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Nuclear polarization corrections in μ-^{4}He atoms*

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A variety of approximations are investigated which allow us to estimate rather simply the dominant dipole part of the Coulomb-excited nuclear polarization corrections to the μ-^{4}He atom in S states. In particular, the unretarded dipole approximation provides an effective upper limit to this correction and lowest-order Coulomb corrections have been calculated in this approximation. We find that -3.1 meV $\pm 20\%$ is a reasonable estimate for the complete result.

[NUCLEAR STRUCTURE Polarization corrections; muonic atoms; ^{4}He.]

I. INTRODUCTION

The recent elegant experiment by Bertin *et al.*[1] combines the diverse ingredients of lasers, atomic and nuclear physics, and quantum electrodynamics. Muons stopped in a helium target can form metastable ions in the $2S$ state; these were excited by a laser to the $2P\frac{3}{2}$ state and the x rays from the subsequent decay to the $1S$ state were detected. A measurement of the resonant laser energy produced $\Delta E(2P\frac{3}{2}-2S\frac{1}{2})=1527.4\pm0.9$ meV. A variety of theoretical ingredients are required in order to calculate ΔE. A minor role is played by the well-known atomic fine-structure splitting (~10% of ΔE), while vacuum polarization provides the bulk of the result (~110% of ΔE). Somewhat larger than the fine structure is the effect of the nuclear finite size (~−20%). On the scale of 1% or less the Lamb shift[2] and nuclear polarization enter. In spite of the overwhelming size of the vacuum polarization contribution, the limits on the uncertainty of theoretical calculations are probably set by the nuclear finite size, characterized by the mean-square radius $\langle \vec{r}^2\rangle$, and polarization corrections, since the dominant pieces of the vacuum polarization can be calculated very accurately. Recent calculations by Rinker[3] and Borie[4] provide an accurate survey of the many contributions. Rinker finds a theoretical result $\Delta E=1813.1-102.0\langle\vec{r}^2\rangle\pm1$ meV, which predicts $\langle\vec{r}^2\rangle^{1/2}=1.674\pm0.004$ fm. Very recently, Sick, McCarthy, and Whitney[5] analyzed new elastic electron scattering data and deduced $\langle\vec{r}^2\rangle^{1/2}=1.674\pm0.012$ fm, in excellent agreement with Rinker's prediction. The main uncertainty in Rinker's error estimate arose from the polarization correction ΔE_p, and similar considerations apply to the analysis of muonic transitions in other elements as well. Clearly, it is important to understand the polarization corrections as well as possible.

A variety of techniques have been used to calculate ΔE_p.[3,6-10] Basically, the polarization correction involves all processes where the hadron (nucleus) is virtually excited and deexcited by photon exchange with the lepton (muon). In practice, with the exception of Ref. 8, this usually means only the dominant Coulomb force is taken into account.[3,6,7,9] In light muonic atoms the nonrelativistic approximation is generally made for the lepton as well, since binding energies are typically kilovolts compared to the muon rest mass of 105 MeV. This does not imply[11,12] that intermediate-state muon energies are also small and we will comment on this later. In addition, Coulomb distortion effects on the muon wave function are not large and are sometimes ignored.[8]

One of the more popular exercises[7,13] recently has been to investigate the form of the effective potential which arises from two-photon exchange for large separation r of the lepton and hadron. The effective long-range polarization potential is found to be $V_p^{lr}(r)\sim-\alpha\alpha_E/2r^4$, where α is the fine structure constant and α_E is the nuclear electric polarizability. This result illustrates a number of important points: (1) while the potential is not localized in the nucleus, it falls off rapidly outside the nucleus; (2) the dominant nuclear physics ingredient in any calculation is likely to be the electric polarizability or something closely related to it; (3) because of (1), the polarization corrections affect primarily the S states; (4) attempts to calculate ΔE_p from V_p^{lr} for S states must involve a cutoff if a finite answer is to be obtained; (5) the potential is attractive.

Our primary objective in this work will be to shed some light on the results of previous calculations of ΔE_p, which are quite different in approach; in particular, we will investigate the virtual dipole excitations in the nonrelativistic approximation, which are known[3] to dominate the overall Coulomb part of ΔE_p. Within this framework an effective upper limit for the polarization correction for S-

state energies can be achieved using the unretarded dipole approximation, which has been somewhat useful in calculating dispersion corrections for low-energy electron scattering.[14,15] Keeping only first-order Coulomb distortion effects we will find for S states

$$\Delta E_p^{UD} = -\sqrt{2\mu}\,\frac{\alpha}{\pi}|\,\phi(0)|^2[\sigma_{-3/2} + Z\alpha\sqrt{2\mu}\,(\sigma_{-2}^l + a\sigma_{-2})]\,,$$

where

$$\sigma_{-\lambda} \equiv \int_{\omega_{\rm th}}^{\infty} \omega^{-\lambda}\sigma_{\rm abs}^{\gamma}(\omega)\,d\omega\,;$$

$$\sigma_{-\lambda}^{l} = \int_{\omega_{\rm th}}^{\infty} \omega^{-\lambda}\sigma_{\rm abs}^{\gamma}(\omega)\ln[Z\alpha(2\mu/\omega)^{1/2}]d\omega\,.$$

[These are Eqs. (26a) and (26c) of Sec. III.] In this expression μ is the muon-nucleus reduced mass, $\phi(0)$ is the muon wave function at the nucleus, $\sigma_{\rm abs}^{\gamma}(\omega)$ is the total photoabsorption cross section of a nucleus for a photon with energy ω, Z is the proton number, and a is a state-dependent constant. Since σ_{-2} is essentially proportional to α_E and $\sigma_{-3/2}$ is a closely related quantity, observation (2) above is confirmed. Both quantities may be deduced from recent experiments.[17,18] This equation is one of our primary results; note that no cutoff is needed. More accurate numerical results than those given by Eq. (26) will be obtained by relaxing the unretarded approximation and will confirm the results of Rinker[3] and Bernabeu and Jarlskog.[8]

