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Low-energy theorem for Compton scattering and the Drell-Hearn-Gerasimov sum rule: Exchange currents*

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The low-energy theorem for Compton scattering is derived using a nonspecific nuclear model which includes exchange currents and relativistic corrections. The Drell-Hearn-Gerasimov sum rule is also derived using this model and the subtraction problem is addressed.

[NUCLEAR REACTIONS Compton scattering, exchange currents, relativistic corrections, sum rules.]

I. INTRODUCTION

One of the more remarkable theorems in electromagnetic physics is the low-energy theorem for Compton scattering (LET) of photons.[1,2] This theorem, proven many years ago for arbitrary systems which possess minimal invariance properties, states that the charge, mass, and magnetic moment determine the photon scattering amplitude in the limit of long wavelengths. All that is needed in order to prove the theorem are Lorentz invariance and gauge invariance of the Compton amplitude, a nondegenerate (except for magnetic quantum numbers) ground state, and sufficient analyticity of matrix elements to allow long-wavelength expansions to be made. In addition, time reversal (or parity) invariance is assumed. As stated by Krajcik and Foldy,[3] this theorem has an "impeccable lineage."

Concomitant with this theorem are two sum rules, which require additional assumptions. We may write the amplitude for the *forward* scattering of photons of energy ω in the form

$$f(\omega)=\vec{\epsilon}\cdot\vec{\epsilon}\,'f_1(\omega)+i\omega f_2(\omega)\vec{S}\cdot\vec{\epsilon}\,'\times\vec{\epsilon}\,, \tag{1}$$

where $\vec{\epsilon}$ and $\vec{\epsilon}\,'$ are the initial and final photon polarization vectors, $\vec{S}$ is the spin operator of our system (nucleus), and f_1 and f_2 are the spin-independent and spin-flip amplitudes, respectively. Crossing symmetry requires that $f_1(-\omega)=f_1(\omega)$ and $f_2(-\omega)=f_2(\omega)$, while the low-energy theorem specifies that

$$f_1(0)=\frac{Z^2}{M_B}\,, \tag{2a}$$

$$f_2(0)=\left(\frac{\mu}{S}-\frac{Z}{M_B}\right)^2. \tag{2b}$$

We have extracted the factors of nucleon charge, e, from our amplitude and written f in terms of the proton number Z, nuclear *total* mass M_B, and the total nuclear magnetic moment μ. The maximum spin component is S.

If one assumes that the amplitudes f_1 and f_2 satisfy dispersion relations, it is possible to derive sum rules. The sum rule for f_1 is known to require a subtraction because of the low-energy theorem; we will ignore this amplitude in what follows. Assuming that $f_2(\omega)$ approaches a constant at infinite photon energy, we may write

$$f_2(0)=f_2(\infty)+\frac{1}{4\pi^2\alpha S}\int_{\omega_{\rm th}}^{\infty}(\sigma_P-\sigma_A)\frac{d\omega'}{\omega'}\,, \tag{3}$$

where P and A refer to polarized photons with helicities parallel and antiparallel to the nuclear spin in its maximum spin state, respectively. The photon absorption cross sections σ are integrated from threshold, $\omega_{\rm th}$. If[4] $f_2(\infty)=0$, the use of the low-energy theorem produces the Drell-Hearn-Gerasimov (DHG) sum rule[5-7]:

$$\int_{\omega_{\rm th}}^{\infty}(\sigma_P-\sigma_A)\frac{d\omega'}{\omega'}=4\pi^2\alpha S\left(\frac{\mu}{S}-\frac{Z}{M_B}\right)^2. \tag{4}$$

The existence of the unsubtracted dispersion relation clearly has important consequences, since it immediately produces a sum rule.[8]

Several years ago, it was pointed out[4] that the use of ordinary Foldy-Wouthuysen[9] electromagnetic interactions for a composite system of "fundamental" particles violates Eq. (4) and that perhaps a subtraction is necessary. It was shown by Refs. 3, 9–13 that the difficulty with the calculation of Ref. 4 was the neglect of relativistic effects on the wave function of the moving nucleus,[14] or equivalently, that the matrix elements of the charge and current operators of the composite system do not have proper Lorentz transformation properties[15-20]; the latter properties are necessary for the proofs of both the LET and the DHG sum rule. Nevertheless, the subtraction question is separate from the problem of doing the physics correctly. There clearly exist models which give the correct

LET but require a subtraction; these models probably are not physical, however.

It is relevant to our discussion to point out that the nonrelativistic model (plus relativistic corrections) which is conventionally used, and which we will use, does not generate a Compton amplitude that has sufficiently good analytic properties to allow a conventional dispersion relation to be written.[21-23] The additional (spurious) singularities are sufficiently far from threshold that they probably do not affect the calculation of the DHG sum rule, however.

In Sec. II, we discuss the matrix elements of the charge, current, and Compton seagull operators, meson exchange contributions to these operators, and Lorentz invariance. In Sec. III, we derive the LET in two different ways, and in Sec. IV we discuss the DHG sum rule in such a way that the subtraction problem is emphasized. Throughout this work we will stress the meson-exchange current aspect of the problem; although the results of this work are not new, we hope to gain insight into exchange currents and their consequences. In addition, we will directly relate the charge-current density commutator algebra to the two problems, since this is how gauge and Lorentz invariance enters the calculation. Relativistic and exchange corrections to the dipole operator also play an important role.

