Dynamics of nuclear fluid.
III. General considerations on the kinetic theory of quantum fluids

Cheuk-Yin Wong and John A. McDonald*

Oak Ridge National Laboratory, † *Oak Ridge, Tennessee 37830*

From the quantum kinetic equation of Bogoliubov, Gurov, Born, and Green, the equations of motion for macroscopic variables are examined in many different ways. First, in the configuration space a hierarchy of generalized fluid-dynamical equations can be obtained by taking the appropriate limits of the quantum kinetic equation. The resultant equations of continuity, of momentum, and of energy are similar in form to those one encounters in classical fluid dynamics with the exception of additional terms proportional to $\hbar^2$. Secondly, the quantum kinetic equation is examined in phase space. The same set of equations of continuity, of momentum, and of energy can be derived by taking the first three moments of the quantum kinetic equation. The exact results we obtained are utilized to form the starting point for many simplifying approximations for the investigation of the dynamics of quantum many-body systems such as the elastic response limit, the hydrodynamical limit, Landau's Fermi-liquid theory, and finally the time-dependent Hartree-Fock and the multideterminant time-dependent Hartree-Fock approximations. The fact that all these different dynamical descriptions can be traced to a common origin provides a unifying viewpoint to the present approach with the quantum kinetic equation.

> NUCLEAR STRUCTURE Dynamics of nuclear fluid, quantum kinetic equation, and generalized fluid-dynamical equation. Elastic response, hydrodynamical limit, Landau's Fermi-liquid theory, time-dependent Hartree-Fock approximation, and multideterminant time-dependent Hartree-Fock approximation.

I. INTRODUCTION

This is the third of a series of articles dealing with the dynamics of nuclear fluid. Other studies concern themselves with the equations governing the dynamics starting with time-dependent Hartree-Fock approximations (TDHF),[1] the generalization of the time-dependent Hartree-Fock approximation to include spin and isospin degrees of freedom.[2] We intend to review here the kinetic theory of quantum fluid on a rigorous microscopic basis, the starting point of which need not be the time-dependent Hartree-Fock approximation of a single Slater determinant. This quantum kinetic theory is based on the work of Bogoliubov and Gurov,[3,4] Born and Green,[5] and others who developed it in the 1940's with applications in superfluidity, superconductivity, and the nuclear liquid drop model in mind. However, their work seems to have been forgotten in the intervening years, at least as far as nuclear applications are concerned.

The main result of Bogoliubov, Gurov, Born, and Green was that, starting from a microscopically exact many-body Schrödinger equation, one can derive, in a clear and well-defined manner, the equations of motion for the macroscopic variables which turn out to be similar in form to those in classical fluid dynamics. To be more precise, one derives equations of the same form as the classical equations, which, in the limit $\hbar \to 0$, are exactly the classical equations. This is hardly a surprising result. It is worthwhile noting that, qualitatively speaking, a many-body system that is made up of quantum mechanical particles is more "fluidlike" than a many-body system of classical particles. Since classical particles have, in theory, discrete and precisely determined position and momentum coordinates, it makes sense to talk about such concepts as local fluid density and local fluid velocity only when there are very many particles in the system. However, for even a single quantum particle, there is a well-defined local fluid density and local fluid velocity and the problem of granularity can be bypassed. Classically, one must introduce a probability distribution in order to define local macroscopic quantities. In quantum mechanics, one begins with a probability distribution. A quantum fluid is in essence a fluid of probability and it possesses a continuous quality which classical fluids inherently lack.

In kinetic theory of classical fluid, one obtains from Liouville's equation a system of coupled equations of motion for the reduced distribution functions. This system of equations, known as the BBGKY[5-8] (Bogoliubov-Born-Green-Kirkwood-

Yvon) equation, is used, among other things, for the derivation of Boltzmann's equation, and the equations of hydrodynamics. Likewise, in quantum many-body theory there is a corresponding system of coupled equations, the quantum kinetic equation of Bogoliubov and Gurov[4] and Born and Green.[5] It is defined either in the configuration space or in the phase space. In the configuration space, the kinetic equation is a hierarchy of equations which couple the reduced density matrices. In the phase space, it is written as a set of coupled equations for the reduced Wigner functions.[9-11] Interpreting the reduced Wigner function as analogous to the reduced distribution function in classical statistical mechanics, the quantum kinetic equation is identical in form to the BBGKY equation, with the exception of the presence of additional terms involving $\hbar$. Because of such close analogy, the procedure used in deriving the generalized fluid-dynamical equation from the BBGKY equation for the classical fluid can be well adopted in our investigation.

