since the target nucleons are identical, the matrix elements of the transition operators involving any one of the A particles also carry all the information as to what are the matrix elements for the other A-1 particles. This permits us to replace the sums over $i=1,\cdots A$ by factors of A.

Consider, as an example, the pickup operator $T_p^{(i)}$. Its matrix element between on-shell states is

$$M_i = \langle \vec{k}_p \phi_p^{(i)} | T_p^{(i)} | \phi_e \vec{k}_e\rangle . \tag{70}$$

The target ground state is completely antisymmetric with respect to exchange of any pair. Furthermore, all the information on the i dependence of $\phi_p^{(i)}$ and $T_p^{(i)}$ is contained in the arguments of its matrix elements, i.e.,

$$\begin{aligned}\langle \vec{k}_p \phi_p^{(i)} | \vec{q}_0, \vec{q}_1, \ldots, \vec{q}_A\rangle &= \phi_d^*\left(\frac{m\vec{q}_0 - m_0\vec{q}_i}{m+m_0}\right)\phi_R^*(\vec{q}_1,\vec{q}_2,\ldots,\vec{q}_{i-1},\vec{q}_{i+1},\ldots,\vec{q}_A) \\ &\times \delta\left[\vec{k}_p - \left(\vec{q}_0 + \vec{q}_i - \sum_{j\neq i}\vec{q}_j\right)\right] ,\end{aligned} \tag{71}$$

where the internal wave functions of the deuteron ϕ_d, and residual nucleus ϕ_R are independent of i. It therefore follows that

$$M_i = f_i M_1 , \tag{72}$$

where f_i is the sign of the permutation exchanging i and 1. If we then define

$$T_p = T_p^{(1)}\sqrt{A} , \tag{73}$$

$$\Lambda_p = \Lambda_p^{(1)} , \tag{74}$$

and similarly for T_0 and Λ_0, the unitarity relation (65) becomes

$$\mathrm{disc}\, T_e = T_e^\dagger \Lambda_e T_e + T_p^\dagger \Lambda_p T_p + T_0^\dagger \Lambda_0 T_0 , \tag{75}$$

where the transition operators for pickup and knockout are

$$T_p = \sqrt{A}\,\tau_p(1+\Gamma_e T_e) , \tag{76}$$

$$T_0 = \sqrt{A}\,\Omega_e \tau_e (1+\Gamma_e T_e) = \frac{1}{\sqrt{A}}\Omega_e T_e . \tag{77}$$

Since both the approximations we wish to investigate are sometimes applied in KMT form [i.e., as approximations to Eq. (21)], it is useful to have

the unitarity results of our three-body model expressed in terms of the auxiliary (KMT) elastic operator $\hat{T}_e = (A-1)T_e/A$. Multiplying Eq. (75) by $(A-1)/A$, we have

$$\mathrm{disc}\hat{T}_e = \left(\frac{A}{A-1}\right)(\hat{T}_e^\dagger \Lambda_e \hat{T}_e + \hat{T}_p^\dagger \Lambda_p \hat{T}_p + \hat{T}_0^\dagger \Lambda_0 \hat{T}_0)\,, \tag{78}$$

where

$$\hat{T}_p = \left(\frac{A-1}{A}\right)T_p = \left(\frac{A-1}{\sqrt{A}}\right)\tau_p(1+\Gamma_e T_e)\,, \tag{79}$$

$$\hat{T}_0 = \left(\frac{A-1}{A}\right)T_0 = \left(\frac{A-1}{\sqrt{A}}\right)\Omega_e\tau_e(1+\Gamma_e T_e)\,. \tag{80}$$

The transition operators in Eqs. (79) and (80) can be expressed in terms of $\hat{T}_e$ by converting the operators τ_e and τ_p into the corresponding KMT-type operators $\hat{\tau}_e$ and $\hat{\tau}_p$. Using Eqs. (16) and (43) together with the identity

$$(1-\Gamma_e\tau_e)(1+\Gamma_e T_e) = (1+\Gamma_e\hat{T}_e)\,, \tag{81}$$

which is easily verified from Eqs. (18) and (20), we can write

$$\hat{T}_p = \left(\frac{A-1}{\sqrt{A}}\right)\hat{\tau}_p(1+\Gamma_e\hat{T}_e)\,, \tag{82}$$

$$\hat{T}_0 = \left(\frac{A-1}{\sqrt{A}}\right)\Omega_e\hat{\tau}_e(1+\Gamma_e\hat{T}_e) = \frac{\Omega_e\hat{T}_e}{\sqrt{A}}\,. \tag{83}$$

Equation (78) is the equivalent of the unitarity relation (75) for our three-body model of the single-scattering optical potential when expressed in terms of KMT-type operators.

The closure and on-shell impulse approximations may now be obtained as approximations to our three-body model and the resulting unitarity relations displayed. The KMT-type transition operators that are employed to formulate both these approximations must be converted to properly normalized physical transition operators before information on the reactive content is deduced from a unitarity relation.

In the closure approximation[5] the Green function in Eq. (24) is approximated by the free propagator for the projectile alone. One therefore replaces $\hat{\tau}_e$ by the operator

$$t_c(E') = v + v\frac{1}{E'+i\epsilon-K}t_c(E')\,, \tag{84}$$

where $E' = E-\Delta$, with Δ a constant which is an "average value" of the target Hamiltonian. Following the KMT method as described in Sec. II, the auxiliary optical potential in the closure approximation is therefore

$$\hat{U}_c = (A-1)t_c\,, \tag{85}$$

and the elastic operator T_e^c is given by

$$T_e^c = \frac{A}{A-1}\hat{T}_e^c\,, \tag{86}$$

where

$$\hat{T}_e^c = \hat{U}_c + \hat{U}_c\Gamma_e\hat{T}_e^c\,. \tag{87}$$

The operator t_c is not simply a one-body operator since it contains a potential involving the nucleon coordinate, but the dependence on this coordinate is simple. Explicitly, the two-body matrix element of $t_c(E)$ in the mixed representation (momentum space for the projectile, coordinate space for the target nucleon) is given by[13]

$$\langle\vec{k}_0\vec{r}_1|t_c(E')|\vec{k}_0'\vec{r}_1'\rangle = \delta(\vec{r}_1-\vec{r}_1')e^{i(\vec{k}_0-\vec{k}_0')\cdot\vec{r}_1}\langle\vec{k}_0|t_c(E')|\vec{k}_0'\rangle\,, \tag{88}$$

where $\langle\vec{k}_0|t_c(E')|\vec{k}_0'\rangle$ is the one-body T matrix obtained by assuming the struck nucleon is held fixed at the origin.

The matrix elements of $\hat{U}_c$ are given by

$$\langle\vec{k}'\phi_e|\hat{U}_c|\phi_e\vec{k}\rangle = (A-1)\int d^3k_0 d^3k_0' d^3r_1 d^3r_1'\langle\vec{k}'\phi_e|\vec{k}_0\vec{r}_1\rangle \times\langle\vec{k}_0\vec{r}_1|t_c(E')|\vec{k}_0'\vec{r}_1'\rangle\langle\vec{k}_0'\vec{r}_1'|\vec{k}\phi_e\rangle\,, \tag{89}$$

where the complete set of mixed representation states for the projectile and particle 1 have been inserted on both sides of t_c. Defining the one-body density matrix

$$\rho(\vec{r}_1,\vec{r}_1') = \int\langle\phi_e|\vec{r}_1,\vec{r}_2,\ldots,\vec{r}_A\rangle \times\langle\vec{r}_1',\vec{r}_2,\ldots,\vec{r}_A|\phi_e\rangle d^3r_2\cdots d^3r_A\,, \tag{90}$$

and applying Eq. (88), Eq. (89) becomes

$$\langle\vec{k}'\phi_e|\hat{U}_c|\phi_e\vec{k}\rangle = (A-1)\langle\vec{k}'|t_c(E')|\vec{k}\rangle\rho(\vec{k}'-\vec{k})\,, \tag{91}$$

where we have defined

$$\rho(\vec{q}) = \int d^3r e^{i\vec{q}\cdot\vec{r}}\rho(\vec{r},\vec{r})\,. \tag{92}$$