II. CHARGES, CURRENTS, AND SEAGULLS

In order to minimize calculational complexity, it is conventional to adopt an ordering scheme when calculating relativistic corrections. The most obvious criterion is to keep terms through some agreed powers of (v/c), where v is a representative velocity in the system we are treating. Alternatively, since $v = p/m$ where p and m are representative momentum and mass, respectively, we may count powers of $(1/m)$. We choose the latter procedure because keeping factors of c is tedious and messy, and much of the past work we will refer to used this convention. In addition, because nuclei are weakly bound, the potential and kinetic energy are roughly equal in magnitude (and opposite in sign), and the nonrelativistic potential will be treated as order $(1/m)$; meson exchange contributions to charge and current operators will be treated on the same footing as the conventional (kinetic) contributions to these operators. In particular the charge density $\rho(\vec{x})$ has the usual leading term ρ_0, of order $(1/m)^0$, in addition to relativistic corrections[16-20] of orders $(1/m^2)$ and higher arising from both kinetic and (in general) exchange terms. The current consists of convection, magnetization, and exchange parts of order $(1/m)$, while corrections are of order $(1/m^3)$ and higher. The potential V consists of terms of order $(1/m)$, V_0, the nonrelativistic, static terms, and nonstatic terms of order $(1/m^3)$, and higher, ΔV. The nonstatic parts of V produce corrections of order $(1/m^2)$ to the wave function of our system.[14]

We begin by examining the S matrix for the one (Ref. 25) and two (Ref. 26) photon processes. For absorption of a photon with momentum $\vec{k}$ and energy ω we have

$$S_\gamma = \frac{-ie}{(2\omega)^{1/2}}(2\pi)^4\delta^4(P_f - P_i - k)\langle f\vec{P}_f|\hat{J}_\mu(0)\epsilon^\mu|i\vec{P}_i\rangle , \tag{5}$$

where e is the (positive) fundamental charge ($\alpha \equiv e^2/4\pi$, the fine structure constant), P_f and P_i are the final and initial total (nuclear) momenta, and ϵ^μ is the photon polarization vector. We use the conventions and metric of Ref. 27. The important physics is contained in the matrix element of the four-current operator evaluated at $\vec{x} = t = 0$, $\hat{J}_\mu(0)$, which we have written in a arbitrary frame of reference. We will emphasize two properties of this matrix element: (1) gauge invariance or current conservation, and (2) (approximate) Lorentz invariance.

The two-photon Compton amplitude can be constructed easily[24,25] and the scattering of a photon of initial momentum $k^\mu = (\omega, \vec{k})$ and polarization $\epsilon^\mu(\vec{k})$ and final momentum and polarization k' and ϵ' is determined by

$$S_{\gamma\gamma} = -\frac{i(2\pi)^4}{(4\omega\omega')^{1/2}}\delta^4(k' + P_f - P_i - k)e^2\epsilon_\mu(\vec{k})\epsilon_\nu(\vec{k}')T^{\mu\nu} , \tag{6a}$$

$$T^{\mu\nu} = \sum_n \frac{\langle i|J^\nu(-\vec{k}', 2\vec{P}_f + \vec{k}')|n\rangle\langle n|J^\mu(\vec{k}, 2\vec{P}_i + \vec{k})|i\rangle}{-\omega_{ni} + \omega + i\epsilon} + \sum_n \frac{\langle i|J^\mu(\vec{k}, 2\vec{P}_f - \vec{k})|n\rangle\langle n|J^\nu(-\vec{k}', 2\vec{P}_i - \vec{k}')|i\rangle}{-\omega'_{ni} - \omega' + i\epsilon} + B^{\mu\nu}(k, k') \tag{6b}$$

which is crossing symmetric ($\mu \leftrightarrow \nu$, $k \leftrightarrow -k'$). We have denoted by ω_{ni} the energy difference of intermediate and initial states and the prime indicates different intermediate momenta in the two "dispersive" terms. The "seagull" amplitude $B^{\mu\nu}$ arises from direct two-photon interactions with nucleons. In addition, we have anticipated[25] that matrix elements of the current may be written in the form

$$\langle f\vec{P}_f|\hat{J}^\mu(0)|i\vec{P}_i\rangle \equiv \langle f|J^\mu(\vec{q},\vec{s})|i\rangle\,, \quad (7a)$$

$$q = P_f - P_i\,, \quad (7b)$$

$$s = P_f + P_i\,, \quad (7c)$$

where the nuclear states $|f\rangle$ and $|i\rangle$ are in their center-of-mass system ($\vec{P}\equiv 0$) and J^μ is a function of *internal* coordinates only. Clearly any dependence of J^μ on the time components of q^μ and s^μ may be eliminated by using energy conservation and this will be discussed shortly.

The two most important relationships we will use follow from the assumed gauge invariance of the one- and two-photon amplitudes. We require that

$$q^\mu J_\mu(\vec{q},\vec{s}) = 0\,, \quad (8a)$$

$$k^\mu T_{\mu\nu} = k'^\nu T_{\mu\nu} = 0\,, \quad (8b)$$

where we have removed the initial and final states from (8a) for convenience. Because of crossing symmetry only one of the two identities (8b) is useful. Combining (8a) and (8b) we find the "current algebra" which holds for both the real and virtual Compton amplitudes

$$J^\nu(-\vec{k}', 2\vec{P}_f + \vec{k}')\rho(\vec{k}, 2\vec{P}_i + \vec{k}) - \rho(\vec{k}, 2\vec{P}_f - \vec{k})J^\nu(-\vec{k}', 2\vec{P}_i - \vec{k}') + k_\mu B^{\mu\nu} = 0 \quad (8c)$$

which relates the commutators of J^ν and ρ to the seagull amplitude $B^{\mu\nu}$. This relation *alone* is a powerful constraint on sum rules and low-energy theorems.[22,24]

