Subtraction techniques in three-particle scattering*

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A general description of subtraction techniques in scattering theory is given. Applications with the objective of achieving reduced integral equations with nonsingular kernels are applied to both the Faddeev and the Alt, Grassberger, and Sandhas forms of the three-particle equations with and without three-body forces. In the Faddeev case a method similar to one recently proposed by Karlsson is found which is analogous to well-known procedures in the two-particle case. This analogy is shown to be fairly weak when three-particle forces are included. In the absence of two-particle bound states phase-space type integral equations are found which permit the generation of approximate 3-to-3 amplitudes which satisfy unitarity and possess the correct representations of the double-scattering poles as well as the proper connectedness structure. A generalization of the so-called structure invariant perturbation theory to include three-body forces is established using the Alt, Grassberger, and Sandhas equations.

[NUCLEAR REACTIONS Scattering theory, three-particle scattering, unitary approximations.]

I. INTRODUCTION

Subtraction techniques appear repeatedly in a variety of guises throughout the literature on scattering integral equations. A typical member of the latter class can be represented formally as

$$T = B + KT . \tag{1.1}$$

Often the kernel K factors into the product of a term proportional to B and a propagator and this leads to special simplification. The kernel K can, however, always be decomposed into the sum of a reduced kernel K_R and an essentially arbitrary quantity K_S, namely,

$$K = K_R + K_S . \tag{1.2}$$

If we let Γ_R denote the solution of

$$\Gamma_R = 1 + K_R \Gamma_R , \tag{1.3}$$

then

$$T = \Gamma_R B + \Gamma_R K_S T , \tag{1.4}$$

at least formally. Equations (1.1)–(1.4) characterize what we refer to as a *subtraction technique.*

K_S is usually chosen to fulfill a dual role. It should be simple enough, or possess some special property, so that the exact solution of Eq. (1.4) can be regarded as well within one's computational capacities. Given this stipulation the solution of Eq. (1.3) is then usually more difficult. However, the second function of K_S is to represent some dominant feature(s) of the original kernel K so that either the solution of (1.3) is a somewhat simpler task than the direct consideration of (1.1) or it is feasible to contemplate approximate solutions.

In the extreme situation when K_S so nearly approximates K that K_R can be regarded as a very small perturbation it is often more efficient to exploit what might be called the two-potential realization of the subtraction procedure. This is obtained by interchanging the roles of K_R and K_S in Eq. (1.3) and Eq. (1.4) so that we have

$$\Gamma_S = 1 + K_S \Gamma_S$$

and

$$T = \Gamma_S B + \Gamma_S K_R T .$$

We emphasize that K_S retains its role of a "solvable" kernel while K_R continues to be regarded as somewhat intractable yet comparatively unimportant.

Karlsson[1] has recently introduced a subtraction technique for reducing the partial wave analyzed Faddeev equations for three-particle scattering to two systems of integral equations analogous to Eq. (1.3) and Eq. (1.4). One set, corresponding to (1.3), involves only two-variable equations albeit with nonsingular kernels, while the other set consists only of one-variable equations. The off-shell structure of the three-particle amplitudes is inferred from these equations. Representations of both the on- and off-shell amplitudes are obtained in terms of functions which have no discontinuities across the three-particle unitary cut although they may possess discontinuities with respect to the branch cuts generated by any possible two-particle bound-state poles.

This is reminiscent of those techniques which

have been used to deal with the simpler problem of removing the fixed-point singularities which appear in the kernels of two-particle scattering integral equations.[2] In both cases K_S is chosen so that K_R is nonsingular. In the two-particle case the class of kernels considered in Ref. 2 are such that the partial wave amplitudes of the kernel of Eq. (1.4) are separable. Similarly, in Ref. 1 Eq. (1.4) reduces to a set of one-dimensional equations.

These types of subtraction techniques have proven useful in the two-particle case in primarily two respects. First, the reduction to a real nonsingular integral equation simplifies the numerical work in computing the off-shell and on-shell scattering amplitudes. Second, the explication of the off-shell structure afforded by the representation in terms of real amplitudes has served to suggest approximations for those two-body off-shell amplitudes which are used as input into multiparticle scattering integral equations.[3] The possibility of establishing similar attributes for on- and off-shell multiparticle scattering amplitudes would appear to be of considerable interest and this provides the primary motivation for the present investigation. In this regard, we recall that in many formulations of N-particle scattering the input into the scattering integral equations consists of the off-shell N_L-particle amplitudes, where $2 \leq N_L \leq N-1$.

In Sec. II we consider a simple modification of the subtraction technique proposed in Ref. 1 which enhances the resemblance to the two-particle case even further. The analysis is confined solely to the physical situation considered in Ref. 1, namely, when the interaction is considered to consist only of pairwise forces.

The development of similar techniques in Sec. III but including a three-particle potential shows that the presence of such a force destroys the analogy with the two-particle case rather thoroughly. We find that the direct generalization of the method introduced in Sec. II is a variant of a subtraction technique introduced by Brayshaw.[4] Another technique which is somewhat more convenient for studying those amplitudes which correspond to two-particle channels is also introduced. The latter method turns out to be the generalization of the so-called structure-invariant perturbation theory.[5]

II. PAIRWISE INTERACTIONS

In the absence of three-body forces the Faddeev equations for three-particle scattering can be written in the form

$$M_{\beta\alpha}(E+) = t_\beta(E+)\delta_{\beta\alpha} + t_\beta(E+)\sum_{\lambda=1}^{3}\bar{\delta}_{\beta\lambda}G_0(E+)M_{\lambda\alpha}(E+) \tag{2.1a}$$

$$= t_\beta(E+)\delta_{\beta\alpha} + \sum_{\lambda=1}^{3} M_{\beta\lambda}(E+)G_0(E+)\bar{\delta}_{\lambda\alpha}t_\alpha(E+) \tag{2.1b}$$

which we take to be defined upon the space of zero total linear momentum. When the total three-particle cm energy satisfies $E>0$, which is the physical domain of 3-to-3 scattering, the kernels of the integral equations (2.1) are singular due to the vanishing of the denominator of the on-shell matrix elements of the propagator

$$G_0(E+) = (E - H_0 + i0)^{-1}.$$

H_0 denotes the three-body kinetic energy operator excluding the energy of the center of mass. Branch cut singularities may also be generated by the possible bound-state poles of the two-particle transition operators $t_\alpha(E+)$. The index α on the latter operators refers to the noninteracting particle and in Eqs. (2.1) $\bar{\delta}_{\alpha\beta} = 1 - \delta_{\alpha\beta}$.