II. GENERAL FORMALISM

Our primary assumptions in this work will be: the nonrelativistic treatment of both the muon and nucleus, and the ignoring of all but the static Coulomb interaction between the two systems. The first step[18] is to separate the Coulomb interaction into a piece H_0 which is elastic with respect to the nucleus and a piece ΔH_c which generates only nuclear transitions. The first piece is treated to all orders by including it as part of the unperturbed lepton Hamiltonian, and it generates the usual static hydrogenic spectrum modified by the nuclear charge distribution plus recoil corrections.[18] The second piece ΔH_c generates nuclear transitions and is treated perturbatively; it contributes to the energy in second- and higher-order perturbation theory. Because both the nuclear finite size and the nuclear polarization generate small corrections, it is sufficient to restrict ourselves to a second-order treatment of ΔH_c and to ignore the nuclear finite size while doing so. With these assumptions the polarization correction in the lepton-nucleus center-of-mass frame becomes

$$\Delta E_p = \sum_{N\neq 0}\langle 0'\,|\,\Delta H_c|\,N\rangle \times\left[\sum_n \frac{|\,n\rangle\langle n|}{\epsilon_0-\epsilon_n-\omega_N}\right]\langle N|\,\Delta H_c\,|\,0'\rangle\,, \quad (1)$$

where we have labeled by $|\,N\rangle$ each internal nuclear state which has energy ω_N with respect to the nuclear ground state, and by $|\,n\rangle$ each lepton state in the center of mass which has an energy ϵ_n. In addition, $|\,0'\rangle$ is simultaneously the ground state of the nucleus $|\,0\rangle$ and the unperturbed atomic state $|\,i\rangle$, which we denote by $\phi(\vec{r})$ in coordinate space; the latter state has an energy ϵ_0. We have written the lepton intermediate state $|\,n\rangle$ in a way that emphasizes that the bracket contains the Coulomb Green's function.[19-22] Defining $E_N=\omega_N-\epsilon_0$ (>0), the Green's function is denoted $\hat{G}(-E_N)$ and we may rewrite Eq. (1) in the form

$$\Delta E_p = \sum_{N\neq 0}\langle 0'\,|\,\Delta H_c\,|\,N\rangle\,\hat{G}(-E_N)\langle N|\,\Delta H_c\,|\,0'\rangle\,. \quad (2)$$

A more useful form may be obtained by noting that the nuclear matrix element $\langle N|\,\Delta H_c\,|\,0\rangle$ is just the lepton transition potential $\Delta V_N(\vec{r})$, where $\vec{r}$ is the vector from the nuclear center of mass to the lepton. This leads to

$$\Delta E_p = \sum_{N\neq 0}\langle i\,|\Delta V_N(\vec{r})\,G_c(-E_N;\vec{r},\vec{r}\,')\,\Delta V_N(\vec{r}\,')\,|\,i\rangle \quad (3)$$

in an obvious notation which emphasizes the lepton coordinate. We wish to evaluate ΔE_p for four special cases: (a) ignore the Coulomb attraction in the lepton states and use the nonrelativistic equivalent of the nuclear model of Bernabeu and Jarlskog (denoted BJ); (b) ignore Coulomb effects and use dipole nuclear states only; (c) ignore Coulomb effects and use the *unretarded* dipole approximation; (d) work in the unretarded dipole approximation and include first-order Coulomb distortion effects.

We begin our discussion by ignoring Coulomb effects in the Green's function G_c; in this limit G_c is essentially the nonrelativistic free Green's function for complex momentum. We find that $G_c \to G_0 = -\mu\exp(-\kappa_N|\,\vec{r}-\vec{r}\,'\,|)/2\pi|\,\vec{r}-\vec{r}\,'\,|$, where $\kappa_N \equiv (2\mu E_N)^{1/2}$. The first observation is that κ_N is a number which varies roughly from $\frac{1}{4}\to\frac{1}{2}$ over the region of the intermediate nuclear spectrum which can be expected to dominate the polarization corrections; furthermore, the exponential is small unless $\vec{r}$ and $\vec{r}\,'$ are *roughly* equal. Clearly the latter situation becomes a better and better approximation as E_N increases. Therefore, as a rough approximation we may write $G_0 \cong \lambda\delta^3(\vec{r}-\vec{r}\,')$ and, integrating with respect to $\vec{r}$, we find $\lambda=-1/E_N$. Substituting this result into Eq. (3), we obtain the semiclassical approximation

$$\Delta E_p = \langle i \mid V_{sc}(r) \mid i\rangle \,, \tag{4a}$$

$$V_{sc}(r) = \sum_{N\neq 0} \langle 0 | \Delta H_c(\vec{r}) \frac{|N\rangle\langle N|}{E_0 - E_N} \Delta H_c(\vec{r}) | 0\rangle \,, \tag{4b}$$

which has the classic form of a polarization potential and is the "A" term of Ref. 9. Rather than proceeding along these lines, it is profitable to Fourier transform the matrix element in Eq. 3 (using G_0). We also write ΔH_c in the form

$$\Delta H_c(\vec{r}) = -\alpha \int \frac{d^3\vec{r}_N}{|\vec{r}-\vec{r}_N|} \hat{\rho}(\vec{r}_N)$$
$$= -\frac{\alpha}{2\pi^2} \int \frac{d^3\vec{q}}{\vec{q}^2} e^{i\vec{q}\cdot\vec{r}} \hat{\rho}(\vec{q}) \,, \tag{5a}$$

$$\hat{\rho}(\vec{q}) = \int d^3 r_N \, e^{-i\vec{q}\cdot\vec{r}_N} \hat{\rho}(r_N) \,, \tag{5b}$$

where $\hat{\rho}(\vec{r})$ is the nuclear charge operator, and its Fourier transform satisfies $\hat{\rho}(\vec{q}=0) \equiv Z$. We also define $\int \exp(i\vec{q}\cdot\vec{r})\, \phi(\vec{r})\, d^3\vec{r}$ to be $\tilde{\phi}(\vec{q})$. We obtain

$$\Delta E_p = -\frac{(4\pi)^2\alpha^2}{(2\pi)^9} \sum_{N\neq 0} \int d^3\vec{q}\, d^3\vec{q}'\, d^3\vec{q}'' \frac{\langle 0 | \hat{\rho}(-\vec{q}') | N\rangle \langle N | \hat{\rho}(\vec{q}'') | 0\rangle}{E_N + \vec{q}^2/2\mu} \frac{1}{\vec{q}'^2} \frac{1}{\vec{q}''^2} \tilde{\phi}^\dagger(\vec{q}'-\vec{q})\, \tilde{\phi}(\vec{q}-\vec{q}'') \,. \tag{6}$$