The requirements of special relativity enter in in a natural way through the transformational properties of the charge-current matrix elements. What results from analyzing these properties[15,18-20] are equations involving commutators of $\vec{J}$ and ρ with the "boost" operator and the Hamiltonian. Our requirements here are far less detailed and we will not need the complete machinery of special relativity. Our basic requirement is a knowledge of the structure of the $\vec{s}$ dependence of $\rho(\vec{q},\vec{s})$. This necessitates a digression. The $\vec{s}$ dependence in the absence of exchange effects was examined in Ref. 16 and arose from two primary causes: (1) the momentum dependence of the spin-orbit charge density (to be examined below), which is a component of the Foldy-Wouthuysen charge operator, and (2) the effect of relativity on the wave function of a moving system introduced into the density operator throught the transformation Eq. (7a). In particular, the wave function of a slowly moving composite system has the form[13,14,17,18]

$$|i\vec{P}\rangle \simeq [1 - i\chi(\vec{P})]|i\rangle e^{i\vec{P}\cdot\vec{R}}\,, \quad (9a)$$

where χ is an operator of order $(1/m^2)$[i.e.$(v/c)^2$] which vanishes for $\vec{P}=0$ and $\vec{R}$ is the usual nonrelativistic definition of the center-of-mass coordinate (see Refs. 17, 18). This wave function is an eigenfunction of the momentum operator with eigenvalue $\vec{P}$ and of the Hamiltonian H with eigenvalue $E_i(\vec{P})$ given to order $(1/m)^3$ by

$$E_i(\vec{P}) = M_t + \epsilon_i + \frac{\vec{P}^2}{2M_t} - \frac{\vec{P}^2\epsilon_i}{2M_t^2} - \frac{\vec{P}^4}{8M_t^3} \simeq (\vec{P}^2 + M_B^2)^{1/2}\,, \quad (9b)$$

where $M_B = M_t + \epsilon_i$ is the sum of the masses of the nuclear constituents ($M_t = \sum_i m_i$) and the (negative) binding energy ϵ_i. Equation (9b) will be important later.

Using Eq. (9a) we also have

$$J_\mu(\vec{q},\vec{s}) \simeq J^0_\mu(\vec{q},\vec{s}) + i(\chi(\vec{P}_f)J_\mu(\vec{q},\vec{s}) - J_\mu(\vec{q},\vec{s})\chi(\vec{P}_i))\,, \quad (10)$$

where J^0_μ arises from the Foldy-Wouthuysen charge and current densities. This equation is the equivalent of $J_\mu = e^{i\chi}J^{\rm FW}_\mu e^{-i\chi}$ in the language of Ref. 3. In the absence of exchange currents one finds[16]

$$\rho(\vec{q},\vec{s}) = \left(1 - \frac{\vec{s}\cdot\vec{q}}{8M_t^2}\vec{s}\cdot\vec{\nabla}_q\right)\rho_0(\vec{q}) + \Delta\rho(\vec{q}) + \left[h_0, \frac{\vec{s}\cdot\vec{\nabla}_q}{2M_t}\rho_0(\vec{q})\right] + \frac{\vec{s}\cdot\vec{J}_0(\vec{q})}{2M_t} + \frac{i}{4M_t}\vec{s}\times\vec{q}\cdot\{\vec{J}', \rho_0(\vec{q})\}\,, \quad (11)$$

where h_0 is the *internal* nonrelativistic nuclear Hamiltonian ($h_0|i\rangle \simeq \epsilon_i|i\rangle$), $\vec{J}_0$ is the usual nonrelativistic internal current operator, $\vec{J}'$ is the *internal* angular momentum operator, and $\Delta\rho$ is the $\vec{s}$-independent relativistic correction to ρ_0. The operators h_0 and $\vec{J}_0$ are order $(1/m)$, while ρ_0 and $\Delta\rho$ are of order $(1/m)^0$ and $(1/m)^2$, respectively. If one introduces pion-exchange currents (and presumably other types, as well) one can show explicitly[18] that the only effect is to modify $\Delta\rho$ and to introduce the meson-exchange potential in h_0 and the nonrelativistic meson-exchange current in $\vec{J}_0$. The proof is algebraically complex, but the result is hardly surprising. Equation (11) is an important part of the rest of this paper.

The operator $\Delta\rho(\vec{q})$ is also interesting in that it defines the relativistic corrections to the dipole operator. Defining in the usual way

$$\vec{D} = -i\vec{\nabla}_q[\rho_0(\vec{q}) + \Delta\rho(\vec{q})]_{\vec{q}=0}\,, \quad (12a)$$

we find[17]

$$\vec{D} = \sum_{i=1}^{A}\left[e_i\vec{x}_i' - \frac{(2\mu_i - e_i)}{4m_i^2}\vec{\sigma}(i)\times\vec{\pi}_i + \frac{Z}{2M_t}\frac{\vec{\sigma}(i)\times\vec{\pi}_i}{m_i} + \frac{Z}{2M_t}\{\vec{x}_i', \vec{\pi}_i^2/2m_i\} - \frac{Z}{2M_t}\vec{x}_i'\sum_{j\neq i}V_{ij}\right] + \Delta\vec{D}^M\,, \quad (12b)$$

where V_{ij} is the two-body nonrelativistic (momentum-independent) potential and $\Delta\vec{D}^M$ is the meson exchange contribution to $\vec{D}$ in the "standard" representation.[18] We have also defined the usual relative coordinates and momenta $\vec{x}_i' = \vec{x}_i - R$ and $\vec{\pi}_i = \vec{p}_i - m_i\vec{P}/M_t$ for the ith particle in terms of canonical coordinates and momenta $\vec{x}_i$ and $\vec{p}_i$. This quantity ($\vec{D}$) is useful in the derivation of low-energy theorems, because it determines the long-wavelength limit of the current operator. Using Eq. (8a) with $\vec{s} \equiv 0$, we find

$$\vec{q}\cdot\vec{J}(\vec{q}) = [h, \rho(\vec{q})] \tag{13}$$

and a single derivative with respect to $\vec{q}$, and subsequently setting $\vec{q}$ to zero yields

$$\vec{J}(0) = i[h, \vec{D}]\,. \tag{14}$$

If we *assume* that $\vec{D}$ is simply given by the first term in Eq. (12b), we have Siegert's theorem.[29] In these expressions h is the complete internal nuclear Hamiltonian and includes relativistic corrections to both the potential and kinetic energies.