We are concerned solely with those subtraction procedures which result in a reduced kernel free of the propagator singularity. The properties of $t_\alpha(z)$, where z is a complex parametric energy, are very important pursuant to this end. In Ref. 5 it is shown that $t_\alpha(z)$ can always be decomposed into its so-called essential (e) and residual (r) parts

$$t_\alpha(z) = t^e_\alpha(z) + t^r_\alpha(z). \tag{2.2}$$

These operators possess the following properties pertinent to the present investigation with respect to the appropriate two-particle subspaces:
(i) $t^e_\alpha(z)$ contains the complete contribution of the possible bound-state poles of $t_\alpha(z)$;
(ii) both $t^e_\alpha(z)$ and $t^r_\alpha(z)$ are, separately, bounded on the cut in z from $z=0$ to $z=+\infty$ and for negative z as well except in the region of the (possible) bound-state poles in the case of $t^e_\alpha(z)$;
(iii) $t^{e,r}_\alpha(\pm) = t^{e,r}_\alpha(\mp)^\dagger$, where $\pm$ refers to $z = E \pm i0$, E real;
(iv) $\Delta t^e_\alpha \equiv t^e_\alpha(+) - t^e_\alpha(-) = \Delta t_\alpha$ for all real E;
(v) $t^r_\alpha(z) = G_0(z)^{-1}\tau(z)G_0(z)^{-1}$, where $\tau(z)$ is nonsingular for real z and $\Delta t^r = \Delta\tau = 0$;
(vi) the partial wave projections of $t^e_\alpha(z)$ can always be regarded as being of finite rank.

Let us for the moment ignore any possible spin-orbit coupling. Then the partial wave projection of a half-shell partial wave amplitude can be written in the factorized form

$$t_l\left(k', k; \frac{k^2}{2m}\right) = f_l(k', k)t_l\left(k, k; \frac{k^2}{2m}\right), \tag{2.3}$$

where, as a consequence of off-shell unitarity, $f_l(k', k)$ is real. If $f_l(k', k)$ is nonsingular, which

is the case if $t_l(k,k;k^2/2m)\neq 0$, then for positive parametric energies one can take

$$t_l^e\left(k'',k';\frac{k^2}{2m}\right)=f_l(k'',k)t_l\left(k,k;\frac{k^2}{2m}\right)f_l(k',k). \tag{2.4}$$

Next we state these properties of the two-particle operators in terms of the operators defined on the full three-particle space. Let $|E,\eta\rangle$ be any positive energy state for which $G_0(E+)^{-1}|E,\eta\rangle=0$; η refers to any other variables needed to specify this state. Then from property (v) we infer that

$$t_\alpha^r(E+)|E,\eta\rangle=\langle E,\eta|t_\alpha^r(E+)=0 \tag{2.5}$$

which in turn implies the half-shell identities

$$t_\alpha(E+)|E,\eta\rangle=t_\alpha^e(E+)|E,\eta\rangle, \tag{2.6a}$$

$$\langle E,\eta|t_\alpha(E+)=\langle E,\eta|t_\alpha^e(E+). \tag{2.6b}$$

Let $\vec{p}_\alpha$ denote the momentum of particle α relative to the c.m. of particles β,γ, while the relative momentum of the latter two is denoted by $\vec{q}_\alpha$. Three convenient choices of orthonormal bases on the $(\vec{P}=0)$ three-particle Hilbert space are $\{|\vec{q}_\alpha,\vec{p}_\alpha\rangle_\alpha\}$, $\alpha=1,2,3$. Then the matrix elements of $t_\alpha(E+)$ are

$$\begin{aligned}{}_\alpha\langle\vec{q}_\alpha',\vec{p}_\alpha'|t_\alpha(E+)|\vec{q}_\alpha,\vec{p}_\alpha\rangle_\alpha&=\delta(\vec{p}_\alpha'-\vec{p}_\alpha)\langle\vec{q}_\alpha'|\hat{t}_\alpha(E-\vec{p}_\alpha{}^2/2\mu^\alpha+i0)|\vec{q}_\alpha\rangle.\end{aligned} \tag{2.7}$$

It is convenient to define an energy shell projector

$$\mathcal{S}_\alpha(E)\equiv\int d\vec{q}_\alpha\int d\vec{p}_\alpha|\vec{q}_\alpha,\vec{p}_\alpha\rangle_\alpha\frac{\delta(q_\alpha-\sqrt{s_\alpha})}{s_\alpha}\times\theta(s_\alpha){}_\alpha\langle\vec{q}_\alpha,\vec{p}_\alpha|, \tag{2.8}$$

where

$$s_\alpha=2\mu_\alpha\left(E-\frac{1}{2\mu^\alpha}\vec{p}_\alpha{}^2\right),$$

and

$$\mu_\alpha=m_\beta m_\gamma(m_\beta+m_\gamma)^{-1},$$

$$\mu^\alpha=m_\alpha[(m_\beta+m_\gamma)/M].$$

A positive parametric energy projector $\mathcal{S}^{(+)}(E)$ with respect to $t_\alpha(E+)$, cf. Eq. (2.7), is obtained from Eq. (2.8) by replacing $(s_\alpha)^{-1}\delta(q_\alpha-\sqrt{s_\alpha})$ by unity.

The generalization of Eq. (2.3) on the three-particle space is then

$$t_\alpha(E+)|E,\eta\rangle=f_\alpha\mathcal{S}_\alpha(E)t_\alpha(E+)|E,\eta\rangle, \tag{2.9}$$

where f_α is the operator representation of the half-shell function on the three-particle space. In the special case of no spin-orbit coupling the matrix elements of f_α are simply

$${}_\alpha\langle\vec{q}_\alpha',\vec{p}_\alpha'|f_\alpha|\vec{q}_\alpha,\vec{p}_\alpha\rangle_\alpha=\delta(\vec{p}_\alpha'-\vec{p}_\alpha)\sum_{l,m}f_l^\alpha(q_\alpha',q_\alpha)Y_l^m(\vec{q}_\alpha',\vec{\xi})Y_l^m(\vec{q}_\alpha,\vec{\xi})^*.$$

In general the half-shell function which is diagonal with respect to the total relative two-particle angular momentum is a real matrix in the spin indices.[6] Finally, we suppose that Eq. (2.4) holds so that

$$t_\alpha^e(E+)\mathcal{S}_\alpha^{(+)}(E)=f_\alpha\mathcal{S}_\alpha(E+)t_\alpha(E+) \tag{2.10a}$$

$$=t_\alpha(E+)\mathcal{S}_\alpha(E+)f_\alpha^\dagger. \tag{2.10b}$$

This is legitimate so long as f_α is well defined which we suppose to be the case for manipulative purposes. Those instances when f_α possesses poles are best treated as limiting cases in the final equations.