One of the primary approximations used in Refs. 6 and 8 (besides the replacement $G_c \to G_0$) results from the recognition that the momentum components of the atomic wave function ϕ are confined to reasonably small values. For the 1S state, for example, $\tilde{\phi}(\vec{q}) \sim \beta/(\vec{q}^2+\beta^2)^2$, with $\beta = Z\alpha\mu$ roughly an MeV in size for μ-He. All the other energy and momentum scales in Eq. (6) are considerably larger, and thus $\tilde{\phi}(\vec{q}) \cong 0$ unless $\vec{q} \cong 0$. The expression for $\tilde{\phi}$ above is an adequate representation for a δ function provided β is small and we may approximate $\tilde{\phi}(\vec{q})$ by $\phi(0)(2\pi)^3\delta^3(\vec{q})$ for any S state. For lack of a better name, we will call this low $-Z$ approximation the *wave function approximation*, which simplifies Eq. (6) to the form

$$\Delta E_p = -2\frac{\alpha^2}{\pi} | \phi(0)|^2 \sum_{N\neq 0} \int \frac{d^3\vec{q} |\langle N | \hat{\rho}(\vec{q}) | 0\rangle|^2}{\vec{q}^4(\omega_N + \vec{q}^2/2\mu)} \tag{7}$$

after dropping ϵ_0 compared with ω_N; this is the nonrelativistic version of the model of Bernabeu and Jarlskog.[8] The comparison is most easily made by dropping all magnetic (transverse), retardation, and other relativistic effects in the results of Ref. 8 and rewriting Eq. (7) in the form

$$\Delta E_p = -8\alpha^2 | \phi(0)|^2 \int_0^\infty \frac{dq}{q^2} \int_{\omega_{\rm th}}^\infty \frac{W_c(q,\omega)}{\omega + q^2/2\mu} d\omega \,, \tag{8a}$$

where

$$W_c(q,\omega) = \sum_{N\neq 0} |\langle N | \hat{\rho}(\vec{q}) | 0\rangle|^2 \delta(\omega - \omega_N) \tag{8b}$$

is the usual inelastic Coulomb response function obtainable from electron scattering[24,25] above the inelastic threshold $\omega_{\rm th}$. We will evaluate Eq. (8) using a crude model in Sec. IV.

Equations (7) and (8) demonstrate the fact that the convergence of the q integral may be drastically altered by approximations and that the model dependence of the result (ΔE_p) depends in a significant way on the extent to which the q dependence of W_c is needed to cut off the q integral. Because we are dealing only with inelastic virtual transitions, the threshold behavior of W_c is determined by dipole states and $W_c \sim \vec{q}^2$ for small $\vec{q}^2$; thus there are no small-q (infrared) problems with Eq. (8). A natural approximation would be to ignore the $\vec{q}^2$ dependence of the denominator. This is the same as Eq. (4b) after the wave function approximation and places the burden of convergence on W_c. Clearly, results obtained using this approximation could be quite model dependent. Furthermore, for small μ some damping must be provided by W_c or the nonrelativistic approximation will be completely inadequate for lepton intermediate states. This is the case for electrons.[11,12] For muons, μ is sufficiently large that the denominator in Eq. (8c) provides most of the convergence needed in the nonrelativistic regime.

Experience[3,26,27] has shown that dipole excitations are the most important in calculating polarization corrections. For nuclear transitions from spinless ground states to 1^- states, angular momentum considerations lead to

$$\langle N | \hat{\rho}(\vec{q}) | 0\rangle = i\vec{q}\cdot\vec{D}_{N0} F_N(\vec{q}^2) \,, \tag{9}$$

where $F_N(0)=1$ and $\vec{D}$ is the nuclear dipole operator. The *unretarded dipole approximation* consists of neglecting F_N; although it is clearly incorrect for large $\vec{q}^2$, it guarantees the correct threshold properties. Furthermore, it relates the matrix element of $\hat{\rho}$ to photoabsorption, since at low photon energies the unretarded dipole approximation is excellent and current continuity[28] then relates current matrix elements to $\vec{D}_{N0}$. An alternative derivation of the same result in coordinate space is instructive. We expand $|\vec{r}-\vec{r}_N|^{-1}$ according to its angular momentum content and keep only $l=1$ components. We find $|\vec{r}-\vec{r}_N|^{-1}\cong\hat{\vec{r}}\cdot\hat{\vec{r}}_N r_</r_>^2$, where $r_<$ and $r_>$ are the smaller and larger of $|\vec{r}|$ and $|\vec{r}_N|$. If we keep only those terms where the lepton is *outside* the nucleus, we can approximate

$$\frac{1}{|\vec{r}-\vec{r}_N|}\cong\vec{r}\cdot\vec{r}_N/r^3=-\vec{r}_N\cdot\vec{\nabla}(1/r)\,, \qquad (10)$$

and it is easy to show using Eq. (5) that this is identical to the *unretarded dipole approximation*. Introducing Eq. (10) into V_{sc}, Eq. (4b), leads immediately to V_p^{lr}, since $\alpha_E\equiv 2\alpha\sum_{N\neq 0}|\langle N|D_z|0\rangle|^2/\omega_N$.