Information concerning the intermediate states which are implicit in Eq. (91) can be obtained through the unitarity relation for t_c. Defining $g_c(E') \equiv (E'+i\epsilon-K)^{-1}$, applying the operator unitarity theorem to Eq. (84) and taking the limit $\epsilon\to 0$ we obtain

$$\langle\vec{k}'\phi_e|\mathrm{disc}\hat{U}_c|\phi_e\vec{k}\rangle = (A-1)\langle\vec{k}'\phi_e|t_c^\dagger\hat{\Lambda}_c t_c|\phi_e\vec{k}\rangle\,. \tag{93}$$

(See the Appendix for a discussion of a limit of this type.) The projector $\hat{\Lambda}_c$ only involves the coordinates of the projectile, viz.,

$$\hat{\Lambda}_c(E) = \int d^3k_0 |\vec{k}_0\rangle \delta(E' - k_0^2/2\mu_0) \langle \vec{k}_0| . \quad (94)$$

Because g_c is a one-body operator, the state of the target particles is not restricted by the on-shell condition. To identify the reaction amplitudes we apply the operator unitarity theorem to Eq. (87) and use Eq. (93) to get

$$\text{disc}\hat{T}_e^c = \hat{T}_e^{c\dagger}\Lambda_e\hat{T}_e^c + \frac{A}{A-1}\hat{T}_0^{c\dagger}\hat{\Lambda}_c\hat{T}_0^c , \quad (95)$$

where

$$\hat{T}_0^c = \frac{A-1}{\sqrt{A}} t_c(1 + \Gamma_e\hat{T}_e^c) = \hat{T}_e^c/\sqrt{A} . \quad (96)$$

Converting to a relation for the physical transition operators by multiplying Eq. (95) by $A/(A-1)$ yields

$$\text{disc}T_e^c = \left(\frac{A-1}{A}\right) T_e^{c\dagger}\Lambda_e T_e^c + T_0^{c\dagger}\hat{\Lambda}_c T_0^c , \quad (97)$$

where we have used Eq. (86) and the definition

$$T_0^c = \sqrt{A} t_c(1 + \Gamma_e\hat{T}_e^c) = T_e^c/\sqrt{A} . \quad (98)$$

This permits us to move the apparently extraneous $(-1/A)$ piece of the elastic term over to the reactive part giving

$$\text{disc}T_e^c = T_e^{c\dagger}\Lambda_e T_e^c + T_0^{c\dagger}(\hat{\Lambda}_c - \Lambda_e)T_0^c . \quad (99)$$

Equation (99) is the unitarity relation for the elastic scattering amplitude arising from a closure approximation to the KMT pseudo-optical potential.

We can now use Eqs. (98) and (99) to explicate the reactive content of the closure approximation.

Because t_c is a two-body operator, when it is applied to the initial state $|\phi_e\vec{k}_e\rangle$, only the coordinate of a single nucleon in the target is affected. Since the dependence of t_c on the struck particle is extremely simple, the final state of the nucleus can be easily obtained. We expand the target wave function in a parentage expansion. Taking matrix elements of Eq. (27) gives

$$\langle \vec{p}_1, \ldots, \vec{p}_A | \phi_e\rangle = \sum_\lambda \phi^\lambda(\vec{p}_1)\phi_R^\lambda(\vec{p}_2, \ldots, \vec{p}_A) . \quad (100)$$

(This state should be antisymmetrized but we will not consider the effect of this here.) Writing the coordinates of the target nucleons, $2, \ldots, A$, compactly as ξ, the relevant "reactive" matrix elements are

$$\langle \vec{k}_0'\vec{p}_1'\xi | T_0^c | \phi_e\vec{k}_e\rangle = \sqrt{A}\sum_\lambda \langle \vec{k}_0'\vec{p}_1' | t_c(1 + \Gamma_e\hat{T}_e^c) | \phi^\lambda\vec{k}_e\rangle \phi_R^\lambda(\xi) . \quad (101)$$

The operator $1 + \Gamma_e\hat{T}_e^c$ converts the projectile's plane wave into a distorted wave state $\chi_{k_e}^{c(+)}$. Inserting a complete set of plane waves for the projectile and particle 1 on the right of t_c and using Eq. (88) yields the explicit expression

$$\langle \vec{k}_0'\vec{p}_1'\xi | T_0^c | \phi_e\vec{k}_e\rangle = \sqrt{A}\sum_\lambda \phi_R^\lambda(\xi) \int \langle \vec{k}_0' | t_c(E) | \vec{k}_0\rangle \delta(\vec{k}_0' + \vec{p}_1' - \vec{k}_0 - \vec{p}_1) \times \chi_{k_e}^{c(+)}(\vec{k}_0)\phi^\lambda(\vec{p}_1) d^3k_0 d^3p_1 . \quad (102)$$

The δ function from $\hat{\Lambda}_c$ at the intermediate states in Eq. (99) restricts the magnitude of $\vec{k}_0'$ to be equal to the projectile's initial momentum k_e (or to some value slightly shifted from it) but it permits any values for p_1' and ξ.

The physical interpretation of this matrix element is straightforward. It is the matrix element for knockout of a nucleon from the target, conserving momentum at the projectile-nucleon scattering vertex but not necessarily conserving energy in the final state. The knockout amplitude appears as a distorted wave impulse approximation (DWIA) with the initial distortion given by the auxiliary optical potential in the closure approximation. Although the Fermi motion and recoil of the struck nucleon have not been ignored, the definition of t_c is such that its matrix elements in Eq. (102) do not involve the momentum of the struck nucleon. There is no distortion in the final state.

A second approximation which is commonly employed is an approximation suggested by KMT.[2] They take $\hat{\tau}_e \approx \bar{t}$ where

$$\bar{t} = v + v\frac{1}{\bar{E} + i\epsilon - K - K_1}\bar{t} , \quad (103)$$

where $\bar{E} = E - E_e$ is the projectile's initial kinetic energy. The total kinetic energy of the pair is given by $K + K_1 = K_{01} + k_{01}$ where K_{01} is the kinetic energy of the projectile-nucleon center of mass and k_{01} is their relative kinetic energy. It is common to neglect the spread in the kinetic energy of the pair c.m. and replace $\bar{E} - K_{01}$ by its initial value when the Fermi motion of the struck nucleon is ignored, viz., $\bar{E}(1 + m_0/m)^{-1}$. [For typographical convenience we will define $x = m_0/m$ and $\eta = (1 + x)^{-1}$]. This eliminates the energy shift in the Green function which couples the effective two-body collision energy to the Fermi motion. It is essentially a three-body effect and cannot always be ignored in certain important cases. If we make this replacement then the T matrix in Eq. (103) is replaced by the T matrix defined by the equation

$$t_{IA}(\eta\bar{E}) = v + v\frac{1}{\eta\bar{E} + i\epsilon - k_{01}} t_{IA}(\eta\bar{E}) . \quad (104)$$

This equation is a two-body problem in the relative

coordinate alone. The auxiliary optical potential matrix element implied by Eq. (104) is

$$\langle \vec{k}'\phi_e | \hat{U}_{IA} | \phi_e \vec{k} \rangle = (A-1) \int d^3p_1 \rho(\vec{p}_1 - \vec{k}' + \vec{k}, \vec{p}_1) \times \langle \vec{k}' - x\eta(\vec{k} + \vec{p}_1) | t_{IA}(\eta\bar{E}) | \eta(\vec{k} - x\vec{p}_1) \rangle . \quad (105)$$

If the dependence of t_{IA} on the momentum p_1 is suppressed by setting $p_1 = 0$ in the arguments of t_{IA}, then this becomes

$$\langle \vec{k}'\phi_e | \hat{U}_{IA} | \phi_e \vec{k} \rangle \approx (A-1) \langle \vec{k}' - x\eta\vec{k} | t_{IA}(\eta\bar{E}) | \eta\vec{k} \rangle \rho(\vec{k}' - \vec{k}) . \quad (106)$$

The structure of this equation should be compared with the structure of Eq. (91). Further approximations are often made in order to reduce Eq. (106) to a simply calculable form in coordinate space. (One such approximation is the assumption that t_{IA} is local, i.e., only a function of the difference of its momentum arguments.) We will not consider these here but will investigate the implications of the full impulse approximation form (105).