Again for manipulative convenience we write Eqs. (2.1) is an obvious matrix notation where we also suppress the dependence upon E:

$$M=t+t\bar{\delta}G_0M \tag{2.1a'}$$

$$=t+MG_0\bar{\delta}t. \tag{2.1b'}$$

Then in the notation of Sec. I we take

$$K_S=f\mathcal{S}t\bar{\delta}G_0 \tag{2.11a}$$

so that

$$K_R=t^R\bar{\delta}G_0, \tag{2.11b}$$

where

$$t^R=t-f\mathcal{S}t,$$

and we see as a consequence of Eq. (2.9) that the propagator singularity in K_R has been neutralized. Other choices for K_S achieve this aim as well, e.g., $t^e\bar{\delta}G_0$ or $t^e\mathcal{S}^{(+)}\bar{\delta}G_0$ in the case Eqs. (2.10) are not valid. Corresponding to Eq. (1.4) we have with the kernels (2.11)

$$M=\Gamma_Rt+\Gamma_Rf\mathcal{S}t\bar{\delta}G_0M, \tag{2.12}$$

where Γ_R is defined by Eq. (1.3) and Eq. (2.11b) and we note that in this instance $\mathcal{S}\Gamma_R=\mathcal{S}$. So if we multiply (2.12) on the left by $\Gamma_Rf\mathcal{S}$ we obtain

$$\Gamma_Rf\mathcal{S}M=\Gamma_Rf\mathcal{S}t+\Gamma_Rf\mathcal{S}t\bar{\delta}G_0M$$

which when subtracted from (2.12) yields

$$M=\Gamma_Rf\mathcal{S}M+\Gamma_Rt^R. \tag{2.13}$$

If we introduce a half-shell quantity F defined by

$$F\mathcal{S}\equiv\Gamma_Rf\mathcal{S} \tag{2.14}$$

and the resolvent $\mathcal{R}=\Gamma_R-1$ corresponding to K_R, Eq. (2.13) can be rewritten as

$$M=F\mathcal{S}M+\mathcal{R}(\bar{\delta}G_0)^{-1}. \tag{2.15}$$

Half on-shell (2.15) becomes

$$M|E,\eta\rangle = F\mathcal{S}M|E,\eta\rangle . \tag{2.16}$$

By definition (2.14) $F\mathcal{S}$ possesses at most discontinuities across the branch cuts generated by any possible two-particle bound-state poles. In the absence of the latter F is real and (2.16) is an obvious solution of the half-shell form of the general discontinuity relation satisfied by M,

$$\Delta M = M(\pm)(1+\bar{\delta})\Delta G_0 M(\mp) + [M(\pm)G_0(\pm)\bar{\delta}+1]\Delta t_b[1+\bar{\delta}G_0(\mp)M(\mp)], \tag{2.17}$$

provided only that F satisfies the on-shell condition

$$\mathcal{S}F\mathcal{S} = \mathcal{S} \tag{2.18a}$$

or, equivalently,

$$\langle E,\eta'|F_{\beta\alpha}(E+)|E,\eta\rangle = \delta_{\beta\alpha}\langle E,\eta'|E,\eta\rangle . \tag{2.18b}$$

The sum of the discontinuities of t across any possible bound-state poles is denoted by Δt_b and, of course, in the absence of these poles $\Delta t_b = 0$. Even in the latter circumstance (2.16) is remarkable in that it is not a form which is necessarily implied by (2.17) as we shall see in Sec. III. On the other hand, if $\Delta t_b \neq 0$ the restrictions implied by (2.17) on F are certainly much more complex. In either case, by (2.14) the half-shell function satisfies the nonsingular integral equation

$$F_{\beta\alpha}(E+)|E,\eta\rangle = \delta_{\beta\alpha}f_\alpha(E)|E,\eta\rangle + \sum_{\lambda=1}^{3}[K_R(E+)]_{\beta\lambda}F_{\lambda\alpha}(E+)|E,\eta\rangle . \tag{2.19}$$

As in the analogous two-particle case the on-shell matrix elements of M can be expressed in terms of F. Inserting (2.16) into the kernel term of the on-shell version of (2.1a′) we obtain

$$(\mathcal{S}M\mathcal{S}) = \mathcal{S}t^e\mathcal{S} + \mathcal{S}t^e\bar{\delta}G_0F(\mathcal{S}M\mathcal{S}) \tag{2.20}$$

which is an integral equation for the on-shell matrix elements corresponding to 3-to-3 scattering. It is instructive to express (2.20) in an explicit form in terms of the canonical Faddeev amplitudes $T = M(1+\bar{\delta})$ with respect to an arbitrary initial state $|E,\eta\rangle$:

$$\begin{aligned}{}_\beta\langle \vec{q}'_\beta, \vec{p}'_\beta|\mathcal{S}_\beta(E)T_\beta(E+)|E,\eta\rangle &= {}_\beta\langle \vec{q}'_\beta, \vec{p}'_\beta|\mathcal{S}_\beta(E)t_\beta(E+)|E,\eta\rangle \\ &+ \sum_{\gamma=1}^{3}\int d\vec{q}''_\gamma \int d\vec{p}''_\gamma\, {}_\beta\langle \vec{q}'_\beta, \vec{p}'_\beta|A_{\beta\gamma}(E+)|\vec{q}''_\gamma, \vec{p}''_\gamma\rangle \\ &\times \frac{\delta(q''_\gamma - \sqrt{s''_\gamma})}{s''_\gamma}\,\theta(s''_\gamma)\,{}_\gamma\langle \vec{q}''_\gamma, \vec{p}''_\gamma|T_\gamma(E+)|E,\eta\rangle ,\end{aligned} \tag{2.21}$$

where

$$A_{\beta\gamma}(E+) = \sum_{\lambda=1}^{3}\mathcal{S}_\beta(E)t^e_\beta(E+)\bar{\delta}_{\beta\lambda}G_0(E+)F_{\lambda\gamma}(E+) .$$