Our primary task is to investigate the dipole contributions to ΔE_p. A common and very useful approximation is to assume[3,14] that $F_N(\vec{q}^2)$ is a universal (this is, transition-independent) function, $F(\vec{q}^2)$; this approximation has some limited theoretical validity[29] and we will use it. Furthermore, angular momentum constraints allow us to write $\sum_{N\neq 0}|\langle N|\vec{D}\cdot\vec{q}|\,0\rangle|^2=\sum_{N\neq 0}|\langle N|D_z|\,0\rangle|\,\vec{q}^2$, and consequently

$$\Delta E_p^D\cong -8\alpha^2|\,\phi(0)|^2\sum_{N\neq 0}|\,D_{N0}^z|^2\int_0^\infty\frac{dqF^2(q^2)}{\omega_N+q^2/2\mu}\,. \qquad (11)$$

Dropping the $\vec{q}^2$ term in the denominator is equivalent to calculating the matrix element of $V_p^{lr}(r)$ in the wave function approximation. Numerical results for Eq. (11) will be calculated in Sec. IV. This equation is the analog of Rinker's calculation if we neglect Coulomb distortion and relativistic effects. A very useful result is obtained if we make the unretarded dipole approximation. Using $\int_0^\infty dq\,[\omega_N+q^2/2\mu]^{-1}=\pi[\mu/2\omega_N]^{1/2}$, Eq. (11) can then be written in the form

$$\Delta E_p^{UD}\cong -8\pi\alpha^2|\,\phi(0)|^2(\tfrac{1}{2}\mu)^{1/2}\sum_{N\neq 0}|\,D_{N0}^z|^2/\omega_N^{1/2}\,, \qquad (12)$$

and a further simplification results if we note that the cross section for photoabsorption in the unretarded dipole approximation is given by

$$\sigma_{abs}^\gamma(\omega)=4\pi^2\alpha\sum_{N\neq 0}|\,D_{N0}^z|^2\,\omega\,\delta(\omega-\omega_N)\,, \qquad (13a)$$

and consequently

$$\sigma_{-n}\equiv\int_{\omega_{th}}^\infty\sigma_{abs}^\gamma(\omega)\,\omega^{-n}\,d\omega=4\pi^2\alpha\sum_{N\neq 0}|\,D_{N0}^z|^2\,\omega_N^{1-n} \qquad (13b)$$

so that

$$\Delta E_p^{UD}=-\sqrt{2\mu}\,\alpha\,|\,\phi(0)|^2\sigma_{-3/2}\,/\pi \qquad (14)$$

and

$$\Delta E_p^D=-\frac{4\mu\alpha}{\pi^2}|\,\phi(0)|^2\int_{\omega_{th}}^\infty\frac{\sigma_{abs}^\gamma(\omega)\,d\omega}{\omega}\int_0^\infty\frac{dqF^2(q^2)}{q^2+2\mu\omega}\,. \qquad (15)$$

Because the integrand in Eq. (11) is positive definite and $F^2\leq 1$ for any nonpathological form factor, Eq. (14) is essentially an *upper limit* for the dipole part of ΔE_p within the framework of our approximations; that is, $|\Delta E_p^D|<|\Delta E_p^{UD}|$.

III. COULOMB CORRECTIONS

In the previous section we derived the nonrelativistic, no Coulomb distortion results corresponding to those of Refs. 3 and 8 as well as an effective upper limit for the dipole case. In this section we will derive the Coulomb distortion corrections of order $(Z\alpha)$ and $Z\alpha\ln(Z\alpha)$ to the unretarded dipole polarization correction for S states. We use Eqs. (3), (5), and (10) in the dipole approximation and project out the $l=1$ part of the lepton Green's function to obtain

$$\Delta E_p^{UD}=4\pi\alpha^2\sum_{N\neq 0}|\,D_{N0}^z|^2\int_0^\infty dr\int_0^\infty dr'\,\phi(r)\,\frac{g(-\omega_N;r,r')}{rr'}\,\phi(r')\,, \qquad (16)$$

where the dipole Green's function $g(-E;r,r')$ satisfies the equation (with $l=1$)

$$\left[-E+\frac{1}{2\mu}\frac{d^2}{dr^2}-\frac{l(l+1)}{2\mu r^2}+\frac{Z\alpha}{r}\right]g_l(-E;r,r')=\delta(r-r')\,. \qquad (17)$$

At no extra cost the multipole Green's function g_l can also be obtained for all l by applying the usual conditions of regularity at the origin and at infinity, continuity at $r=r'$, and a discontinuity of first derivatives at $r=r'$ of amount 2μ in order to obtain the right-hand side of Eq. (17). The solutions of the homogeneous equation are the Whittaker functions $M_{\kappa,l+1/2}(r/\beta)$ and $W_{\kappa,l+1/2}(r/\beta)$, which are regular at the origin and infinity, respectively. We have defined $\kappa=Z\alpha(\mu/2E)^{1/2}$ and $\beta=(8\mu E)^{-1/2}$, assuming that E is positive. Using the Wronskian[30] $MdW/dr-WdM/dr=-\Gamma(2l+2)/\beta\Gamma(l+1-\kappa)$, we immediately obtain

$$g_l(-E;r,r')=-2\mu\beta\,\frac{\Gamma(l+1-\kappa)}{\Gamma(2l+2)}\times M_{\kappa,l+1/2}(r_</\beta)\,W_{\kappa,l+1/2}(r_>/\beta)\,, \qquad (18)$$

where $r_<$ and $r_>$ are the lesser and greater of r and

r'. Given this relation and explicit forms for the multipole transition charge density, the integral in Eq. (3) can be done numerically in a brute force manner. Our interest, however, is directed at a variety of approximation schemes.