The nuclear charge and current operators are usually[18,20] derived using a procedure which consists of the following (or its equivalent): (1) perform a nonrelativistic reduction of the equations of motion of individual nucleons interacting with mesons (e.g., a Foldy-Wouthuysen reduction)[28]; (2) use the meson equations of motion to connect meson vertices; this produces exchange operators; (3) the same procedure produces the meson-exchange potentials by means of a "renormalization" scheme. The nonexchange or kinetic parts of ρ and $\vec{J}$ are well known,[15] and, in general, rather complicated. We will show in Sec. III that only terms in the current operator of orders $1/m$, $1/m^3$, ω/m, and ω/m^2 are required, and in the charge operator of orders $(1/m)^0$ and $1/m^2$. Such terms are generated by the following contributions to the nuclear Hamiltonian:

$$H_N = \sum_{i=1}^{A} \frac{(\vec{p}_i - e_i\vec{A}_i)^2}{2m_i} - \frac{(\vec{p}_i - e_i\vec{A}_i)^4}{8m_i^3} + e_i\phi_i - \frac{\mu_i\vec{\sigma}(i)\cdot\vec{B}_i}{2m_i} + \frac{(2\mu_i - e_i)}{8m_i^2}\vec{\sigma}(i)\times\{(\vec{p}_i - e_i\vec{A}_i);\vec{E}_i\} \tag{15}$$

which is obviously gauge invariant. The quantities ϕ_i, $\vec{A}_i$, $\vec{B}_i$ and $\vec{E}_i$ are the external electromagnetic scalar and vector potentials, magnetic and electric fields at the position of the ith nucleon with charge e_i, mass m_i, magnetic moment μ_i, Pauli spin operator $\vec{\sigma}(i)$, and momentum $\vec{p}_i$. Since the Hamiltonian is gauge invariant, it will generate amplitudes in perturbation theory which are gauge invariant. To H_N in Eq. (15) must be added the potential terms and exchange charges and currents. The derivation of such quantities must be performed for each specific type of exchange; the reader is referred to Refs. 18–20 for examples. The strong interaction Hamiltonian may then be written in the nuclear c.m. frame

$$h = \sum_i (\vec{\pi}_i^{\,2}/2m_i - \vec{\pi}_i^{\,4}/8m_i^3) + V_0 + \Delta V(\vec{P} = 0)\,, \tag{16}$$

where ΔV is of the same order $(1/m^3)$ as the second term. With the kinetic energy parts, we may immediately verify those parts of Eq. (14) independent of potential. The isoscalar potential-dependent parts of this equation may be verified, for example, using the scalar and vector meson exchange potentials, charges, and currents derived in Ref. 20 ($\Delta\vec{D}^M = 0$, in this case).

The last term in Eq. (15) presents a slight problem; since $\vec{E} = -\vec{\nabla}\phi - \dot{\vec{A}}$, the current is proportional to q^0. As we noted earlier, it is possible to replace a term of the form $q^0\vec{H}$ by a commutator, namely $[H_0, \vec{H}]$, where H_0 is the nonrelativistic Hamiltonian. This is useful in discussing the Lorentz transformational properties,[15] and is the standard representation used in describing exchange currents. In the problem under discussion it will also be necessary to transform the two-photon amplitude, as well. This is easily accomplished by writing the spin-orbit current schematically as $\vec{J}_{so} = q^0\vec{H} \equiv [H_0, \vec{H}] + [q^0\vec{H} - [H_0, \vec{H}]]$ for the appropriate q^0 in the two-photon amplitude. The extra (bracketed) term cancels the denominator in Eq. 6(b), and to order $(1/m^3)$ simply modifies the seagull terms $B^{\mu\nu}$. *Only the seagull operators then depend on ω and ω'.* In this representation the Hamiltonian in Eq. (15) generates charges, currents, and seagulls in the form[16–19]

$$\rho(\vec{q}) = \sum_i e_i e^{i\vec{q}\cdot\vec{x}_i'} - i\vec{q}\cdot\sum_i \frac{(2\mu_i - e_i)}{8m_i^2}\vec{\sigma}(i)\times\{\vec{\pi}_i, e^{i\vec{q}\cdot\vec{x}_i'}\}, \tag{17a}$$

$$\vec{J}(\vec{q},\vec{s}) = \sum_i \frac{e_i}{2m_i}\{\vec{\pi}_i, e^{i\vec{q}\cdot\vec{x}_i'}\} - \sum_i \frac{e_i}{8m_i^3}\{\vec{\pi}_i^{\,2}, \{\vec{\pi}_i, e^{i\vec{q}\cdot\vec{x}_i'}\}\} + i\sum_i \frac{\mu_i}{2m_i}\vec{\sigma}(i)\times\vec{q}e^{i\vec{q}\cdot\vec{x}_i'} + \frac{\vec{s}}{2M_t}\rho(\vec{q}) + [H_0, \vec{H}(\vec{q})] + \text{order}(s, q/m^3), \tag{17b}$$

$$B^{mn} = \sum_i \left[\delta^{mn} \frac{e_i^2}{m_i} e^{i(\vec{k}-\vec{k}')\cdot \vec{x}_i'} - \frac{e_i^2}{4m_i^3} \left(\{\vec{\pi}_i^2, e^{i(\vec{k}-\vec{k}')\cdot \vec{x}_i'}\}\delta^{mn} + \tfrac{1}{2}\{\{e^{i\vec{k}\cdot\vec{x}_i'}, \pi_i^m\}, \{\pi_i^n, e^{-i\vec{k}'\cdot\vec{x}_i'}\}\}\right)\right]$$
$$+ i(\omega+\omega')\sum_i \frac{(2\mu_i - e_i)}{4m_i^2} e_i \sigma^k(i)\epsilon_{kmn} e^{i(\vec{k}-\vec{k}')\cdot\vec{x}_i'} + [J_n(-\vec{k}'), H_m(\vec{k})] + [J_m(\vec{k}), H_n(-\vec{k}')], \quad (17c)$$

$$B^{j0} = -i\sum_i \frac{(2\mu_i - e_i)}{2m_i^2} e_i (\vec{\sigma}(i)\times\vec{k}')_j e^{i(\vec{k}-\vec{k}')\cdot\vec{x}_i'}, \quad (17d)$$

$$B^{0j} = i\sum_i \frac{(2\mu_i - e_i)}{2m_i^2} e_i (\vec{\sigma}(i)\times\vec{k})_j e^{i(\vec{k}-\vec{k}')\cdot\vec{x}_i'}, \quad (17e)$$

$$B^{00} = \text{order } (\omega^2/m^3)$$

$$\vec{H}(\vec{k}) = -i\sum_i \frac{(2\mu_i - e_i)}{8m_i^2} \{\vec{\sigma}(i)\times\vec{\pi}_i, e^{i\vec{k}\cdot\vec{x}_i'}\}. \quad (17f)$$