Equations (2.21) are a set of phase-space type of integral equations which reduce after a partial wave decomposition to a set of one-dimensional integral equations with finite domains of integration. Finally, we remark that since $\mathcal{S}_\beta(E)t^e_\beta(E+) = \mathcal{S}_\beta(E)t_\beta(E+)\mathcal{S}_\beta(E+)f^\dagger_\beta$ is always true, given the on-shell two-particle amplitudes, any real two-particle half-shell function satisfying $\mathcal{S}f\mathcal{S} = \mathcal{S}$, and any real, connected F_C such that $F\mathcal{S} = (f + F_C)\mathcal{S}$ satisfies (2.18), Eqs. (2.21) generate, provided that $\Delta t_b = 0$, a unitary, properly connected 3-to-3 scattering amplitude which possesses the correct representations of the double-scattering poles.

Equation (2.1b′), which we rewrite as

$$M = t + MK^{tr} ,$$

can be analyzed in a similar manner. In this instance we choose

$$K^{tr}_S = G_0\bar{\delta}t\mathcal{S}f^\dagger$$

so that

$$K^{tr}_R = G_0\bar{\delta}t^R .$$

Corresponding to (2.15) one finds that

$$M = M\mathcal{S}F^{tr} + (\bar{\delta}G_0)^{-1}\mathcal{R}^{tr} , \tag{2.22}$$

where

$$\mathcal{S}F^{tr} = \mathcal{S}f^\dagger\Gamma^{tr}_R ,$$

Γ^{tr}_R is defined by (1.3) with K_R replaced by K^{tr}_R, and $\mathcal{R}^{tr}$ is the resolvent corresponding to the latter kernel. F^{tr} is evidently subject to the constraint

$$\mathcal{S}F^{tr}\mathcal{S} = \mathcal{S} .$$

Noting that

$$\mathcal{R}^R = (\bar{\delta}G_0)^{-1}\mathcal{R} = \mathcal{R}^{tr}(\bar{\delta}G_0)^{-1} ,$$

where $\mathcal{R}^R$ satisfies

$$\begin{aligned}\mathcal{R}^R &= \bar{\delta}G_0t^R\bar{\delta}G_0 + \bar{\delta}G_0t^R\mathcal{R}^R \\ &= \bar{\delta}G_0t^R\bar{\delta}G_0 + \mathcal{R}^Rt^RG_0\bar{\delta} ,\end{aligned}$$

we can combine Eqs. (2.15) and (2.22) into the single symmetrical expression

$$M = F\mathcal{S}M\mathcal{S}F^{tr} + (\bar{\delta}G_0)^{-1}\mathcal{R}^R(\bar{\delta}G_0)^{-1} \tag{2.23}$$

Equation (2.23) can be regarded as the three-particle generalization of an analogous expression for the two-particle transition operator.[5] When $\Delta t_b = 0$ the resemblance is even more striking for then (2.23) constitutes a representation for M which manifestly satisfies the off-shell unitarity relation (2.17) when F, F^{tr}, and $\mathcal{R}^R$ are real.

Our principal results which consist of Eqs. (2.19), (2.21), and (2.23) have their formal counterparts in Ref. 1 in terms of somewhat differently defined quantities. The equations in Ref. 1 corresponding to Eqs. (2.21) involve the partial wave amplitudes of ${}_\beta\langle \vec{q}'_\beta, \vec{p}'_\beta | T_\beta(E+) | E, \eta\rangle$, where the magnitude of $\vec{q}_\beta$ is again taken to be equal to $\sqrt{s'_\beta}$ but for the extended range $-\infty < s'_\beta < 2\mu_\beta E$. For $s'_\beta > 0$, these amplitudes coincide with the physical on-shell amplitudes; however, for $s'_\beta < 0$ one has introduced an analytic continuation of the amplitudes ordinarily defined by Eqs. (2.1).[7]

The continuation is defined in Ref. 1 in the course of executing the subtraction technique. Instead of (2.11a) one chooses

$$K_S = f^c t^c \bar{\delta} G_0 .$$

Here f^c involves an analytic continuation of the two-particle half-shell function. For instance, the partial wave amplitude of ${}_\alpha\langle \vec{q}'_\alpha, \vec{p}'_\alpha | f^c_\alpha | \vec{q}_\alpha, \vec{p}_\alpha\rangle_\alpha$ is defined as $f^\alpha_l(q'_\alpha, \sqrt{s_\alpha})$ for all $-\infty < s_\alpha < 2\mu_\alpha E$. $t^c_\alpha(E+)$ differs from $t_\alpha(E+)$ only in that the former is obtained from the latter by replacing the magnitude of $\vec{q}'_\alpha$ by $\sqrt{s'_\alpha}$ where it enters into the partial wave amplitudes of ${}_\alpha\langle \vec{q}'_\alpha, \vec{p}'_\alpha | t_\alpha(E+) | \vec{q}_\alpha, \vec{p}_\alpha\rangle_\alpha$. Assuming that these (two-particle) continuations have been well defined the same manipulations which led to (2.16) can be carried out only now the definitions of F and the effective meaning of $\mathcal{S}_\alpha$ are changed in an obvious way. Similar remarks apply to Eq. (2.23) and to Eqs. (2.21). The latter still reduce to a set of one-dimensional integral equations but now possess infinite rather than finite domains of integration. Rather more information is implied by these counterparts of Eqs. (2.21) than is needed to determine the physical 3-to-3 scattering amplitudes.