In this regard, particularly useful is identity 6.669.4 of Ref. 23 which allows us to write

$$g_l(-E;r,r') = -2\mu\sqrt{rr'}\int_0^\infty dx\,\exp[-(r+r')\cosh(x)/2\beta] \times \coth^{2\kappa}(\tfrac{1}{2}x)\,I_{2l+1}\left(\frac{\sinh x\sqrt{rr'}}{\beta}\right). \tag{19}$$

Our primary interest lies with the 1S and 2S states and for these states the following general identity for S states is simple and useful:

$$\phi_{ns}(r) \equiv \phi(0)\,\hat{P}_\lambda e^{-\lambda r}|_{\lambda=\lambda_n}\,, \tag{20a}$$

$$\hat{P}_\lambda = \sum_{k=0}^{n-1} a_k^n\left(\frac{\partial}{\partial\lambda}\right)^k, \tag{20b}$$

where $\lambda_n = Z\alpha\mu/n$, $a_0^n \equiv 1$, and $a_1^n = (n-1)\lambda_n$. In addition $a_k^n \sim (Z\alpha)^k$. Inserting Eq. (19) into Eq. (16) and using identity (20) for both initial and final states (λ' and λ, respectively), we obtain for $l=1$

$$\Delta E_p^{UD} = -8\pi\alpha^2|\phi(0)|^2\hat{P}_\lambda\hat{P}_{\lambda'}\sum_{N\neq 0}|D_{N0}^z|^2 I_N\,, \tag{21a}$$

$$I_N = \frac{\beta}{3}\int_0^\infty dx\,\frac{\coth^{2\kappa}(\frac{1}{2}x)}{\sinh(x)}z^2(x)\,{}_2F_1(2,2,4;z(x))\,, \tag{21b}$$

where $z(x) = \sinh^2(x)/[2\beta\lambda+\cosh(x)][2\beta\lambda'+\cosh(x)]$. The Gauss's hypergeometric function may be expressed in the elementary form $-6[2z+(2-z)\log(1-z)]/z^3$. Although the integral is difficult, our interest extends only as far as the terms of order $Z\alpha$ and $Z\alpha\ln(Z\alpha)$. The possible logarithmic terms are a serious complication since a power series expansion in $Z\alpha$ would not exist. The $Z\alpha$-dependence of I_N resides in three places: in κ, λ, and λ'. The derivatives implicit in $\hat{P}$ cannot change the order of a given term since the coefficient of $(\partial/\partial\lambda)^k$ is $(Z\alpha)^k$, although a term such as $(Z\alpha)^k\ln Z\alpha$ can be converted to $(Z\alpha)^k$. We first observe that the integral (in z) becomes quite elementary when $\kappa=\lambda=\lambda'=0$ and we find that $I_N = 2\beta$ in this limit and reproduces Eq. (12). For small κ (and $\lambda=\lambda'=0$) a power series exists and the first term I_κ has the form

$$I_\kappa = \beta\kappa\int_0^1 dx\,\frac{\ln x}{x^2}\left[2x+(1+x^2)\ln\left|\frac{1-x}{1+x}\right|\right] = 2\beta\kappa(2\ln 2-1)\,, \tag{22}$$

where the analytic result follows from integrations by parts and Spence function identities[31]; since the Spence functions cancel from the final result, more elementary means presumably exist for obtaining the integral. For $\kappa=0$, the integral (21b), I_λ, may be shown to possess a logarithmic dependence on λ and λ'. Consequently great care must be exercised, and one finds by a very tedious application of integration by parts and partial fractions that

$$I_\lambda = 2\beta\{1+2\beta(\lambda+\lambda')\ln[2\beta(\lambda+\lambda')]+O(Z\alpha)^2\}\,. \tag{23}$$

For the two special cases, $\lambda=\lambda'\neq 0$ and $\kappa=0$, as well as $\lambda=\lambda'=0$ and $\kappa\neq 0$, the integral (21b) was evaluated numerically for a range of small values of the parameters and compared to the results (22) and (23). The differences were quadratic in the parameters and the coefficient of the parabola was not large, indicating that the expansion is probably a good one. We thus obtain

$$I_N = 2\beta\{1+2\beta(\lambda+\lambda')\ln[2\beta(\lambda+\lambda')] + \kappa(2\ln 2-1)\}+O(Z\alpha)^2\,, \tag{24}$$

and performing the λ derivatives we find

$$\Delta E_p^{UD} = -8\pi\alpha^2(\tfrac{1}{2}\mu)^{1/2}|\phi(0)|^2\sum_{N\neq 0}\frac{|D_{N0}^z|^2}{\omega_N^{1/2}}(1+\Delta_N)\,, \tag{25a}$$

$$\Delta(1S) = 4\beta\lambda_1\ln(4\beta\lambda_1)+2\beta\lambda_1(2\ln 2-1) = 4\beta\lambda_1\ln(8\beta\lambda_1)-2\beta\lambda_1\,, \tag{25b}$$

$$\Delta(2S) = 8\beta\lambda_2\ln(4\beta\lambda_2)+5\beta\lambda_2+4\beta\lambda_2(2\ln 2-1) = 8\beta\lambda_2\ln 8\beta\lambda_2+\beta\lambda_2\,. \tag{25c}$$

Using Eq. (13) we obtain

$$\Delta E_p^{UD} = -\sqrt{2\mu}\,\frac{\alpha}{\pi}|\phi(0)|^2\{\sigma_{-3/2}+Z\alpha\sqrt{2\mu}\,(\sigma_{-2}^l+a\sigma_{-2})\} \tag{26a}$$

$$a(1S) = \tfrac{1}{8}\,,\quad a(2S) = \ln 2-\tfrac{1}{2}\,, \tag{26b}$$

$$\sigma_{-n}^l = \int_{\omega_{\text{th}}}^\infty \sigma_{\text{abs}}^\gamma(\omega)\,\omega^{-n}\ln(Z\alpha(2\mu/\omega)^{1/2})d\omega \equiv \sigma_{-n}\ln[Z\alpha(2\mu/\bar{\omega})^{1/2}]\,, \tag{26c}$$

where the last relation defines $\bar{\omega}$. We will see in Sec. IV that $\bar{\omega}\cong 30$ MeV and thus the logarithm in Eq. (26c) has the value -3.25. Consequently the a term is quite small and similar in size for the 1S and 2S states. Using the fact that $|\phi(0)|^2 = (\mu\alpha Z)^3/\pi n^3$, we see that the polarization corrections for the 1S and 2S states should differ almost exactly by a factor of 8; this result agrees with Rinker. In addition we see in Eq. (26) the natural emergence of the wave function approximation as a low-Z limit.