Applying the operator unitarity theorem to Eq. (104) and taking the limit $\epsilon \to 0$ gives

$$\mathrm{disc} t_{IA} = t_{IA}^{\dagger} \hat{\Lambda}_{IA} t_{IA} , \quad (107)$$

where

$$\hat{\Lambda}_{IA} = \int d^3k_0 d^3p_1 | \vec{k}_0 \vec{p}_1 \rangle \delta\left(\eta\bar{E} - \frac{\eta}{2m_0} (\vec{k}_0 - x\vec{p}_1)^2 \right) \langle \vec{k}_0 \vec{p}_1 | . \quad (108)$$

The δ function in $\hat{\Lambda}_{IA}$ restricts the intermediate states to the plane wave states of the projectile and struck nucleon which have a relative kinetic energy equal to their initial relative kinetic energy, assuming the struck nucleon is at rest. The reactive amplitudes are constructed by building a unitarity relation for $\hat{T}_e^{IA}$ and then multiplying by $(A-1)/A$ as in the case of the closure approximation discussed above. The result is

$$\mathrm{disc} T_e^{IA} = T_e^{IA\dagger} \Lambda_e T_e^{IA} + T_0^{IA\dagger} (\hat{\Lambda}_{IA} - \Lambda_e) T_0^{IA} , \quad (109)$$

with

$$T_0^{IA} = \sqrt{A} t_{IA} (1 + \Gamma_e \hat{T}_e^{IA}) = T_e^{IA} / \sqrt{A} . \quad (110)$$

The reactive amplitudes then take the form

$$\langle \vec{k}_0' \vec{p}_1' \xi | T_0^{IA} | \phi_e \vec{k}_e \rangle = \sqrt{A} \sum_{\lambda} \phi_R^{\lambda}(\xi) \int \langle \eta(\vec{k}_0' - x\vec{p}_1') | t_{IA}(\eta\bar{E}) | \eta(\vec{k}_0 - x\vec{p}_1) \rangle \delta(\vec{k}_0' + \vec{p}_1' - \vec{k}_0 - \vec{p}_1) \chi_{k_e}^{IA(+)}(\vec{k}_0) \phi^{\lambda}(\vec{p}_1) d^3k_0 d^3p_1 . \quad (111)$$

The physical interpretation of this matrix element is straightforward. It is the nucleon knockout amplitude to the quasifree states ignoring the binding energy of the struck particle. The amplitude is a standard DWIA matrix element with the final state distortion turned off. The impulse T matrix has the two-body kinematics appropriate to quasifree knockout. When Eq. (111) is employed in the unitarity relation (109), the projection operator $\hat{\Lambda}_{IA}$ restricts the matrix elements of t_{IA} to be half-shell.

As with the closure approximation, we see that the unitarity relation Eq. (109) has a correction to the projection operator in the reactive part. However, because $\hat{\Lambda}_{IA} - \Lambda_e$ is not a true projection operator, this relation is *not* a model consistent unitarity relation in the sense that each term does not represent a distinguishable asymptotic state of the model system.

The difference between Eqs. (99) and (109) and our model unitarity relation, Eq. (75), is that in the reactive contributions of the latter the intermediate states are described by genuine projection operators. This difference stems from a property of the three-body model operator $\hat{\tau}_e$ which is not preserved by the closure or impulse approximation to $\hat{\tau}_e$. We recall from Sec. II that the introduction of the exact $\hat{\tau}_e$ through Eq. (13) effectively solves part of the two-body elastic scattering equation for the transition operator T_e. Thus intermediate states of $\hat{\tau}_e$ include propagation of the target ground state, thereby avoiding direct treatment of the projection operator inherent in the Watson operator τ_e. Proper counting of the elastic channel intermediate states in T_e and a proper unitarity relation for T_e is then guaranteed if the KMT method of Eqs. (20) and (21) is employed. Any approximation to $\hat{\tau}_e$ must contain elastic channel intermediate states if that proper and model consistent unitarity relation is to be maintained. In our three-body model we have assumed that the target has a good single particle structure generated by the potential u_i. It is the presence of this potential in the Green function, Eq. (25), which allows elastic intermediate states for $\hat{\tau}_e$, as evidenced by the first term of Eq. (58).

However, the closure [Eq. (84)] and impulse [Eq. (104)] approximations to $\hat{\tau}_e$ cannot contribute elastic intermediate states, as evidenced by Eqs. (93) and (107).

Our three-body formulation of the optical potential also shows how this unitarity defect of the closure and impulse approximations can be avoided. The integral equations (41) and (42) for the Watson operator τ_e yield the projectile-nucleon T matrix t_p as the first term when the equations are iterated. Thus the operators t_c and t_{IA} can be used as closure and impulse approximations to τ_e rather than to $\hat{\tau}_e$. Furthermore, since in the Watson method the elastic intermediate states are inserted when the Lippmann-Schwinger Eq. (18) for T_e is solved, proper unitarity will be preserved. A calculation following this prescription for the example of the closure approximation is summarized by the equations

$$\overline{T}_e^c = A t_c (1 + \Gamma_e \overline{T}_e^c) , \tag{112}$$

$$\mathrm{disc}\,\overline{T}_e^c = \overline{T}_e^{c\dagger} \Lambda_e \overline{T}_e^c + \overline{T}_0^{c\dagger} \hat{\Lambda}_c \overline{T}_0^c , \tag{113}$$

where

$$\overline{T}_0^c = \sqrt{A}\, t_c (1 + \Gamma_e \overline{T}_e^c) = \overline{T}_e^c / \sqrt{A} . \tag{114}$$

We note that a comparison of the standard Lippmann-Schwinger Eqs. (8) and (13) for τ_e and $\hat{\tau}_e$ suggests that $\hat{\tau}_e$ may be more accurately approximated by a two-body free T matrix than may τ_e, because of the projected intermediate states in the latter. However, the present work shows that within a three-body formulation, the two-body component given by the first term of the iterated series is the same for both operators. [See Eqs. (35), (36) and (41), (42)]. In the absence of calculational comparisons, the implications of the unitarity equation suggest that the Watson operator τ_e is more suited to approximation by a two-body T matrix than is the KMT operator $\hat{\tau}_e$. This point will be discussed further below.

VI. SUMMARY AND CONCLUSIONS

In this paper we develop a three-body model for the single-scattering term of the optical potential in multiple-scattering theory. This model assumes that a distinguishable projectile scatters from a target of identical nucleons and that the target wave function has a good single particle structure. The formulation can accommodate the kinematical three-body effects arising from the transformation from projectile-nucleus c.m. frame to projectile-nucleon c.m., from the Fermi motion of the struck nucleon, and from its separation energy. It can also include the intermediate state distortion effects due to the potential which binds the struck nucleon to the rest of the nucleus. The result is that the effective interaction describing the scattering of the projectile from a bound nucleon is a three-body T matrix.

The operator introduced by Watson to describe the scattering of a projectile from a single bound nucleon is a many-body operator. The struck nucleon may interact with any of the other bound nucleons during the scattering with the projectile thus leading to further excitations. Consistent with the assumed diluteness of the nucleus and the single-scattering nature of the first order term, we replace the interaction of the struck nucleon with the other nucleons in the nucleus by an effective single particle potential. This potential plays two roles, binding the struck nucleon to the rest of the nucleus and providing intermediate state distortions for the motion of the struck nucleon. This replacement is the only approximation we make.

The on-shell intermediate states present in Watson's optical potential operator provide the absorptive channels since elastic intermediate states are expressly excluded. In exploring the consequences of our model, we have omitted explicit treatment of the bound excited states of the target nucleus, and have concentrated on the proper handling of the more difficult (and more important) continuum and rearrangement states. The bound inelastic states may be easily included by making the optical potential a matrix on these states, and then, if desired, formally eliminating all states but the ground state.