The exemplar of all subtraction techniques in three-particle scattering theory is that introduced by Alt, Grassberger, and Sandhas[8] in connection with their formalism which is phrased in terms of the operators

$$U_{\beta\alpha}(E+) = \bar{\delta}_{\beta\alpha} G_0(E+)^{-1} + \sum_{\lambda,\gamma=1}^{3} \bar{\delta}_{\beta\lambda} M_{\lambda\gamma} \bar{\delta}_{\gamma\alpha} . \tag{2.24}$$

In contrast to the $M_{\beta\alpha}$ the matrix elements of the preceding operators are directly related to the scattering amplitude for the processes $\alpha \to \beta$ for $\alpha, \beta = 0, 1, 2, 3$, where the zero index refers to the channel with three free particles. In connection with this Eqs. (2.1) are also defined (trivially) for zero indices with the stipulation that $t_0 \equiv 0$. The combination of Eq. (2.23) and Eq. (2.24) yields the general, although apparently uninteresting, representation

$$U = \bar{\delta} G_0^{-1} + \bar{\delta} F \mathcal{S} M \mathcal{S} F^{tr} \bar{\delta} + G_0^{-1} \mathcal{R}^R G_0^{-1} ,$$

while from Eqs. (2.1) we infer the integral equation

$$U = \bar{\delta} G_0^{-1} + \bar{\delta} t G_0 U . \tag{2.25}$$

The Alt *et al.* class of subtraction techniques is defined by decomposing t into two parts. The decomposition (2.2) corresponding to which we take $K_S = \bar{\delta} t^e G_0$ yields as the realization of Eq. (1.4)

$$U = U^r + U^r G_0 t^e G_0 U , \tag{2.26a}$$

where

$$U^r = \bar{\delta} G_0^{-1} + \bar{\delta} t^r G_0 U^r \tag{2.26b}$$

is an integral equation with a nonsingular kernel which possesses no discontinuities across any of the unitary cuts whether or not there are any two-particle bound states. Equation (2.26a) reduces to a set of one-dimensional integral equations for the so-called vertex matrix elements of $U_{\beta\alpha}$ after a partial wave decomposition in virtue of property (vi) of t^e; arbitrary matrix elements of $U_{\beta\alpha}$ can be determined by using (2.26a) as a quadrature rule. Clearly this subtraction technique is well defined whether the half-shell functions are or not. Equations (2.26) form the basis of the so-called structure invariant perturbation theory.[5] The generalization of Eqs. (2.26), and therefore of all of the associated remarks, to the case where three-particle forces are included is considered in Sec. III.

It is instructive to restate Eqs. (2.26) in terms of the M operators. This statement is facilitated by the definition of Z^r as the solution of

$$Z^r = 1 + t^r \bar{\delta} G_0 Z^r \tag{2.27}$$

so that

$$U^r = \bar{\delta} G_0^{-1} + \bar{\delta} Z^r t^r \bar{\delta} .$$

Then one finds that M satisfies

$$M = Z^r t + Z^r t^e \bar{\delta} G_0 M . \tag{2.28}$$

Thus the subtraction technique introduced by Brayshaw[4] is merely the M-operator realization of Eqs. (2.26) at least in this instance where there are no three-particle forces.

Finally, if instead of the decomposition defined by (2.2) we take in Eqs. (2.26) $f \delta t$ in place of t^e so that t^r is replaced by t^R we obtain the U-operator realization of previous the subtraction technique which culminated in Eq. (2.23). The equivalence of the two sets of equations is provided by (2.28) in which Z^r is replaced by Γ_R and t^e by $f \delta t$. Although these altered forms of Eqs. (2.26) are not especially useful for practical calculation they do illustrate the point that any subtraction involving a decomposition of the two-particle transition operators is very efficiently formulated in terms of the Alt *et al*. equations.

III. GENERAL THREE-PARTICLE INTERACTIONS

We next suppose that the total interaction consists of a three-particle potential V_4 in addition to the pair potentials V_α, $\alpha = 0, 1, 2, 3$ with $V_0 \equiv 0$. Whether or not $V_4 \neq 0$ the Alt *et al.*[8] operators are defined quite generally in terms of the total Green function $G(E+) = (E - H + i0)^{-1}$ by means of the representation

$$G(E+) = \delta_{\beta\alpha} G_\alpha(E+) + G_\beta(E+)\, U_{\beta\alpha}(E+) G_\alpha(E+), \quad (3.1)$$

where the α-channel Green function is denoted by $G_\alpha(E+) = (E - H_0 - V_\alpha + i0)^{-1}$. Equations (3.1) ensure the direct relationship between the channel matrix elements of $U_{\beta\alpha}$ and the scattering amplitudes for the processes $\alpha \to \beta$. We remark that for manipulative purposes Eqs. (3.1) are most efficiently utilized when expressed in the (four-dimensional) matrix form

$$G(1 + \bar{\delta}) = \hat{G} + \hat{G} U \hat{G}, \quad (3.1')$$

where $(\hat{G})_{\beta\alpha} = \delta_{\beta\alpha} G_\alpha$.

It is a simple matter to deduce the integral equations satisfied by U from (3.1′) and the resolvent identities for G. One finds

$$U = \bar{\delta} G_0^{-1} + V_4 + [\bar{\delta} t + V_4(1 + G_0 t)] G_0 U \quad (3.2a)$$

$$= \bar{\delta} G_0^{-1} + V_4 + U G_0 [(t G_0 + 1) V_4 + t \bar{\delta}]. \quad (3.2b)$$

These equations differ slightly from equations for U derived elsewhere.[9] However the two sets can easily be placed into congruence by means of the identities

$$G_\beta U_{\beta\alpha} = \bar{\delta}_{\beta\alpha} + G_\alpha U_{\alpha\alpha},$$

$$U_{\beta\alpha} G_\alpha = \bar{\delta}_{\beta\alpha} + U_{\beta\beta} G_\beta$$

which follow from the manifest channel degeneracy of Eqs. (3.1). We point out that Eqs. (3.2) form a closed set with respect to the nonzero-indexed operators. That is, a submatrix of Eq. (3.2a), for example, consists of a set of equations which couple together only those $U_{\beta\alpha}$ for $\alpha, \beta \neq 0$. The fact that Eqs. (3.2) can be regarded as closed in this sense is a consequence of the fact that all of the sums over indices are weighted by t_0 and therefore receive no contribution from the $t_0 = 0$ terms. As in the case of only pairwise forces one finds that the zero-index operators can be determined from the nonzero-index operators by using Eqs. (3.2) as quadrature rules.