In Sec. IV we will estimate Coulomb corrections to forms of ΔE_p which did not involve the unretarded dipole approximation. This will be done by using the factor $(1+\Delta_N)$ inside sums over states. Although this is only correct in Eq. (25), it is probably a reasonably good approximation in general.

IV. RESULTS AND DISCUSSION

We will discuss and evaluate the various approximations we have developed in the opposite order we presented them. As we discussed in the introduction and as emphasized by Bernabeu and Jarlskog,[10] the value of the electric polarizability or, equivalently, σ_{-2} in the unretarded dipole approximation,[32] is crucial in determinations of ΔE_p. Early experimental determinations[16] of σ_{-2} for ^{4}He ($73\pm4\times10^{-4}$ fm^2/MeV) are in agreement with recent efforts[17] ($72\pm4\times10^{-4}$ fm^2/MeV), where all five photodisintegration channels were separately detected, measured, and summed. It was found that only the well-measured (γ,p) and (γ,n) channels contribute significantly to σ_{-2}, and the contribution from multipoles higher than $E1$ is probably very small. Rinker has developed a simple parametrization of the photoabsorption cross section and his form leads to $\sigma_{-2}=75\times10^{-4}$ fm^2/MeV. His value for σ_{-1} is 23.7×10^{-2} fm^2, while Arkatov *et al.*[17] find $24.5\pm1.5\times10^{-2}$ fm^2. For reasons of convenience we will use Rinker's parametrization in what follows.

Although the semiclassical unretarded dipole approximation to ΔE_p [7] is divergent for S states, a reasonable assumption for a cutoff produces[10] a value ~−7 meV for ΔE_p. Our unretarded dipole approximation [Eq. (14)] yields −4.76 meV without Coulomb corrections ($\sigma_{-3/2}=4.14\times10^{-2}$ fm^2 MeV$^{-1/2}$) and −4.19 meV with Coulomb corrections ($\bar\omega=30.1$ MeV) using Eq. (26). This is a reduction of about 12% and agrees with the estimate of Ref. 10 based on the results of Ref. 9.

We must remember that Rinker's calculation used the Dirac Hamiltonian H_D for the muon and a Breitlike equation[33] for the muon-nucleus system; that is, the complete Hamiltonian was taken to be $H=H_D+H_N+\Delta H_C$, where H_N is the (nonrelativistic) nuclear Hamiltonian. Aside from the transverse electromagnetic interaction not included in ΔH_C, this differs[33] somewhat from the approach of Ref. 8. We can estimate roughly the effect of relativistic kinematics in the wave function approximation by writing $\phi(r)\cong\phi_{\rm NR}(0)(1,0,0,0)$ where the latter construction is the four-component spinor and $\phi_{\rm NR}$ is the nonrelativistic muon wave function. The Green's function becomes $[E-(\vec\alpha\cdot\vec p+\beta m)-\omega_N]^{-1}$ $=(E-\omega_N+\vec\alpha\cdot\vec p+\beta m)/[(E-\omega_N)^2-\vec p^2-m^2]\to G_N'(\vec p^2)$ $=-(1-\omega_N/2\mu)/(-\omega_N^2/2\mu+\vec p^2/2\mu+\omega_N)$, where we have used $\langle\vec\alpha\rangle=0$, $\langle\beta\rangle=1$, $E=m$, and have replaced the muon mass m by μ the reduced mass. We may utilize this result by replacing $(q^2/2\mu+\omega_N)^{-1}$ by $-G_N'(\vec q^2)$ in Eq. (11); for $\omega_N\ll2\mu$ they are identical. The denominator in the final expression above can vanish if $\omega_N>2\mu$ and could produce an imaginary part of the amplitude for the scattering of zero kinetic energy muons. This is clearly impossible and is caused by Brown's disease,[34] a defect of relativistic Breit-like equations.[33] In our case $\omega_N\cong\bar\omega\ll2\mu$, so the problem is more conceptual than of numerical importance. As before, the unretarded dipole approximation can be evaluated and the q integral produces $\pi[\mu/2\omega_N]^{1/2}(1-\omega_N/2\mu)^{1/2}\cong\pi[\mu/2\omega_N]^{1/2}(1-\omega_N/4\mu)$. The correction factor modifies Eq. (14) to the form

$$\Delta E_p^{UD'}=-\sqrt{2\mu}\,\frac{\alpha}{\pi}|\phi(0)|^2(\sigma_{-3/2}-\sigma_{-1/2}/4\mu)\,. \qquad (14')$$

With the Rinker parametrization we find $\sigma_{-1/2}/(4\mu\sigma_{-3/2})=0.083$ and this correction agrees well with the unretarded dipole result obtained using $G_N'(\vec q^2)$ in Eq. (12) and including Coulomb corrections, −3.82 meV. These relativistic corrections are therefore not particularly large.