We present a pair of coupled integral equations for the single-scattering optical potential in our model. These equations have a structure similar to the AGS version of the Faddeev equations[10] for a three-body problem having only two interactions, the projectile-nucleon interaction and the nucleon-core interaction. The projectile-core interaction is absent as we are writing equations for the optical potential. This interaction is obtained by summing over all the nucleons in the nucleus and its effect obtained by solving the LS equation with the optical potential.

Since the equations have Faddeev structure, they permit the exact inclusion of the "two-particle" continuum states without the difficulties of unspecified boundary conditions or noncompact kernels. They should be amenable to the numerical techniques which have become well established in work on the three-nucleon problem.

The Green function in the original LS equation for the Watson single-scattering optical potential contains a projection operator which prohibits intermediate state propagation of the target ground state. The effects of this projection operator appear in the context of our model as a modification

of the kernel compared to the standard three-body equations. This extra term in the kernel cancels exactly the on-shell intermediate state contribution which corresponds to the target ground state. This cancellation allows us to have a three-body-like model which is still consistent with the many-body structure of the amplitude. The intermediate state corresponding to the struck nucleon bound to the core is present in all A three-body problems. If these A three-body problems were superposed injudiciously, considerable overcounting would result. Our study of the unitarity relation for this model demonstrates that the modification of the Faddeev equations which we obtain prevents this overcounting. This happens because the elastic intermediate states are all produced by solving the LS equation and they are therefore totally excluded from the states used to construct the optical potential. This is maintained in our model but not in all approximations to it.

In the KMT approach an auxiliary optical potential is defined by removing the projection operator from the Watson single-scattering term. This auxiliary optical potential is then used in an LS equation to produce an auxiliary transition operator which can be shown to be exactly equal to the physical transition operator when multiplied by the factor $A/(A-1)$. The exact equivalence has been shown to hold if the exact single-scattering operators are used. We demonstrate that the equivalence also holds in our three-body approximation. The resulting KMT single-scattering operator satisfies three-body integral equations of the same type as does the Watson single-scattering operator, but without the overcounting modification in the kernel. The processes included in the three-body model are illustrated in Fig. 1.

If full three-body calculations are to be done, it is not clear whether Eqs. (41), (42) (Watson form) or (35), (36) (KMT form) have an advantage. The major numerical difficulty in solving either set exactly is in the three-body nature of the equations. If, however, one considers iterating the equations, then the weakening of the kernel of Eq. (42) by the subtraction of the pole, suggests that Eqs. (41), (42) for the Watson optical potential may converge faster than Eqs. (35), (36) for the KMT optical potential. The weakening of the kernel in this manner is similar to the quasiparticle method of Alt, Grassberger, and Sandhas[10] which has proved very successful in providing rapidly convergent corrections to three-body calculations. Unfortunately, counterexamples are known where a pole subtraction worsens the convergence.[14] Firm statements will have to await realistic calculations. Our current estimation of the three-body experience, however, suggests that the free two-body T matrix

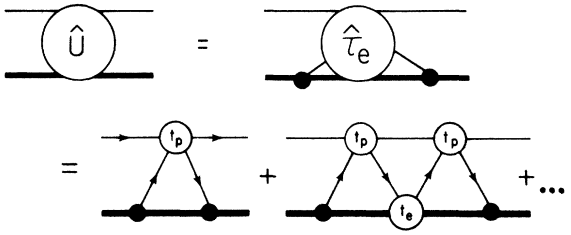

FIG. 1. A diagrammatic representation of the optical potential in the three-body model proposed. The heavy black line indicates the nucleus, the heavy black dot the struck nucleon-core wave function, and the open circle a two-body T matrix. The terms shown correspond to the first few iteration of Eqs. (35) and (36) for the auxiliary optical potential of the KMT approach. The iterations of Eqs. (41) and (42) for the Watson optical potential can be represented by the above diagram if t_e is replaced by $t_e - G_0^{-1}\Gamma_e G_0^{-1}$.

will be a better approximation to the Watson single-scattering optical potential than it will to the KMT single-scattering optical potential in contrast to previous expectations.[15] This is because the presence of the Q operator tends to cancel the distorting effect of the binding potential leaving an operator which is closer to a free T matrix.

Considerable effort has been devoted to the derivation of the unitarity properties of the model. Unitarity relations assist in understanding the physical basis of the optical potential's absorptive part by allowing us to examine the reaction mechanisms which are implicit in its construction and to check if a consistent and experimentally reasonable treatment of these reactions is given. This can provide a valuable guide to the validity of the approximations. Our principal result is Eq. (65) which displays explicitly the structure of the absorptive part of the cross section in terms of the inelastic states of the model and the associated inelastic transition operators. Equations (63) and (64) give the inelastic transition operators consistent with the model. The inelastic states obtained are nucleon knockout and pickup. The transition operators for these reactions have a quasi-distorted-wave form. The initial distorted wave is the elastic scattering wave function generated by the model optical potential. The departure from standard distorted wave approximations lies in the way the final state distortion is handled.

In a distorted wave approximation, the initial distortion is followed by a single scattering given by a projectile-nucleon potential or T matrix. The outgoing bound pair or the two free particles may scatter from the residual nucleus producing final distorted waves. No single-scattering optical model can provide the full final state distortion to the intermediate reactive states. In single scattering, only one particle at a time can be extracted

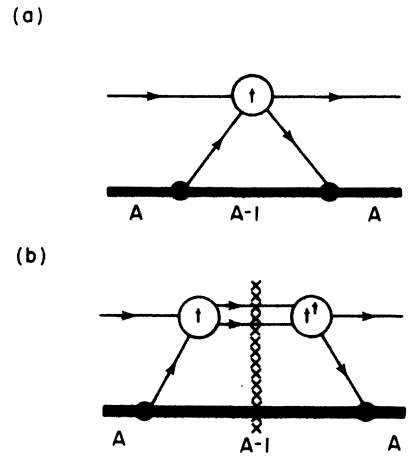

FIG. 2. (a) A diagrammatic representation of a single-scattering folded-T-matrix approximation for the optical potential. (b) The imaginary part of diagram (a) using unitarity on the effective interaction. The contribution to unitarity is obtained by taking the DW matrix element of this operator. [See Eq. (48).] The hatched line indicates an on-shell intermediate state.

from the nucleus. The nucleus must return to its ground state before a second particle can be seen. Once the first particle has been extracted, the projectile is not allowed to interact with the rest of the nucleus at all. The extracted particle may only interact with the rest of the nucleus via its binding potential. Since most of the important final state distortion in knockout or pickup comes from the imaginary part of the particle-core optical potential, which in turn arises from knockout of an additional particle, this cannot be present in the reactions implicit in single scattering. Our three-body model is characterized by the fact that it includes as much scattering as is possible in the context of the three-body reaction mechanism consistent with single scattering.

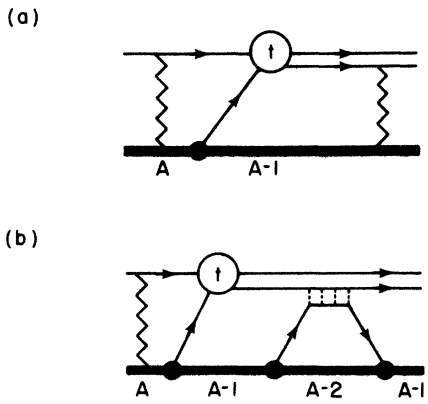

FIG. 3. (a) A diagrammatic representation of part of a DW amplitude for knockout. The zig-zag line indicates a particle-nucleus scattering. (b) Part of the diagram in (a) with the final state knocked-out nucleon-nucleus scattering expanded by using single scattering for the optical potential. Observe the presence of four-body intermediate states.

This argument is sketched diagrammatically in Fig. 2 for an optical potential given by the folding of a two-body T matrix. In Fig. 2(a) we show the single-scattering approximation schematically. In Fig. 2(b) the T matrix is opened to reveal the intermediate states. The hatched line cutting the diagram indicates the presence of an on-shell knockout state. Figure 3(a) shows the diagram for the knockout of a single nucleon in a distorted wave approximation. In Fig. 3(b) the final state distortion of the outgoing nucleon is opened up to reveal the lowest order contribution to the distortion in a single-scattering impulse approximation. This clearly brings up a second nucleon and would only be present if double-scattering terms were included in the construction of the original optical potential.