In this equation and elsewhere we will ignore the fact that e_i and μ_i are actually form factors. The leading contributions to B^{00} are generated by terms like $\vec{E}^2/m^3$ which arise in the FW expansion.[30] In nonrelativistic order B^{00} terms do not occur in meson-exchange contributions.[31] Note that the last term in Eq. (17a) produces the second term in Eq. (12b). To all the operators, ρ, $\vec{J}$, and B in Eq. (17) must be added exchange parts.

III. LOW-ENERGY THEOREM

We proceed with the development of the low-energy theorem in the lab frame in two ways. The first method is due to Low[1] and was the original method used to prove the theorem. We wish to keep all terms in the long-wavelength limit of the Compton amplitude of orders (ω/m^2) and $(1/m^3)$. We choose to work in transverse gauge which means that $\epsilon_0 = \epsilon_0' = 0$ and that $\vec{k}\cdot\vec{\epsilon} = \vec{k}'\cdot\vec{\epsilon}' = 0$; thus we will calculate $\epsilon_n(\vec{k}) T^{mn}\epsilon_n'(\vec{k}') \equiv T$. We follow previous work in separating the contributions to T^{mn} into ground and excited intermediate states. The ground state part has the form

$$T_{\rm GR}^{mn} \simeq \frac{1}{\omega}\langle i | J_n(-\vec{k}') + \frac{k_n}{M_t}\rho(-\vec{k}') | i\rangle \times \langle i | J_m(\vec{k}) | i\rangle \left(1 + \frac{\omega}{2M_B}\right) + \text{crossed term}, \quad (18)$$

where the current has been separated into internal and convection parts and the denominator has been expanded using $\omega_{ii} \simeq \omega^2/2M_B - \omega^4/8M_t^3$ from Eq. (9b). To the order we wish to work, only nonrelativistic currents of order $(1/m)$ will be needed in Eq. (18). To first order in photon momentum, the following multipole expansion is an identity[24] which encompasses all parts of the current: magnetization, convection, and exchange:

$$J_m(\vec{q}) = i[h_0, D_m] - \tfrac{1}{2} q_n [h_0, Q_{mn}] - i(\vec{q}\times\vec{\mu})_m. \quad (19)$$

The three terms are $E1$, $E2$, and $M1$, while Q_{mn} is the quadrupole tensor which will play no role in our final results, and $\vec{\mu}$ is the magnetic moment (including spin, orbital, and exchange parts). Clearly the $E1$ and $E2$ parts vanish and using $\omega - \omega' \simeq (\vec{k}-\vec{k}')^2/2M_B$ obtained from (9b) and energy conservation we find

$$T_{\rm GR}^{mn} \simeq \omega \sum_{i'} \langle i_f | (\hat{k}'\times\vec{\mu})_n | i'\rangle\langle i' | (\hat{k}\times\vec{\mu})_m | i\rangle + \frac{i\omega Z}{M_t}\langle i_f | (\hat{k}\times\vec{\mu})_m | i\rangle \hat{k}_n - (m \leftrightarrow n, \hat{k}\leftrightarrow\hat{k}') \quad (20)$$

which depends only on magnetic moments.

The magnetic quantum numbers have been explicitly introduced in Eq. (20) and using the Wigner-Eckart theorem[32] allows us to simplify things. We work in the spin space of our system which allows us to replace

$$\vec{\mu} \to \bar{\mu}\vec{S}, \quad \bar{\mu} \equiv \mu/S, \quad (21)$$

where $\vec{S}$ is the total spin operator of the nucleus and μ is the total magnetic moment. Using the commutation relation of the spin operators then allows us to simplify Eq. (20) and we obtain without *explicit* reference to the currents

$$T_{\rm GR} = \epsilon_m T_{\rm GR}^{mn}\epsilon_n'$$
$$= i\,\omega\bar{\mu}^2\vec{S}\cdot(\vec{\epsilon}'\times\hat{k}')\times(\vec{\epsilon}\times\hat{k}) - i\frac{Z\bar{\mu}\omega}{M_t} \times [(\vec{\epsilon}'\cdot\hat{k})\vec{S}\cdot\vec{\epsilon}\times\hat{k} - \vec{\epsilon}\cdot\hat{k}'(\vec{S}\cdot\vec{\epsilon}'\times\hat{k}')]. \quad (22)$$

The excited state and seagull parts may be obtained using both forms of Eq. (8b). We have

$$T^{00} \equiv \hat{k}_m T^{mn}\hat{k}_n'$$
$$= \sum_n \frac{\langle i | \rho(-\vec{k}', 2\vec{k}-\vec{k}') | n\rangle\langle n | \rho(\vec{k},\vec{k}) | i\rangle}{\omega - \omega_{ni}} + \text{crossed term} + B^{00}. \quad (23)$$

As we noted earlier, B^{00} vanished to order (ω/m^2) and the introduction of exchange currents should not alter this. Such terms should appear in order (ω^2/m^3). Dropping B^{00} we see that the excited state part of Eq. (23) contributes only through order (ω^2) and may be dropped [since B^{00} arises from the excited state spectrum (pair plus meson terms), it also has this form]. The ground state part may be evaluated using Eqs. (11), (19), and the multipole expansion

$$\rho_0(\vec{q}) + \Delta\rho(\vec{q}) \simeq Z + i\vec{q}\cdot\vec{D} - q^m q^n Q_{mn} + \cdots. \quad (24)$$