The $M_{\beta\alpha}$ operators can be introduced just as before [cf. Eq. (2.24)], namely

$$U = \bar{\delta} G_0^{-1} + \bar{\delta} M \bar{\delta}, \quad (3.3)$$

except that the operators $M_{0\alpha}$ and $M_{\beta 0}$ are not necessarily trivial as they were in Sec. II. The integral equations for the M operators are obtained by inserting the expression (3.3) for U into Eqs. (3.2) and using the fact that $\bar{\delta}^{-1} = (\bar{\delta} - 2)/3$ in four dimensions. After some manipulation it is found that all zero-indexed operators can be eliminated from the sums over indices appearing in these equations with the result that the M operators satisfy for all $\alpha, \beta = 0, 1, 2, 3$,

$$M_{\beta\alpha} = t_\beta \delta_{\beta\alpha} + (1 + t_\beta G_0) \frac{V_4}{3} \left(\frac{1}{3} + G_0 \sum_{\lambda=1}^{3} M_{\lambda\alpha} \right) + t_\beta G_0 \sum_{\lambda=1}^{3} \bar{\delta}_{\beta\lambda} M_{\lambda\alpha} \quad (3.4b)$$

$$= t_\beta \delta_{\beta\alpha} + \left(\frac{1}{3} + \sum_{\lambda=1}^{3} M_{\beta\lambda} G_0 \right) \frac{V_4}{3} (1 + G_0 t_\alpha) + \sum_{\lambda=1}^{3} M_{\beta\lambda} \bar{\delta}_{\lambda\alpha} G_0 t_\alpha. \quad (3.4b)$$

We see that if $\alpha, \beta \neq 0$ Eqs. (3.4) couple only nonzero-index operators together. The latter can be used to determine the zero-index operators by utilizing Eqs. (3.4) as quadrature rules. Only the operators $M_{\beta\alpha}$ for $\alpha, \beta \neq 0$ enter into the statement of the 3-to-3 scattering amplitude but it is clear from (3.3) that $M_{0\alpha}$ and $M_{\beta 0}$ are needed to describe every other scattering process.

The preceding results indicate that just as in Sec. II the M operators are particularly useful for the description of the 3-to-3 process. On the other hand, the U operators lend themselves more readily to the characterization of scattering processes with no more than one channel with three free particles at least within the context of the subtraction techniques considered here. The generalization of these techniques to the case at hand follows along the lines of Sec. II. However, in order to avoid notational confusion we segregate the description of this development from the set of equations introduced thus far in this section.

A. M operators

In this subsection we confine ourselves solely to the $M_{\beta\alpha}$ operators defined by Eqs. (3.4) for

$\alpha, \beta \neq 0$. All equations which are written in a matrix notation in the channel indices refer to this 3×3 channel space. This opportunity is presented to us because of the closure properties of Eqs. (3.4) which were referred to previously. If we write

$$\mathcal{V} \equiv \frac{V_4}{3}(1+\bar{\delta}),$$

then Eqs. (3.4) can be written in the compact matrix form

$$M = t + (1+tG_0)\mathcal{V}(\tfrac{1}{3}+G_0M) + tG_0\bar{\delta}M \quad , \tag{3.5a}$$

$$= t + (\tfrac{1}{3}+MG_0)\mathcal{V}(1+G_0t) + M\bar{\delta}G_0t \, . \tag{3.5b}$$

The generalization of the discontinuity relation (2.17) can be deduced from Eqs. (3.5):

$$\begin{aligned}\Delta M = {} & M(\pm)(1+\bar{\delta})\Delta G_0 M(\mp) \\ & + \{M(\pm)G_0(\pm)[\mathcal{V}G_0(\pm)+\bar{\delta}] + [\mathcal{V}G_0(\pm)/3+1]\} \\ & \times \Delta t_b\{[1+G_0(\mp)(\mathcal{V}/3)] + [\bar{\delta}+G_0(\mp)\mathcal{V}]G_0(\mp)M(\mp)\} \, .\end{aligned} \tag{3.6}$$

However, Eqs. (2.17) and (3.6) are identical provided only that $\Delta t_b = 0$, namely, that there exist no two-particle bound states, viz.,

$$\Delta M = M(\pm)(1+\bar{\delta})\Delta G_0 M(\mp), \quad \Delta t_b = 0 \, . \tag{3.6'}$$

Let us next apply the same sort of analysis which led to Eq. (2.15) in the case $V_4 = 0$. In order to define an appropriate K_S when the latter condition is not true we employ a device very similar to that introduced by Brayshaw[4] in that we decompose $\mathcal{V}$ into subtracted ($\mathcal{V}_S$) and remainder ($\mathcal{V}_R$) parts:

$$\mathcal{V} = \mathcal{V}_S + \mathcal{V}_R \, , \tag{3.7a}$$

where

$$\mathcal{V}_S \equiv f\mathcal{S}\mathcal{V} \, , \tag{3.7b}$$

so that $\mathcal{S}\mathcal{V}_R = 0$. Also we write

$$t = t^S + t^R \, , \tag{3.8a}$$

where

$$t^S = f\mathcal{S}t = t^e\mathcal{S}^{(+)} \, . \tag{3.8b}$$

Then corresponding to Eqs. (2.11) and to the notation of Sec. I we define

$$K_{S,R} \equiv [t^{S,R}\bar{\delta} + \mathcal{V}_{S,R} + t^{S,R}G_0\mathcal{V}]G_0 \tag{3.9a}$$

$$\equiv \hat{K}_{S,R}G_0 \, . \tag{3.9b}$$

Evidently $\mathcal{S}K_S = K_S$ and $\mathcal{S}K_R = 0$ but $\hat{K}_R\mathcal{S} \neq 0$ as a consequence of the $\mathcal{V}$ terms so that K_R is still a singular kernel. Again exploiting the notation of Sec. I and our decompositions for t and $\mathcal{V}$ we see that the Born term in Eqs. (3.5) can be written as

$$B = B_S + B_R \, ,$$

where

$$B_{S,R} = t^{S,R} + \tfrac{1}{3}\mathcal{V}_{S,R} + \tfrac{1}{3}t^{S,R}G_0\mathcal{V} \, .$$

Then corresponding to (1.4) and (2.12) we have

$$M = \Gamma_R B + \Gamma_R K_S M \, , \tag{3.10}$$

where Γ_R is defined by Eq. (1.3) and Eqs. (3.9). Multiplication of (3.10) on the left by $\Gamma_R f\mathcal{S}$ yields

$$\Gamma_R f\mathcal{S}M = \Gamma_R f\mathcal{S}B + \Gamma_R K_S M$$

which then subtracted from (3.10) yields

$$M = \Gamma_R f\mathcal{S}M + \Gamma_R B_R \, . \tag{3.11}$$

Equation (3.11) differs from (2.13), or equivalently (2.15), in two important respects. First, due to the presence of the three-particle potential K_R is still singular and therefore $\Gamma_R f\mathcal{S}$ is not a real operator. Second, while $\mathcal{S}\Gamma_R B_R = \mathcal{S}B_R = 0$ we have $\Gamma_R B_R \mathcal{S} \neq 0$ so that the second term on the right-hand side of (3.11) does not vanish half on-shell. This is, of course, correlated with the fact that $\Gamma_R f\mathcal{S}$ is not a real operator in general.