In our model, after the incident projectile is distorted in the optical field and interacts with the target nucleon extracting it from the nucleus, any number of projectile-nucleon interactions may take place possibly forming a projectile-nucleon bound state. The nucleon may also interact with the nucleus via the binding potential any number of times. All these interactions are summed to all orders and in every possible sequence, in the solution of our three-body integral equations for the optical potential.

We show how both the closure and impulse approximations can be obtained as approximations to our three-body model. Each approximation ignores the binding interaction of the struck nucleon in calculating the optical potential operator. They differ in the kinematics used to construct the projectile-nucleon two-body T matrix. The unitarity methods developed above are applied to these approximations to extract the reactions which are implicit in them and to identify the associated transition operators. Since both methods ignore the pole in the off-shell two-body T matrices, no singularities corresponding to pickup are present. The only reaction present in the absorptive parts in the closure and impulse approximations is knockout.

In the closure approximation, the allowed knockout states have the projectile's kinetic energy restricted to its incident value (or to some fixed value slightly lower). The energy states of the target nucleons are unrestricted. If the transition operator acting on the nucleons of the target only produces low lying excited states of the target, this approximation should be reasonable.

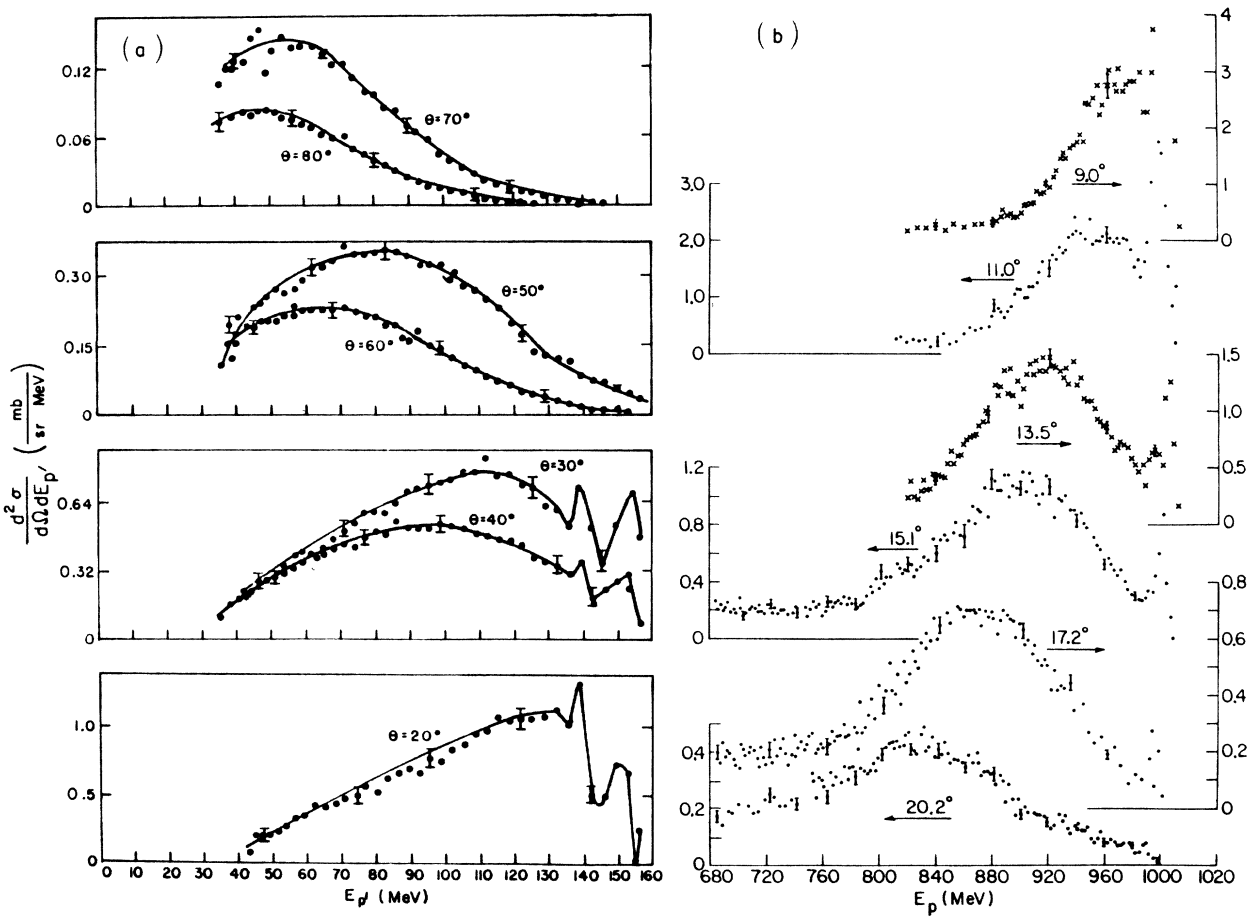

FIG. 4. Inelastic proton spectra at various angles for $p+{}^{12}C$. (a) 160 MeV [Wall and Roos, Phys. Rev. 150, 811 (1966)]. (b) 1000 MeV [Corley *et al.*, Nucl. Phys. A184, 437 (1972)].

In the impulse approximation, the allowed knockout states contain the projectile and a knocked out nucleon. Their relative kinetic energy is restricted to that corresponding to free scattering from a target nucleon which is initially at rest and unbound. These are exactly the states of quasifree knockout at all energies and angles.

In both the closure and impulse approximations, the transition operator to the knockout states has a distorted wave character. The incoming projectile is distorted by the relevant optical potential and the knockout is produced by a two-body T matrix. In the case of the impulse approximation this is a half-shell T matrix of the commonly used type. In both cases there is no final state distortion.

In conventional distorted wave calculations of knockout, final state distortion must be included for both particles to fit the data adequately. This suggests that both the closure and impulse approximations will overestimate the knockout amplitudes and therefore overestimate the imaginary part of the optical potential. In practice, this overestimate may compensate for those inelastic processes which are not included in the approximate optical potential.

The relative adequacy of the closure and impulse approximations may be evaluated by a consideration of the experimental reaction data. This is the particular value of the extraction of implicit reaction mechanisms; it allows one to check the consistency of one's implicit physical assumptions by comparison with other data.

As an example we consider the inelastic scattering of protons from ^{12}C at 160 and 1000 MeV. Singles spectra for the inelastic scattering are shown in Fig. 4 (from Ref. 4). If these singles spectra are integrated over angle as a function of energy loss, one obtains the energy spectra shown in Fig. 5. At 160 MeV the spectrum is quite flat and does not peak at small energy loss. The angular integration is over the forward hemisphere in the lab, which is the region relevant for quasifree knockout. A significant fraction of the cross section is observed. From geometrical considerations, taking a black nucleus of radius $1.2\times A^{1/3}$ gives a total reaction cross section of about 250

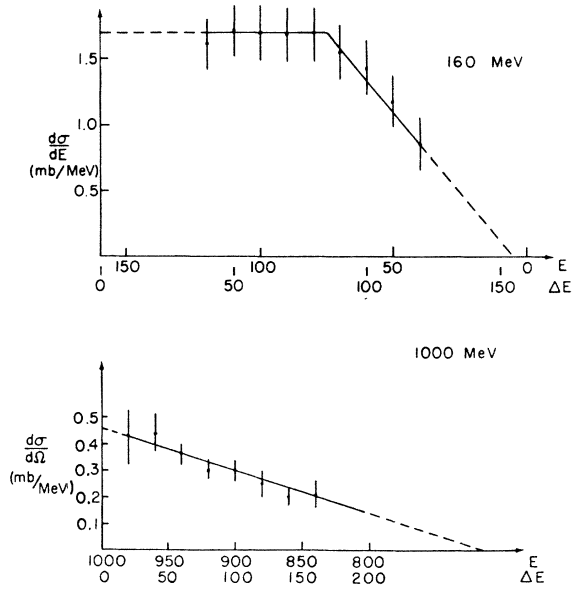

FIG. 5. Angle integrated inelastic proton spectra for scattering from ^{12}C. Dashed lines indicate extrapolations of the data.

mb. About 120 mb of that is seen between energy losses of 40 and 120 MeV. If the curves are extrapolated as indicated by the dotted lines in Fig. 5(a) one obtains about 200 mb of reaction cross section. At 1000 MeV the spectrum does tend to fall off with increasing energy loss but not as rapidly as one might hope. Extrapolating as indicated by the dotted lines in Fig. 5(b) one obtains about 70 mb of cross section. The experimental total reaction cross section has been measured as being 222 mb.