The quadrupole contribution is found to vanish to order (ω/m^2). We invoke parity or time reversal invariance to eliminate the electric dipole terms. Only the last two terms in Eq. (11) contribute to f_2 and we find

$$T^{00} \simeq \frac{Z^2}{M_B}(1+\hat{k}\cdot\hat{k}') + 2i\,\frac{Z\omega}{M_t}\left(\bar{\mu} - \frac{Z}{2M_t}\right)\vec{S}\cdot\hat{k}\times\hat{k}'. \quad (25)$$

In order to complete this proof, an invariance argument is needed. The excited state spectrum generates no singular terms if $\omega < \omega_{ni}$. Although time reversal alone suffices to complete proof,[3,33] we will follow Low and also assume parity conservation as well. The most general crossing symmetric form for $(T^{ij}_{\rm ex} + B_{ij})$ is then given by

$$(T^{ij}_{\rm ex} + B_{ij}) = A\delta_{ij} + B\omega S_k\epsilon_{kij} + C\{S_i, S_j\} + \text{order}(\omega^2). \quad (26)$$

Since it follows by inspection that $\hat{k}_m T^{mn}_{\rm GR}\hat{k}'_n \equiv 0$, Eqs. (23), (25), and (26) yield

$$A = \frac{Z^2}{M_B}, \quad B = \frac{2iZ}{M_t}\left(\bar{\mu} - \frac{Z}{2M_t}\right), \quad C = 0 \quad (27)$$

and thus

$$\epsilon_m(T^{mn}_{\rm ex} + B^{mn})\epsilon'_n = \frac{Z^2\vec{\epsilon}\cdot\vec{\epsilon}'}{M_B} + 2i\,\frac{\omega Z}{M_t}\left(\bar{\mu} - \frac{Z}{2M_t}\right)\vec{S}\cdot\vec{\epsilon}\times\vec{\epsilon}' \quad (28)$$

and combining this with Eq. (22) we have the complete low-energy theorem:

$$T = \frac{Z^2\vec{\epsilon}\cdot\vec{\epsilon}'}{M_B} + i\,\omega\bar{\mu}^2\vec{S}\cdot(\vec{\epsilon}'\times\hat{k}')\times(\vec{\epsilon}\times\hat{k}) - i\frac{Z\bar{\mu}\omega}{M_t} \times[(\vec{\epsilon}'\cdot\hat{k})\vec{S}\cdot\vec{\epsilon}\times\hat{k} - \vec{\epsilon}\cdot\hat{k}'(\vec{S}\cdot\vec{\epsilon}'\times\hat{k}')] + \frac{2i\omega Z}{M_t}\left(\bar{\mu} - \frac{Z}{2M_t}\right)\vec{S}\cdot\vec{\epsilon}\times\vec{\epsilon}' \quad (29)$$

which generates Eqs. (1) and (2) when $\hat{k}'\to\hat{k}$. We may also replace M_t by M_B to this order. We summarize our derivation by noting that it used gauge invariance, Eq. (8), multipole expansions and invariance arguments, first-order relativistic corrections to the Compton energy denominators, Eq. (9b), and *relativistic* and *exchange* corrections to the charge operator.

Although the argument above yields the correct result via Low's clever trick, it is instructive to examine how Eq. (28) results from a straightforward expansion of Eq. (6). We expand the energy denominators for $\omega \ll \omega_{ni}$; recoil factors in the energy play no role to order (ω) and will be ignored. Only dipole states enter the intermediate sum and the expansion (14) may be used. We find

$$T^{mn}_{\rm ex} \simeq -i\langle[J^m(0), D^n]\rangle - \omega\langle[D^m, D^n]\rangle. \quad (30)$$

To this must be added $\epsilon_m B^{mn}\epsilon'_n$ to form the complete T. If we ignore exchange effects and use the explicit current, seagull, and dipole operators given in Eqs. (12b) and (17) and evaluate the commutators in Eq. (30), we reproduce those parts of Eq. (29) which do not involve binding (i.e., in M_B) and exchange magnetic moments. For the spin-dependent parts this was the basic procedure used in Refs. 3, 9–12, except that those calculations were organized differently from the present one since the frequency-dependent form of the spin-orbit current was used. In addition the frequency dependence of the denominator of $T_{\rm ex}$ was ignored; a compensating term in $\vec{J}(\vec{q},\vec{s})$ in Eq. (10) was also ignored. Since the two deleted terms can be shown to cancel, no error was made.

Both commutators in Eq. (30) may be evaluated using the gauge invariance relation (8c). We wish to emphasize that this relation holds for both real and virtual photons and consequently ω and $|\vec{k}|$ may be regarded as independent variables. Since we have gone to some trouble to eliminate ω from the currents, it is clear that the ω dependence in $k_\mu B^{\mu\nu}$ must vanish identically. Thus we may write for *those terms* in B which depend on ω $(\simeq\omega')$

$$\omega B^{00}(\omega) - k^i B^{i0}(\omega) = 0, \quad (31a)$$

$$\omega B^{0i}(\omega) - k^j B^{ji}(\omega) = 0 \quad (31b)$$

and indicating by a prime a derivative with respect to ω we find after differentiating with respect to $\vec{k}$, and setting ω, $\vec{k}$, and $\vec{k}'$ to zero

$$\nabla^i_{\vec{k}} B^{00}(0) = B^{i0\prime}(0), \quad (32a)$$

$$\nabla^j_{\vec{k}} B^{0i}(0) = B^{ji\prime}(0). \quad (32b)$$

Since $B^{00}(0) = 0$ we have $B^{i0\prime}(0) = 0$, which obviously holds for Eq. (17d), just as Eq. (32b) holds for Eqs. (17c) and (17e). Furthermore, B^{ij} contributes to T to order (ω/m^2) through the term $\epsilon_m(B^{mn}(0) + \omega B^{mn\prime}(0))\epsilon'_n$ only; the latter term may be determined from Eq. (32b). Equation (8c) with $\nu = 0$ yields after dropping B^{00}

$$\rho(-\vec{k}', 2\vec{k} - \vec{k}')\rho(\vec{k}, \vec{k}) - \rho(\vec{k}, \vec{k} - 2\vec{k}')\rho(-\vec{k}', -\vec{k}') - k^i B^{i0} = 0. \quad (33)$$

If we take derivatives with respect to $\vec{k}$ and $\vec{k}'$, set $\vec{k}, \vec{k}'$, and ω to zero, and use Eqs. (11), (19), and (24) we obtain after some algebra

$$[D^m, D^n] + B_0^{\prime mn} = \frac{2iZ}{M_t}\left(\mu_k - \frac{ZS_k}{2M_t}\right)\epsilon_{kmn}. \quad (34)$$

Crossing symmetry and Eq. (32b) were use to manipulate the seagull term.