Using the dipole *form factor* reduces this result considerably. Various forms may be used. The Goldhaber-Teller model[35,36] of the giant dipole resonance predicts that $F=F_0$, the ground state elastic form factor: that is, $\langle N|\hat\rho(\vec q)|0\rangle$ $=i\vec D_{N0}\cdot\vec q F_0(\vec q^2)$, and sum rules[29] suggest a similar behavior. The transition charge density in this model is proportional to $d\rho_0(r)/dr$, while Rinker uses $r\,d\rho_0(r)/dr$. The latter form is harder to work with, except for a Gaussian $\rho_0(r)$, where one obtains a modified Gaussian form $F(\vec q^2)$ $=F_0(\vec q^2)\,{}_1F_1(-\frac{1}{2},\frac{5}{2};\frac{1}{6}\vec q^2\langle\vec r^2\rangle)$. If the confluent hypergeometric function ${}_1F_1$ is expanded to first order in $\vec q^2$, it has the same effect as a 20% increase in $\langle\vec r^2\rangle$ in the Gaussian $F_0=\exp(-\frac{1}{6}\vec q^2\langle\vec r^2\rangle)$. Indeed, the modified Gaussian results obtained using Eq. (12) are virtually identical to those obtained using[3] a Gaussian F with $\langle\vec r^2\rangle=(1.65\text{ fm})^2\to1.2\langle\vec r^2\rangle$. The Gaussian results for the no-Coulomb, Coulomb-corrected, and Coulomb-plus-relativistic corrections are −3.21, −2.83, and −2.67 meV, respectively, while the corresponding modified Gaussian results are −3.10, −2.73, and −2.58 meV. Fermi distribution form factors calculated using the analytic expressions of Maximon and Schrack[37] and the parameters of Rinker, which produce the same $\langle\vec r^2\rangle$ used above with the Gaussian, generate results virtually identical to the Gaussian ones. We conclude on the basis of these results that models with the same $\langle\vec r^2\rangle$ should produce very comparable results for ΔE_p^D. By varying the size parameter $\langle \vec{r}^2\rangle$ in the Gaussian form factor we also find that ΔE_p is highly sensitive to $\langle \vec{r}^2\rangle$ only for small values of that parameter; it is much less sensitive and therefore less model dependent over a rather wide range of values bracketing the physical values associated with F_0 and falls off as $1/\langle \vec{r}^2\rangle^{1/2}$ for large values.

In developing the unretarded dipole approximation ($\langle \vec{r}^2\rangle = 0$) we showed that the final result was proportional to $\sigma_{-3/2}$. The dipole approximation in the opposite limit $\langle \vec{r}^2\rangle \to \infty$ is proportional to σ_{-2} since only very small values of $\vec{q}^2$ contribute [$F^2 \cong 0$ in Eq. (15) if $\vec{q}^2 \neq 0$]. Clearly the physical situation lies somewhere in between. Rinker parametrized the coefficient of $\sigma^{\gamma}(\omega)$ in Eq. (15) and found an effective value $\lambda(\omega_{\text{th}}/\omega)^{1.7}$. With this form and Rinker's value of λ we obtain -2.41 meV for ΔE_p, in good agreement with Rinker's value of -2.38. Our own results indicate that a fit of the form $\omega^{-\gamma}$ is somewhat rough and that γ is slightly smaller than 1.7.

The calculation of Bernabeu and Jarlskog (BJ) is the most complete attempt to calculate ΔE_p in the absence of Coulomb distortion and at the same time relate the nuclear physics information required for the calculation directly to experimental data. We have done the same thing in Eq. (8) on a less sophisticated and more transparent level. In order to evaluate the integrals in Eq. (8), BJ use a quasielastic model for W_c (and the corresponding transverse form factor) which they claim adequately agrees with available experimental data. They write, for example,

$$W_c(q^2,\omega) = \delta(\omega - q^2/2m_N - \omega_{\text{th}})\,F(q^2)\,, \qquad (27\text{a})$$

$$F(\vec{q}^2) = \sum_{N\neq 0} |\langle N|\,\hat{\rho}(\vec{q})|\,0\rangle|^2 = \langle 0|\,\hat{\rho}(-\vec{q})\hat{\rho}(\vec{q})|\,0\rangle - Z^2F_0^2(\vec{q}^2)\,, \qquad (27\text{b})$$

and assume the usual space-symmetric shell model for the ground state, which produces

$$F(\vec{q}^2) = Z - Z^2F_0^2(\vec{q}^2) + Z(Z-1)\,C(\vec{q},-\vec{q})\,, \qquad (28)$$

with $C(\vec{q}_1,\vec{q}_2) = \langle 0|\,e^{i\vec{q}_1\cdot\vec{x}_1'}e^{i\vec{q}_2\cdot\vec{x}_2'}|\,0\rangle$, where $\vec{x}_1'$ and $\vec{x}_2'$ are the coordinates of nucleons one and two. For the hyperspherical model,[38] or equivalently (for ^{4}He) the harmonic oscillator model,[39] the function C may be directly related to F_0 in the form $C(\vec{q}_1,\vec{q}_2) = F_0(\vec{q}_1^{\,2} + \vec{q}_1^{\,2} - \frac{2}{3}\vec{q}_1\cdot\vec{q}_2)$ and thus $C(\vec{q},-\vec{q}) = F_0(\frac{8}{3}\vec{q}^2)$. Writing $F_0 = \exp(-\frac{1}{6}\vec{q}^2\langle \vec{r}^2\rangle)$ determines F in terms of a single parameter. It is important to observe that the *dipole* states determine the low-q behavior of $F(\vec{q}^2)$. We have $F(\vec{q}^2) \to (\frac{4}{9}\langle \vec{r}^2\rangle)\vec{q}^2$ and $\int W_c(q^2,\omega)\,\omega^n\,d\omega \to \omega_{\text{th}}{}^nF(\vec{q}^2) \to \omega_{\text{th}}{}^n(\frac{4}{9}\langle r^2\rangle)\vec{q}^2$. In particular, we may obtain $\alpha_E = q^{-2}\int (W_c/\omega)\,d\omega|_{q^2=0} = 2\alpha(\frac{4}{9}\langle \vec{r}^2\rangle/\omega_{\text{th}}$. If we choose $\langle \vec{r}^2\rangle$ to fit α_E (74×10^{-4} fm^2/MeV), we have in effect guaranteed that the long-range part of the polarization potential is correct. This value of $\langle \vec{r}^2\rangle$ is (1.08 fm)^2, which is much smaller than the physical value of (1.65 fm).[2] Using the former value for the only parameter in the model we obtain $\Delta E_p' = -3.28$ meV from Eqs. (8) and (27) (all results for the BJ model will be denoted with a prime). Because of the δ function in energy only a single q integral remains in Eq. (8), and 90% of this integral comes from $\vec{q}^2 < 1$ fm^2. In Ref. 8 the parameter $\langle \vec{r}^2\rangle$ was modified for q beyond ~1 fm, which produced a better correspondence with experimental electron scattering data. We have not done this; it appears that this modification would not have a large effect on the result. Indeed, we can neglect completely any deviation from the threshold behavior of $F(q^2)$ and simply use $F(\vec{q}^2) = (\frac{1}{2}\alpha_E\omega_{\text{th}})\vec{q}^2$. This is the unretarded dipole approximation and it produces -3.61 meV for $\Delta E_p'$ from the analytic result