Both sets of inelastic data have knockout character. The positions of the broad peaks observed in Fig. 4 follow the angular dependence of free scattering as shown in Fig. 6. The slight downward shift is primarily due to the binding energy of the struck particle. The magnitudes and shapes of the peaks are fitted reasonably well by distorted wave calculations.[4]

One possible reason for the missing reaction cross section at 1 GeV is pion production. At this energy, a large fraction of the nucleon-nucleon cross section goes into pion production. This means that much of the quasifree scattering process would no longer be a three-body process, but would involve four final state particles. The data of Corley *et al.*[4] did not go to large enough energy loss to see this.

This kind of process can be included in our method by adding another stage to the expansion of the unitarity relation. The nucleon-nucleon potential itself becomes complex and, given a model for pion production, satisfies a discontinuity relation of its own. Extending our procedure, a unitarity relation would be obtained including three- and four-body intermediate states. If one is in an energy region where pion production is described adequately by an isobar model, the additional

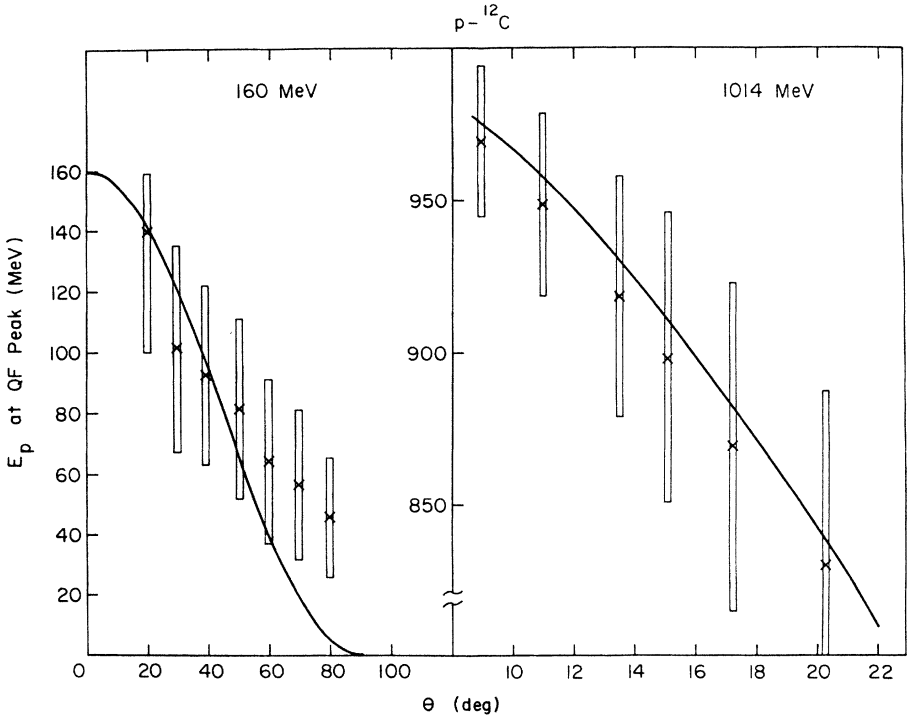

FIG. 6. Energy of peak in proton spectra as a function of angle. Crosses mark the peak positions; the bands indicate the approximate width of the peak. The solid lines indicate the energy-angle relation for free particle scattering.

states could be included by a simple generalization of our three-body procedure. We assume that the 1 GeV data support the hypothesis that quasifree scattering (including inelastic nucleon-nucleon processes) dominates the reaction mechanism. More extensive data investigating this point would be most welcome.

If we accept these data as evidence for the distribution of inelastic strength, then we must conclude that at 160 MeV the average excitation energy of the inelastic states is too large to be well treated by a closure approximation. At 1 GeV the closure approximation may work reasonably well, especially if the missing cross section is occurring at energy losses smaller than those observed rather than larger.

We have also seen that although our three-body model retains the equivalence of the Watson and KMT single-scattering terms, the closure and impulse approximations do not. Arguments given above suggest that these approximations will do better if used for the Watson rather than the KMT single-scattering potential. Combining this result with our analysis of the inelastic data suggests that the impulse approximation for the Watson single-scattering optical potential should be the best of the two-body approximations considered. We note that this approximation was used by Lerner and Redish[16] for constructing the real part of the $p+{}^{16}\mathrm{O}$ optical potential at 65 MeV. When the off-shell dependence of the impulse approximation was retained, i.e., the first iterate to Eq. (41) was used, an excellent result was obtained.

In conclusion, we have constructed a three-body approximation to the single-scattering optical potential in Watson and KMT forms. This model maintains the exact agreement between the two methods as does the exact single-scattering operator. The three-body structure permits the use of unitarity to extract the reactions implicitly responsible for the absorption and their amplitudes. This allows one to consult additional experimental data in order to refine the physical picture inherent in the model and to evaluate approximations to it. The model itself should provide a good basis for further calculations and for extending the description of the theoretical optical model while keeping a closer touch with the information contained in the inelastic data.

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APPENDIX: DISCONTINUITIES ACROSS THE UNITARITY BRANCH CUT

We derive the unitarity relation [Eq. (58)] for the operator $\hat{\tau}_e^{(i)}$ by carefully taking the limit $\epsilon \to 0$ in Eq. (57). This derivation provides an illustrative example of the caution that must be exercised in evaluating expressions of the form $\lim_{\epsilon\to 0} A^\dagger \Delta C A$ when matrix elements of ΔC are not sufficiently connected. Our discussion covers all cases met in Sec. IV and in particular we demonstrate why the limit $\epsilon \to 0$ may safely be taken inside the operator strings of Eq. (56).

From Eq. (57) we must evaluate

$$\mathrm{disc}\hat{\tau}_e^{(i)} = \frac{1}{2\pi i}\lim_{\epsilon\to 0}\Delta\hat{\tau}_e^{(i)} = \frac{1}{2\pi i}\lim_{\epsilon\to 0}\hat{\tau}_e^{(i)\dagger}\Delta G_e^{(i)}\hat{\tau}_e^{(i)}. \tag{A1}$$

Applying the operator unitarity theorem [Eq. (45)] to the resolvent equation for $G_e^{(i)}$ we obtain

$$\Delta G_e^{(i)} = (G_0^\dagger t_e^\dagger + 1)\Delta G_0(1 + t_e G_0). \tag{A2}$$

For convenience we do not label the active target nucleon throughout the remainder of the Appendix. Equation (A1) then reads

$$\mathrm{disc}\hat{\tau}_e = \lim_{\epsilon\to 0}\frac{\epsilon}{\pi}\hat{\tau}_e^\dagger(G_0^\dagger t_e^\dagger + 1)G_0^\dagger G_0(1 + t_e G_0)\hat{\tau}_e, \tag{A3}$$

where we have used

$$\begin{aligned}\Delta G_0(E+i\epsilon) &\equiv G_0(E+i\epsilon)^\dagger - G_0(E+i\epsilon)\\ &= 2i\epsilon G_0(E+i\epsilon)^\dagger G_0(E+i\epsilon).\end{aligned} \tag{A4}$$

In Eq. (A2) it appears as if only eigenfunctions of G_0 appear. The main difficulty is that the structure of Eq. (A1) suggests that the only on-shell intermediate states will be the eigenfunctions of the Hamiltonian contained in the operator $G_e^{(i)}$. When matrix elements are taken, we will see below that such states are present and arise from the singularities of $\Delta G_e^{(i)}$ pinching the real axis. In Eq. (A3) these contributions will be seen to come from the poles of $G_0^\dagger G_0$ (the on-shell breakup states), and the poles of $t_e^\dagger$ and t_e (the on-shell elastic states). However, Eq. (A3) also suggests the possibility that the disconnected part of $\Delta G_e^{(i)}$, represented by the unity parts of the parentheses, allows singularities of $\hat{\tau}_e^\dagger$ and $\hat{\tau}_e$ to contribute. We show, that in our three-body model, this is, in

fact, the case if the nucleon pickup channel is open.