Similar manipulations may be performed on Eq. (8c) with $\nu = m$ which produce the analogous result

$$-i[J^m(0), D^n] + B^{mn}(0) = Z\nabla_k^n J^m(0, 2\vec{k})|_{\vec{k}=0} = (Z^2/M_B)\delta^{mn}, \qquad (35)$$

where the last relation follows from an additional piece of information. The only contribution to the right-hand side comes from the nuclear convection current term including relativistic corrections,[34] $J(\vec{q}, \vec{s}) = (\vec{s}/2M_B)\rho(\vec{q})$, which is an obvious modification of the usual nonrelativistic current term. Adding T^{ex} to B and using Eqs. (34), (35), and (21) reproduces Eq. (28) and the low-energy theorem results again.

Although we have skirted the direct use of model charges, currents, and seagulls, we wish to emphasize that all are affected by meson exchanges in a very complicated model-dependent way. The remarkable thing about the low-energy theorem is that the model dependence cancels to the order we have calculated. Nevertheless, using results for exchange effects which have been previously derived, we have shown in two separate ways how the theorem obtains.

IV. DHG SUM RULE

Concomitant with the f_2 low-energy theorem is the DHG sum rule, as we discussed in the Introduction. The technique used to derive the sum rule follows closely the procedure used in developing the LET. The total cross section for photoabsorption may be written in the form

$$\sigma(\omega) = \frac{4\pi^2\alpha}{\omega}\sum_{n\neq 0} |\langle n|\vec{\epsilon}\cdot\vec{J}(\vec{k},\vec{k})|i\rangle|^2\delta(\omega-\omega_{ni}). \qquad (36)$$

Because we are interested in the difference of integrated cross sections for photon helicities parallel and antiparallel to the nuclear spin, the polarization vectors are complex. We quantize the nuclear spin along the z axis, $\hat{z}$, and define our right- and left-circularly polarized photons (positive and negative helicity with respect to $\hat{z}$, respectively) by

$$\hat{e}_{+1} = -\frac{(\hat{e}_x + i\hat{e}_y)}{\sqrt{2}}, \quad \hat{e}_{-1} = (\hat{e}_x - i\hat{e}_y)/\sqrt{2}; \quad \hat{e}_0 = \hat{z} \qquad (37a)$$

with

$$\hat{e}_\lambda^* = (-1)^\lambda \hat{e}_\lambda;\ \hat{e}_{\lambda'}^*\cdot\hat{e}_\lambda = \delta_{\lambda\lambda'};\ \hat{e}_{\lambda'}^*\times\hat{e}_\lambda = i\delta_{\lambda\lambda'}\lambda\hat{e}_0. \qquad (37b)$$

Correspondingly, we replace $\vec{\epsilon}\,'$ by $\hat{e}_{\lambda'}^*$ in the Compton amplitude and $\vec{\epsilon}$ by $\hat{e}_\lambda$. From Eqs. (7b) and Eqs. (36) we can relate σ_P and σ_A, corresponding to photon polarizations +1 and −1, to the elastic forward scattering amplitude f_2:

$$\mathrm{Im}[f_2(\omega)] = \frac{1}{8\pi\alpha\langle\vec{S}\cdot\hat{k}\rangle}[\sigma_P(\omega) - \sigma_A(\omega)] \qquad (38)$$

and with the usual zero-energy dispersion relation we have

$$f_2(0) - f_2(\infty) = \frac{1}{4\pi^2\alpha\langle\vec{S}\cdot\hat{k}\rangle}\int_{\omega_{th}}^{\infty}(\sigma_P - \sigma_A)\frac{d\omega'}{\omega'} \equiv K_P - K_A. \qquad (39)$$

We begin our examination of Eq. (39) by performing the integral over the δ function implicit in σ in Eq. (36). Since ω_{ni} depends on ω through recoil, the integral produces a recoil factor which may be shown to contribute only in higher order than we are calculating. We then find that

$$K_\lambda = \sum_{n\neq 0}\frac{\langle i|e_\lambda^*\cdot\vec{J}(-\vec{k}_n, -\vec{k}_n)|n\rangle\langle n|\hat{e}_\lambda\cdot\vec{J}(\vec{k}_n,\vec{k}_n)|i\rangle}{\epsilon_{ni}^2\langle\vec{S}\cdot\hat{k}\rangle}, \qquad (40)$$

where $\vec{k}_n = \omega_{ni}\hat{k}$. We expand $\vec{J}$ as a function of $\vec{k}_n$, which is presumably small and order $(1/m)$. In order to evaluate K to order $(1/m^2)$ we need keep only the first three terms in the expansion. The nucleus convection current is orthogonal to $\hat{e}$ and may be dropped. The remaining $\vec{s}$ dependence in $\vec{J}$ is of order $(1/m^3)$ and since $\vec{s}$ is $\vec{k}_n \sim (1/m)$ it may be dropped also. We thus expand

$$\vec{J}(\vec{k}_n, 0) \equiv \vec{J}(0) + \omega_{ni}\vec{J}_{(1)}(\hat{k}) + \omega_{ni}^2 J_{(2)}(\hat{k}) + \cdots. \qquad (41)$$