Nonetheless, (3.11) does simplify considerably half on-shell. In fact we find that corresponding to (2.16)

$$\begin{aligned}M|E,\eta\rangle = {} & G_0^{-1}\tilde{\Gamma}_R G_0 f\mathcal{S}M|E,\eta\rangle \\ & + (G_0^{-1}/3)(\tilde{\Gamma}_R - 1)|E,\eta\rangle \, ,\end{aligned} \tag{3.12}$$

where

$$\begin{aligned}\tilde{\Gamma}_R &= G_0\Gamma_R G_0^{-1} \\ &= 1 + (G_0\hat{K}_R)\tilde{\Gamma}_R \, .\end{aligned}$$

We note that the kernel $G_0\hat{K}_R$ is nonsingular so that $\tilde{\Gamma}_R$ is a real operator.

Let us write

$$\tilde{\Gamma}_R = 1 + A_C \, , \tag{3.13a}$$

where A_C is real, connected, and satisfies

$$A_C\mathcal{S} = \tfrac{1}{3}A_C(1+\bar{\delta})\mathcal{S} \, . \tag{3.13b}$$

Then for any such A_C and any two-particle half-shell function f such that $\mathcal{S}f\mathcal{S} = \mathcal{S}$, one easily establishes that when $\Delta t_b = 0$ Eq. (3.12) provides a solution of the half-shell form of the discontinuity relation (3.6').

We can also formulate a unitary on-shell formalism for the 3-to-3 amplitude when $\Delta t_b = 0$ in terms of a generalization of Eq. (2.20). That is, if we use (3.12) in the on-shell form of (3.5a) we obtain

$$\begin{aligned}(\mathcal{S}M\mathcal{S}) = {} & \mathcal{S}\{t^e(1-\tfrac{1}{3}\bar{\delta}) + \tfrac{1}{3}[t^e\bar{\delta} + (1+t^eG_0)\mathcal{V}]\tilde{\Gamma}_R\}\mathcal{S} \\ & + \mathcal{S}\{[t^e\bar{\delta} + (1+t^eG_0)]\tilde{\Gamma}_R G_0 f\}(\mathcal{S}M\mathcal{S}) \, .\end{aligned} \tag{3.14}$$

It is obvious that (3.14) reduces to (2.20) in the limit $V_4 = 0$. In the case $\Delta t_b = 0$ given any $\tilde{\Gamma}_R$ satisfying Eqs. (3.13) and any f such that $\mathcal{S}f\mathcal{S} = \mathcal{S}$ one

easily proves that the solution of the phase-space integral equations (3.14), which can be rewritten in a form similar to (2.21), corresponds to a unitary, properly connected 3-to-3 amplitude which possesses the correct representation of the double-scattering pole.

The representation (3.12) of the general three-particle half-shell amplitude demonstrates that the presence of three-body forces destroys the similarity to the two-particle case which was exhibited by the representations (2.15), (2.16), and (2.23) when $V_4 = 0$. The reason for this is the lack of any intrinsic factorization properties associated with an amplitude generated by a pure three-particle force or which are implied by three-body unitarity. This is best seen by examining Eqs. (3.5)–(3.14) in the limit of no pairwise forces, namely $t_\alpha = 0$, for all α.

The representations of the two-particle amplitudes t which resemble (2.15), (2.16), and (2.23) are direct consequences of two-particle unitarity. As we have seen in Sec. II these intrinsic factorization properties persist in their influence upon the structure of the three-particle amplitudes when $V_4 = 0$. By way of contrast, the connected three-body operator

$$t_4 \equiv V_4 + V_4(E - H_0 - V_4 + i0)^{-1} V_4$$

has no intrinsic factorization properties which follow from the unitarity relation

$$\Delta t_4 = t_4(+)\Delta G_0 t_4(-) .$$

One could continue on from Eq. (3.11) with the development of the representation of M to obtain an expression analogous to although hardly as interesting as Eq. (2.23). In summary, when $V_4 \neq 0$ the subtraction technique considered here has only one of the attributes possessed by similar techniques in the two-particle and the $V_4 = 0$ cases. Namely, we achieve a reformulation in terms of quantities which satisfy nonsingular integral equations.

All of our conclusions regarding the M amplitudes in the $V_4 \neq 0$ case remain unaltered if we were to carry out the same sort of formal analysis but using the specific definition of the subtraction procedure of Ref. 1 which was described in Sec. II.

B. U operators

The method of Ref. 5 is designed to be applicable even if the half-shell operators f possess poles. It is interesting, therefore, to generalize this method to the case when $V_4 \neq 0$.

Let us rewrite Eqs. (3.2) in the form

$$U = \bar{\delta} G_0^{-1} + V_4 + \bar{\delta}\tau G_0 U \tag{3.15a}$$

$$= \bar{\delta} G_0^{-1} + V_4 + U G_0 \tau_{tr} \bar{\delta} , \tag{3.15b}$$

where

$$\tau \equiv t + \bar{\delta}^{-1} V_4 (1 + G_0 t) \tag{3.16a}$$

$$\tau_{tr} \equiv t + (1 + t G_0) V_4 \bar{\delta}^{-1} . \tag{3.16b}$$

We confine Eqs. (3.15) and Eqs. (3.16) and all subsequent equations to the 3×3 channel space which is permissible since Eqs. (3.2) are closed on this space. The subtraction technique which is introduced as a generalization of the method of Ref. 5 appears to be inapplicable to the 4×4 versions of (3.15); we comment upon this later.