It is convenient to define

$$\hat{\tau}_0 = (1 + t_e G_0)\hat{\tau}_e = \Omega_e \hat{\tau}_e , \tag{A5}$$

and using Eqs. (36) and (37) we can write

$$\hat{\tau}_0 = t_p G_0 \hat{\tau}_p + t_e G_0 \hat{\tau}_e . \tag{A6}$$

Equation (A3) now reads

$$\mathrm{disc}\hat{\tau}_e = \lim_{\epsilon \to 0} \frac{\epsilon}{\pi} \hat{\tau}_0^\dagger G_0^\dagger G_0 \hat{\tau}_0 . \tag{A7}$$

The utility of introducing the operator $\hat{\tau}_0$ is that Eq. (A6) demonstrates that the extra singularity mentioned above is the bound state pole of t_p (and $t_p^\dagger$). It will contribute if the pickup channel is open. To take the limit in Eq. (A7) explicitly, we form matrix elements and study the intermediate state integrations. We work in the center of mass of the full $(A+1)$-body system and introduce two equivalent momentum representations of the nucleon knockout state. The relative momentum of the projectile and the active target nucleon is $\vec{q}_p$, while the momentum of the center of mass of this pair relative to the residual nucleus is $\vec{k}_p$. The other coupling scheme is defined by specifying $\vec{q}_e$ as the relative momentum of the active target nucleon and the residual nucleus, while $\vec{k}_e$ is the momentum of the center of mass of this pair relative to the projectile. We use units such that $\hbar/(\text{nucleon mass}) = 1$, and will assume that the momenta are scaled to include reduced mass factors. If the nucleon knockout state is on shell the momenta satisfy $E = k_e^2 + q_e^2 + E_R = k_p^2 + q_p^2 + E_R$ where E is the total energy, and E_R is the internal energy of the residual nucleus.

The matrix element form of Eq. (A7) will read

$$\langle f | \mathrm{disc}\hat{\tau}_e(E) | i \rangle = \lim_{\epsilon \to 0} \frac{\epsilon}{\pi} \int \frac{\langle f | \hat{\tau}_0^\dagger(E + i\epsilon) | \phi_R, \vec{k}_e \vec{q}_e \rangle \langle \vec{k}_e \vec{q}_e \phi_R | \hat{\tau}_0(E + i\epsilon) | i \rangle}{(E - i\epsilon - k_e^2 - q_e^2 - E_R)(E + i\epsilon - k_e^2 - q_e^2 - E_R)} d^3k_e d^3q_e , \tag{A8}$$

where, for the present, we suppress any detailed information about the initial and final states. Finite contributions are identified by explicating those pole singularities of the integrand which can pinch the contour and so lead to terms of order $1/\epsilon$. The singularity structure of the scattering amplitudes in Eq. (A8) is obtained from the matrix element form of Eq. (A6). We have

$$\begin{aligned} \langle \vec{k}_e \vec{q}_e \phi_R | \hat{\tau}_0(E + i\epsilon) | i \rangle = & \int d^3q_p' \frac{\langle \vec{q}_p | t_p(E + i\epsilon - k_p^2 - E_R) | \vec{q}_p' \rangle \langle \vec{k}_p \vec{q}_p' \phi_R | \hat{\tau}_p(E + i\epsilon) | i \rangle}{(E + i\epsilon - k_p^2 - q_p'^2 - E_R)} \\ & + \int d^3q_e' \frac{\langle \vec{q}_e | t_e(E + i\epsilon - k_e^2 - E_R) | \vec{q}_e' \rangle \langle \vec{k}_e \vec{q}_e' \phi_R | \hat{\tau}_e(E + i\epsilon) | i \rangle}{(E + i\epsilon - k_e^2 - q_e'^2 - E_R)} . \end{aligned} \tag{A9}$$

Because the residual nucleus does not break up, Eq. (A9) has the structure of three-body scattering theory for which the pole singularities in the momenta $\vec{k}_e, \vec{q}_e$ are given by the work of Faddeev. These same singularities are known to be solely responsible for the unitarity relations.[9,17] We sketch how these results are applied to our particular case.

When the energy arguments of the two-body T matrices are in the vicinity of the energies of the two-body bound states, the pole structure is represented by

$$\langle \vec{q}_p | t_p(E + i\epsilon - k_p^2 - E_R) | \vec{q}_p' \rangle \sim \frac{\langle \vec{q}_p | f_p \rangle \langle f_p | \vec{q}_p' \rangle}{E + i\epsilon - k_p^2 - E_p} + \text{nonsingular terms} , \tag{A10}$$

where $E_p = E_R + e_p$, and e_p is the deuteron internal energy. A similar relation holds for the matrix elements of t_e by replacing the subscript p by e. The target ground state energy is E_e. The bound state form factors are related to the corresponding bound state wave functions by

$$\begin{aligned} \langle \vec{q}_p | f_p \rangle &= (E_p - E_R - q_p^2) \langle \vec{q}_p, \phi_R | \phi_p \rangle , \\ \langle \vec{q}_e | f_e \rangle &= (E_e - E_R - q_e^2) \langle \vec{q}_e, \phi_R | \phi_e \rangle . \end{aligned} \tag{A11}$$

Since the energy arguments of the T matrices are not integrated over in Eq. (A9), these poles will be present in the matrix elements of $\hat{\tau}_0$. We can use Eqs. (A10) and (A11) to extract the residues of the amplitude $\hat{\tau}_0$ at these poles. Given the structure of the integrand in Eq. (A8), it is convenient to divide by the factor $(E + i\epsilon - k_e^2 - E_R)$ before extracting the residues. This introduces an extra pole due to the vanishing of this factor, and the corresponding residue is just an on-shell breakup matrix element of $\hat{\tau}_0$. The final result for the pole structure is

$$\frac{\langle \vec{k}_e \vec{q}_e \phi_R | \hat{\tau}_0(E+i\epsilon)|i\rangle}{E+i\epsilon-k_e^2-q_e^2-E_R} = \frac{\langle \vec{q}_p \phi_R | \phi_p\rangle R_p(\hat{k}_p;i)}{(E+i\epsilon-k_p^2-E_p)} + \frac{\langle \vec{q}_e \phi_R | \phi_e\rangle R_e(\hat{k}_e;i)}{(E+i\epsilon-k_e^2-E_e)} + \frac{R_0(\hat{P}_0;i)}{E+i\epsilon-k_e^2-q_e^2-E_R} + \text{nonsingular terms}, \tag{A12}$$

where the residues are given by

$$R_p(\hat{k}_p;i) = \langle \vec{k}_p, \phi_p | \hat{\tau}_p(E+i\epsilon)|i\rangle\,|_{k_p^2=E-E_p}, \tag{A13}$$

$$R_e(\hat{k}_e;i) = \langle \vec{k}_e, \phi_e | \hat{\tau}_e(E+i\epsilon)|i\rangle|_{k_e^2=E-E_e}, \tag{A14}$$

$$R_0(\hat{P}_0;i) = \langle \vec{P}_0, \phi_R | \hat{\tau}_0(E+i\epsilon)|i\rangle|_{P_0^2\equiv k_e^2+q_e^2=E-E_R}. \tag{A15}$$

In Eq. (A15) we have introduced the six-dimensional hyperspherical momentum $\vec{P}_0 = (\vec{k}_e, \vec{q}_e) = (\vec{k}_p, \vec{q}_p)$. In this space the solid angle $\hat{P}_0$ specifies position on the surface $k_e^2+q_e^2 = E-E_R$.