$$\Delta E_p'^{UD} = -\sqrt{2}\mu'\,(\alpha/\pi)\,|\,\phi(0)|^2(\sigma_{-2}\sqrt{\omega_{\text{th}}})\,, \qquad (29)$$

where the term in parentheses is $\sigma_{-3/2}$ for the model of BJ, and $1/\mu' = 1/\mu + 1/m_N$, where m_N is the nucleon mass. We need to examine the reason why the unretarded result above differs so greatly from the analogous result obtained before, -4.76 meV. This difference is almost completely due to a peculiarity of BJ's model; if σ_{-2} is fixed, $\sigma_{-3/2}$ is poorly determined. We note that for the model of BJ, $(\sigma_{-3/2}/\sigma_{-2})^2 = \omega_{\text{th}} = 20.1$ MeV while for the Rinker parametrization, $(\sigma_{-3/2}/\sigma_{-2})^2 = 30.85$ MeV. Rinker's results and our own suggest that one should fix $\sigma_{-1.7}$ for the best results for the dominant dipole contribution. If one fits $\sigma_{-3/2}$ (Rinker parametrization) one obtains $\Delta E_p' = -4.03$ meV and $\Delta E_p' = -3.71$ meV if $\sigma_{-1.7}$ is fit. The latter result is a 13% increase over the $\Delta E_p'$ fit to σ_{-2}.

The dipole part of the function F may be projected out by writing

$$\begin{aligned} F \to& \langle 0|\,\hat{\rho}(-\vec{q}_2)\hat{\rho}(\vec{q}_1)|\,0\rangle - F_0(\vec{q}_1^{\,2})F_0(\vec{q}_2^{\,2}) \\ =& ZF_0(|\,\vec{q}_1 - \vec{q}_2|^2) - Z^2F_0(\vec{q}_1^{\,2})F_0(\vec{q}_2^{\,2}) \\ &- (Z^2 - Z)\,C(\vec{q}_1,-\vec{q}_2)\,, \end{aligned} \qquad (30)$$

and the lth multipole part of F, denoted F_l, is given in the usual way by

$$F_l(q^2) = \tfrac{1}{2}(2l+1)\int_{-1}^{1} P_l(x)\,F(q^2;x)\,dx\,, \qquad (31)$$

where we write $\vec{q}_1^{\,2} = \vec{q}_2^{\,2} = \vec{q}^2$ and $\vec{q}_1\cdot\vec{q}_2 = \vec{q}^2x$. Defining

$$h(x) = (e^x + e^{-x})/x - (e^x - e^{-x})/x^2\,, \qquad (32)$$

we find

$$F_1(\vec{q}^2) = 3e^{-\vec{q}^2\langle \vec{r}^2\rangle/3}[h(\tfrac{1}{3}\vec{q}^2\langle \vec{r}^2\rangle) + h(-\tfrac{1}{9}\vec{q}^2\langle \vec{r}^2\rangle)]\,, \qquad (33)$$

This function F_1 has the same threshold behavior as F. Inserting F_1 into Eq. (8) in place of F, we obtain $\Delta E_p^{D'} = -2.98$ meV if σ_{-2} is fitted while -3.64 meV is obtained if $\sigma_{-3/2}$ is fitted.

These various results may be understood by examining the separate forms for F we have used. All three forms are positive definite and have the same threshold behavior $\sim\vec{q}^2$. The unretarded approximation simply continues this behavior for all $\vec{q}^2$, while the complete form in Eq. (28) smoothly and asymptotically approaches Z as $\vec{q}^2$ increases. The dipole form F_1 rises and then smoothly falls as $9/(\vec{q}^2\langle \vec{r}^2\rangle)$. Thus each succeeding form has a smaller area under it and generates a correspondingly smaller $\Delta E_p'$. Studies indicate that $\Delta E_p'$ is even less sensitive than ΔE_p was to writing $F = \frac{4}{9}\langle r^2\rangle \vec{q}^2 F'(\vec{q}^2)$ and varying the size parameter in simple pole or dipole forms for F' required for the correct asymptotic form. One apparent difference between the calculations of BJ and Rinker is the relatively larger contribution of higher multipoles in the latter calculation (~0.7 meV) than in the former (0.3–0.4 meV), *if* our own model calculation corresponds to what was actually done by BJ. On the other hand, if one uses our previous estimates of approximately a 20% reduction in ΔE_p due to Coulomb and relativistic kinematic effects, there is not an appreciable difference between ($-3.1\times1.2 = -3.7$ meV) and the result one obtains from the BJ model while fitting $\sigma_{-1.7}$.

In summarizing this section, we have estimated an effective upper limit of -3.8 meV for the dipole part of the polarization correction, while our best estimate is -2.6 meV, which is about 7% higher than Rinker's estimate. This discrepancy is not particularly disturbing since our estimates are not precise and the nuclear model we have used is slightly different from Rinker's model. The results of BJ should probably be increased by about 13%, which corresponds to using a physical value for $\sigma_{-1.7}$ in their calculation, rather than σ_{-2}. At the same time Coulomb distortion should lower their result about 12%, leaving their result essentially unchanged overall. In view of uncertainties in the higher (than $E1$) multipoles and in effective form factors, the value[3,8] of -3.1 meV$\pm$20% is probably a reasonable estimate of the nuclear polarization and its uncertainty.

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