The first and third terms are even under $\hat{k}\to-\hat{k}$ and the second odd. Parity conservation eliminates cross terms between the first and second terms when Eq. (41) is used in Eq. (40). This produces

$$K_\lambda = \sum_i\langle i|\vec{\mu}\cdot\hat{e}_\lambda^*\times\hat{k}|i'\rangle\langle i'|\vec{\mu}\cdot\hat{e}_\lambda\times\hat{k}|i\rangle/\langle\vec{S}\cdot\hat{k}\rangle + \langle i|\vec{D}\cdot\hat{e}_\lambda^*\vec{D}\cdot\hat{e}_\lambda - \hat{e}_\lambda^*\cdot\vec{J}_{(1)}(\hat{k})\vec{J}_{(1)}(\hat{k})\cdot\hat{e}_\lambda + \vec{J}(0)\cdot\hat{e}_\lambda^*\vec{J}_{(2)}(\hat{k})\cdot\hat{e}_\lambda + \vec{J}_{(2)}(\hat{k})\cdot\hat{e}_\lambda^*\vec{J}(0)\cdot\hat{e}_\lambda|i\rangle/\langle\vec{S}\cdot\hat{k}\rangle,$$

where the first term was added and subtracted in order to use closure. Using the identity (21) and forming $K_P - K_A$ we find after some manipulation and using Eq. (37b)

$$K_P - K_A = \bar{\mu}^2 + \langle i | [\vec{D}\cdot\hat{\epsilon}^*, \vec{D}\cdot\hat{\epsilon}] + B^{mn\prime}(0)\epsilon_m^*\epsilon_n - B^{mn\prime}(0)\epsilon_m^*\epsilon_n - [\vec{J}_{(1)}\cdot\hat{\epsilon}^*, \vec{J}_{(1)}\cdot\hat{\epsilon}] + [\vec{J}(0)\cdot\hat{\epsilon}^*, \vec{J}_{(2)}\cdot\hat{\epsilon}] + [\vec{J}_{(2)}\cdot\hat{\epsilon}^*, \vec{J}(0)\cdot\hat{\epsilon}] | i\rangle / \langle\vec{S}\cdot\hat{k}\rangle, \quad (42)$$

where $\hat{\epsilon} \equiv \hat{e}_{+1}$. The first three terms are just $f_2(0)$, and using Eq. (34) reproduces Eq. (2b), while the last three commutators are equal to $(\partial/\partial\vec{k}^2)[\vec{J}(-\vec{k})\cdot\hat{\epsilon}^*, \vec{J}(+\vec{k})\cdot\hat{\epsilon}]\,|_{\vec{k}^2=0}$ since terms odd in k will vanish by parity conservation. We thus obtain

$$K_P - K_A = [\bar{\mu} - (Z/M_t)]^2 + \langle i | (\partial/\partial\vec{k}^2)[\vec{J}(-\vec{k})\cdot\hat{\epsilon}^*, \vec{J}(\vec{k})\cdot\hat{\epsilon}]\,|_{\vec{k}^2=0} - (\partial/\partial\omega)B^{mn}(0)\epsilon_m^*\epsilon_n | i\rangle / \langle\vec{S}\cdot\hat{k}\rangle. \quad (43)$$

Since we may write $\bar{\mu} = Z/M_t + \kappa/M_t$, where κ is the anomalous magnetic moment, the first term in Eq. (43) gives the usual DHG sum rule in terms of κ. What then is the rôle of the remaining term?

If we use the ordinary nonrelativistic currents and the seagull term from Eq. (17) one can show after considerable algebra that the remaining term is given by $\langle -\sum_i \kappa_i^2/2m_i^2\vec{\sigma}(i)\cdot\hat{k}\rangle$. This is just the negative of the amplitude f_2 for a collection of *free* nucleons. If one evaluates the forward Compton amplitude for a Dirac particle (spin $\frac{1}{2}$) one finds the result (1) and (2) for *all* energies. One naively expects that weak binding may alter[23,35,36] $f_2(\infty)$ but will not produce $f_2(\infty) = 0$ necessary for an unsubtracted dispersion relation. This statement, of course, has nothing to do with questions of whether the photoproton DHG sum rule requires a subtraction, but is very relevant to the question of whether the nuclear DHG sum rule saturated with *low-energy* photoabsorption data does. In the former case one believes that the anomalous magnetic moment has a dynamical origin and that associated with this is an effective "form factor" which damps the high-energy behavior associated with the anomalous moment derivative coupling. We have assumed a model with a "fundamental" anomalous moment and must assume the consequences $[f_2(\infty) \neq 0]$. Nor does the use of a form factor in the *model* with κ solve the subtraction problem, since such form factors would almost certainly alter the analytic properties of the Compton amplitude and the dispersion relation becomes questionable.[22]

In light of the previous discussion we present a heuristic "proof" that the extra terms in our model DHG sum rule give simply $-f_2(\infty)$. If we examine Eq. (6b) for forward scattering and high energies, a number of simplifications result. We assume that the quasielastic excitation of nucleons dominates (virtual) photoabsorption in this limit if we neglect mesonic effects. This occurs at an energy $\omega^2/2m$. For energies high compared to usual nuclear energies but small compared to $2m$ where the analytic properties become suspect[21] the denominators become $\sim\pm\omega$. Extracting this factor, a commutator of the currents results. The plane wave factors have opposite phase and cancel for those terms ($i=j$) where the commutator does not vanish and the result is a low-order polynomial in ω. Analogously the forward seagull amplitude is a polynomial in ω. The constant term in the commutator vanishes; the first nonvanishing terms have the form $A + B\omega + \cdots$, where B is the last term in Eq. (43), which we can now identify with $-f_2(\infty)$ *in our model calculation.* Thus Eq. (43) is equivalent to Eq. (3), with Eq. (2b) specifying $f_2(0)$. A *model* such as Eq. (17) must be used to calculate $f_2(\infty)$. Although gauge-invariance specifies the charge-charge and charge-current commutator algebra, the current-current algebra is also needed for the sum rule.

Introducing exchange currents in the *usual* way in the Compton amplitude alters analytic properties in a way which *precludes* the usual dispersion relation,[23] making the question of subtractions in this case somewhat moot.

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