We suppose t is decomposed as in Eq. (2.2) and that

$$V_4 = V^e + V^r \tag{3.17a}$$

$$= V^e_{tr} + V^r_{tr} . \tag{3.17b}$$

Equations (3.17) are to be interpreted as (diagonal) matrix equations; e.g., $V_4 \delta_{\beta\alpha} = (V^e_\beta + V^r_\beta)\delta_{\beta\alpha}$. $V^{e,r}$ and $V^{e,r}_{tr}$ remain to be defined. However, with Eq. (3.2) and Eqs. (3.17) we see that τ and τ_{tr} break up into "essential" and "residual" parts:

$$\tau = \tau^e + \tau^r , \tag{3.18a}$$

$$\tau_{tr} = \tau^e_{tr} + \tau^r_{tr} , \tag{3.18b}$$

where

$$\tau^{e,r} = t^{e,r} + \bar{\delta}^{-1} V^{e,r} + \bar{\delta}^{-1} V_4 G_0 t^{e,r} , \tag{3.19a}$$

$$\tau^{e,r}_{tr} = t^{e,r} + V^{e,r}_{tr} \bar{\delta}^{-1} + t^{e,r} G_0 V_4 \bar{\delta}^{-1} . \tag{3.19b}$$

Then, for example, Eq. (3.15a) can be transformed into the integral equation

$$U = U^r(1 + \bar{\delta}^{-1} G_0 V_4) + U^r (G_0 \tau^e G_0) U , \tag{3.20a}$$

where

$$U^r = \bar{\delta} G_0^{-1} + \bar{\delta}\tau^r G_0 U^r . \tag{3.20b}$$

If V^r is such that

$$V^r \mathcal{S} = 0 , \tag{3.21a}$$

then

$$\tau^r \mathcal{S} = 0 \tag{3.21b}$$

and the kernel of Eq. (3.20b) is manifestly nonsingular. One obvious choice for $V^{e,r}$ which is consistent with the conditions (3.17a) and (3.21a) and which is open to us if the two-particle half-shell function is well defined is simply

$$V^e = V_4 \mathcal{S} f^\dagger , \tag{3.22a}$$

$$V^r = V_4 - V^e . \tag{3.22b}$$

With the decomposition (3.22) τ^e exhibits the same type of finite-rank characteristics possessed by t^e.[5] If f is not well defined, we can take in its place in Eqs. (3.22) any convenient two-particle operator $\hat{f}$ with the property $\mathcal{S}\hat{f}\mathcal{S} = \mathcal{S}$ and still obtain all of the desirable features which follow with the

decomposition (3.22).

The formalism is completed by noting that the zero-indexed operators can be calculated, in principle, from the nonzero index operators.[9] For example, it follows from (3.1) that for any $\alpha \neq 0$

$$U_{00} = t_\alpha + (1 + t_\alpha G_0) U_{\alpha\alpha} (G_0 t_\alpha + 1) .$$

However, as noted in Ref. 9 this is not likely to be a convenient way of calculating U_{00} and methods based on the M operators are to be preferred. Nonetheless, this finishes the generalization of Eqs. (2.26) to the case $V_4 \neq 0$ and provides the basis of a structure invariant perturbation theory of the type described in Ref. 5.

Similar results follow when one begins from Eq. (3.15b). In place of Eqs. (3.20) one obtains

$$U = (1 + V_4 G_0 \bar{\delta}^{-1}) U^r_{tr} + U (G_0 \tau^e G_0) U^r_{tr} , \tag{3.23a}$$

where

$$U^r_{tr} = \bar{\delta} G_0^{-1} + U^r_{tr} G_0 \tau^r_{tr} \bar{\delta} . \tag{3.23b}$$

In this case V^r_{tr} is chosen so that $\mathcal{S} V^r_{tr} = 0$. Then $\mathcal{S}\tau^r_{tr} = 0$ and the kernel of (3.23b) is nonsingular. Possible choices for $V^{e,r}_{tr}$ are

$$V^e_{tr} = f \mathcal{S} V_4$$

$$V^r_{tr} = V_4 - V^e_{tr} .$$

Since Eqs. (3.15) hold on the 4×4 channel space the question of the possibility of carrying out the same formal manipulations which led to Eqs. (3.20) and (3.23) is of some relevance. Obviously the same results are obtained given the extended definitions (2.2) and (3.17). When $V_4 = 0$ no problems are encountered with the extended equations (2.26) because $t_0 = 0$ and $(t^e)_0 = (t^r)_0 = 0$. However, both $(V^e)_0$ and $(V^r)_0$, e.g., cannot be zero. However, unless $V^r_0 = 0$, Eq. (3.20b) possesses a singular kernel (in the $U^r_{0\alpha}$ case). But if $V^e_0 = V_4$, which is necessitated by this alternative, then not only are Eqs. (3.20a) no longer closed with respect to nonzero indexed operators but τ^e fails to possess those very finite-rank attributes which motivated this subtraction technique in the first place.

In Sec. II we established the connection between the U operator and Brayshaw[5] subtraction techniques in the case $V_4 = 0$. We next inquire as to the general connection between the two methods. For $V_4 \neq 0$ the intrinsic asymmetry in the generalized U-operator formalism between the treatment of nonzero indexed and zero indexed makes the transformation of the M operators using (3.3) exceedingly cumbersome. It is much simpler to inspect Brayshaw's equations. Employing the decompositions (2.2) and (3.17) one easily obtains from Eqs. (3.5a) for the operators $M_{\beta\alpha}$, with $\alpha, \beta \neq 0$,

$$M = Z^r \Omega + Z^r K^e M , \tag{3.24}$$

where

$$Z^r = 1 + K^r Z^r ,$$

$$K^{e,r} = (t^{e,r} \bar{\delta} + \mathcal{U}^{e,r} + t^{e,r} G_0 \mathcal{U}) G_0 ,$$

and

$$\Omega = t + \tfrac{1}{3} \mathcal{U} + t G_0 \mathcal{U} .$$

Whatever the choices of t^r and $\mathcal{U}^r$ so long as $V_4 \neq 0$, K^r is a singular kernel although $\tilde{K}^r = G_0 K^r G_0^{-1}$ certainly is not if $\mathcal{S}\mathcal{U}^r = 0$. Equation (3.24) is then rewritten in terms of the real operator $\tilde{Z}^r = G_0 Z^r G_0^{-1}$ [cf. Eq. (3.12)]:

$$M = G_0^{-1} \tilde{Z}^r G_0 \Omega + G_0^{-1} \tilde{Z}^r (G_0 \tilde{K}^e G_0) M . \tag{3.25}$$

This last equation can be placed in a form similar to Eqs (3.20) or (3.23) after an off-shell transformation in which an operator equivalent to M on shell is introduced. This dissimilarity in structure as well as the distinct choices of the decomposition of V_4 which are required to achieve an expression in terms of operators satisfying nonsingular integral equations makes the connection between Eqs. (3.24) and the U operator formalism appear to be rather remote when $V_4 \neq 0$.

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