The pole structure of the other factor of the integrand in Eq. (A8) is obtained by reflection through the real axis. Substituting into Eq. (A8) we obtain nine singular integrals. However, only the three integrals in which a pole term is multiplied by its complex conjugate survive in the limit $\epsilon \to 0$. Using

$$\lim_{\epsilon\to 0} \frac{\epsilon}{x^2+\epsilon^2} = \pi\delta(x), \tag{A16}$$

the surviving integrals yield

$$\lim_{\epsilon\to 0}\frac{\epsilon}{\pi}\int d^3k_e d^3q_e \frac{|\langle q_e\phi_R|\phi_e\rangle|^2 R_e^*(\hat{k}_e;f)R_e(\hat{k}_e;i)}{(E-k_e^2-E_e)^2+\epsilon^2} = \int d^3k_e \langle f|\hat{\tau}_e^\dagger(E+i0)|\vec{k}_e\phi_e\rangle\delta(E-k_e^2-E_e) \times \langle \vec{k}_e\phi_e|\hat{\tau}_e(E+i0)|i\rangle, \tag{A17}$$

$$\lim_{\epsilon\to 0}\frac{\epsilon}{\pi}\int d^3k_p d^3q_p \frac{|\langle \vec{q}_p\phi_R|\phi_p\rangle|^2 R_p^*(\hat{k}_p;f)R_p(\hat{k}_p;i)}{(E-k_p^2-E_p)^2+\epsilon^2} = \int d^3k_p \langle f|\hat{\tau}_p^\dagger(E+i0)|\phi_p,\vec{k}_p\rangle\delta(E-k_p^2-E_p) \times \langle k_p\phi_p|\hat{\tau}_p(E+i0)|i\rangle, \tag{A18}$$

$$\lim_{\epsilon\to 0}\frac{\epsilon}{\pi}\int d^3k_e d^3q_e \frac{R_0^*(\hat{P}_0;f)R_0(\hat{P}_0;i)}{(E-k_e^2-q_e^2-E_R)^2+\epsilon^2} = \int d^3k_e d^3q_e \langle f|\hat{\tau}_0^\dagger(E+i0)|\phi_R,\vec{k}_e\vec{q}_e\rangle \times\delta(E-k_e^2-q_e^2-E_R) \times \langle \vec{k}_e\vec{q}_e\phi_R|\hat{\tau}_0(E+i0)|i\rangle. \tag{A19}$$

The integrations over $\vec{q}_e, \vec{q}_p$ in Eqs. (A17) and (A18) have been carried out by employing the normalization condition for the bound state wave functions. Of the remaining terms from the integrand of Eq. (A8), the following example is typical:

$$\lim_{\epsilon\to 0}\frac{\epsilon}{\pi}\left(\frac{R_0^*(\hat{P}_0;f)R_p(\hat{k}_p;i)\langle \vec{q}_p\phi_R|\phi_p\rangle}{(E-i\epsilon-k_p^2-q_p^2-E_R)(E+i\epsilon-k_p^2-E_p)} + \frac{R_p^*(\hat{k}_p;f)R_0(\hat{P}_0;i)\langle\phi_p|\phi_R,\vec{q}_p\rangle}{(E-i\epsilon-k_p^2-E_p)(E+i\epsilon-k_p^2-q_p^2-E_R)}\right) \tag{A20}$$

Setting $x = (E-k_p^2-q_p^2-E_R)$, and $y = (E-k_p^2-E_p)$, and using the limits

$$\lim_{\epsilon\to 0}\frac{\epsilon(xy+\epsilon^2)}{(x^2+\epsilon^2)(y^2+\epsilon^2)} = 0, \qquad \lim_{\epsilon\to 0}\frac{\epsilon^2(x-y)}{(x^2+\epsilon^2)(y^2+\epsilon^2)} = 0, \tag{A21}$$

this term and all remaining terms vanish.

Collecting together these results, the operator form of the unitarity relation for $\hat{\tau}_e$ reads

$$\text{disc}\hat{\tau}_e(E) = \hat{\tau}_e^\dagger(E+i0)\Lambda_e\hat{\tau}_e(E+i0) + \hat{\tau}_p^\dagger(E+i0)\Lambda_p\hat{\tau}_p(E+i0) + \hat{\tau}_e^\dagger(E+i0)\Omega_e^\dagger(E+i0)\Lambda_0\Omega_e(E+i0)\hat{\tau}_e(E+i0), \tag{A22}$$

where the projection operators are

$$\Lambda_e = \int d^3k_e |\phi_e\vec{k}_e\rangle\delta(E-k_e^2-E_e)\langle\vec{k}_e\phi_e|, \tag{A23}$$

$$\Lambda_p = \int d^3k_p |\phi_p\vec{k}_p\rangle\delta(E-k_p^2-E_p)\langle\vec{k}_p\phi_p|, \tag{A24}$$

$$\Lambda_0 = \int d^3k_e d^3q_e |\phi_R\vec{q}_e\vec{k}_e\rangle\delta(E-k_e^2-q_e^2-E_R) \times \langle\vec{k}_e\vec{q}_e\phi_R|. \tag{A25}$$

We have used Eq. (A5) to obtain the form of the last term in Eq. (A22). When the label i of the active target nucleon is restored, one obtains Eq. (58).

We can now address the question of taking the limit inside the operator strings of Eq. (56). Consider first the second term of Eq. (56). The treatment of this term can be covered by following through the above derivation with the states $|i\rangle$ and $\langle f|$ taken to be

$$|i\rangle = (1-\Gamma_e\tau_e^{(i)})(1+\Gamma_e T_e)|\phi_e\vec{k}\rangle, \tag{A26}$$

$$\langle f| = \langle\vec{k}\phi_e|(T_e^\dagger\Gamma_e^\dagger+1)(1-\tau_e^{(i)\dagger}\Gamma_e^\dagger). \tag{A27}$$

The resulting contribution to unitarity will be just Eq. (A22) multiplied by the extra left and right factors. This is the result we have employed in Eq. (62). It is valid only if the extra operators introduced through use of the above states do not alter

the pole singularity structure we have already deduced for the integrand of Eq. (A8). The connectivity structure of the amplitudes in Eq. (A9) ensure this. Pole singularities in the momenta $\vec{k}_e, \vec{q}_e$ can be generated in Eq. (A9) only by having as a final state process an infinite series of repeated interactions internal to a disconnected subsystem.[18] Since the nucleons of the residual nucleus must end up in the eigenstate ϕ_R, the only possible interacting disconnected subsystems in the final state are those already described by t_e and t_p. The taking of the limit inside the second term of Eq. (56) is therefore a valid operation.

Consider now the third term of Eq. (56). Taking only one term of the sum, forming matrix elements, and applying the limit we have

$$\lim_{\epsilon\to 0}\frac{1}{2\pi i}\langle\vec{k}\phi_e|(T_e^\dagger\Gamma_e^\dagger+1)\tau_e^\dagger\Delta\Gamma_e\tau_e(1+\Gamma_eT_e)|\phi_e\vec{k}\rangle=\lim_{\epsilon\to 0}\frac{\epsilon}{\pi}\int d^3k_e\,\frac{\langle\vec{k},\phi_e|(T_e^\dagger\Gamma_e^\dagger+1)\tau_e^\dagger|\phi_e\vec{k}_e\rangle\langle\vec{k}_e\phi_e|\tau_e(1+\Gamma_eT_e)|\phi_e\vec{k}\rangle}{(E-i\epsilon-k_e^2-E_e)(E+i\epsilon-k_e^2-E_e)}\,, \tag{A28}$$

where particle labels are omitted. Since the operators τ_e and T_e end with an interaction of the projectile with a target nucleon, and the target remains in its ground state, there is no possibility of repeated interactions internal to a disconnected subsystem in the intermediate states. The only singularities arise from the denominators. Taking the limit by using Eq. (A16), the result reads (in operator form)

$$\lim_{\epsilon\to 0}\frac{1}{2\pi i}(T_e^\dagger\Gamma_e^\dagger+1)\tau_e^\dagger\Delta\Gamma_e\tau_e(1+\Gamma_eT_e)$$
$$=(T_e^\dagger\Gamma_e^\dagger+1)\tau_e^\dagger\Lambda_e\tau_e(1+\Gamma_eT_e)\,, \tag{A29}$$

which confirms the validity of taking the limit inside the operator strings. This concludes our proof of Eq. (62).

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