Using the quantum kinetic equation, which we introduce in Sec. II, we examine the quantum fluid-dynamical equation in many complimentary ways. In Sec. III, we first consider the quantum kinetic equation in the configuration space. After introducing the polar form of the many-body wave function, we obtain the generalized fluid-dynamical equation by taking the proper limits of different kinds of operations on the quantum kinetic equation. The resultant equations of continuity, of momentum, and of energy are similar in form to those one encounters in classical fluid dynamics with the exception of additional terms proportional to $\hbar^2$. It becomes appropriate to introduce the quantum stress tensor to represent the force terms of quantum origin. Our equations differ from those of Born and Green[5] in that terms of quantum origin are now explicitly exhibited. Next, in Sec. IV, we go to the phase space to write down the quantum kinetic equation, in terms of the Wigner function and the reduced Wigner functions, first given by Irving and Zwanzig.[11] The same hierarchy of fluid-dynamical equations can be derived by taking various moments of the quantum kinetic equation.

Even though the generalized hydrodynamical equations are analogous to the classical hydrodynamical equation, the pattern of behavior of a quantum fluid is governed in an important way by the extra terms of quantum origin. We outline in Sec. V how in some special cases of the random phase approximation when the collective strength is concentrated in one state, the quantum stress tensor is proportional to the first spatial derivatives of the displacement vector and the dynamical equation describes the propagation of elastic waves.

In another local entropy approximation, for the quantum and thermal stress tensors, made possible by an assumption of local equilibrium, the dynamical equations are those of classical hydrodynamics. Following Born and Green[5] and others, we show in Sec. VI how the Navier-Stokes equation can be obtained by expanding the reduced density distribution function and the reduced Wigner function about their equilibrium values. In Sec. VII, we exhibit the connection between the quantum kinetic equation and Landau's transport equation for Fermi liquids at low temperature. Finally, we examine in Sec. VIII the time-dependent Hartree-Fock (TDHF) approximation and the statistical multideterminant time-dependent Hartree-Fock (MDTDHF) approximation as the simplest truncations of the quantum kinetic equation. The comparison of the macroscopic equations of motion obtained from TDHF and MDTDHF indicates that a transition of the original state into states represented by other single-particle determinants gives rise to extra terms in the force density which have the effect of a viscous force when one is interested in phenomena of short enough duration. Section IX concludes the present discussion.

It is worth stressing that the quantum kinetic equation and the corresponding fluid-dynamical equation are general and exact results. For its future application to the nuclear fluid, appropriate approximations and assumptions need to be made to truncate the hierarchy of coupled equations. Besides those approximations discussed here, there may also be other kinds of truncation which allow for the effect of viscosity and correlation between particles. In recent investigations of quantum fluid, Fröhlich[12] rediscovered the quantum kinetic equation and applied it to superconductivity and superfluidity. Investigations along similar lines for the nuclear fluid will be of interest.

II. QUANTUM KINETIC EQUATION

We shall first introduce the quantum kinetic equation in the configuration space; the discussions on the quantum kinetic equation in the phase space will be given in Sec. IV. For a system of N identical particles, we represent a state of the system by a wave function

$$\Psi(\vec{r}_1 \cdots \vec{r}_i \cdots \vec{r}_j \cdots \vec{r}_N, t)$$

$$= \begin{cases} -\Psi(\vec{r}_1 \cdots \vec{r}_j \cdots \vec{r}_i \cdots \vec{r}_N, t) & \text{for fermions}, \\ +\Psi(\vec{r}_1 \cdots \vec{r}_j \cdots \vec{r}_i \cdots \vec{r}_N, t) & \text{for bosons}. \end{cases} \quad (2.1)$$

That is, Ψ is antisymmetric under exchange of particles for fermions and symmetric for bosons. For simplicity, we shall limit our attention to only the spatial coordinates; the spin and isospin co-

ordinates can be easily included later if one so desires. The time development of Ψ is given by the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} \Psi = H\Psi , \qquad (2.2)$$

where H is an N-body Hamiltonian having the form

$$H = \sum_i \left(-\frac{\hbar^2}{2m} \right) \nabla_{r_i}^2 + \sum_{\substack{i<j \\ i=1}} v(\vec{r}_i, \vec{r}_j) . \qquad (2.3)$$

However, the wave function is not the most general way of representing the state of an N-body quantum system. More generally, states are represented by the N-body density matrix $\mathfrak{R}^{(N)}$ $(\vec{r}_1 \cdots \vec{r}_N; \vec{r}_1' \cdots \vec{r}_N')$. For "pure" states, we have

$$\mathfrak{R}^{(N)}(\vec{r}_1 \cdots \vec{r}_N; \vec{r}_1' \cdots \vec{r}_N') = \Psi(\vec{r}_1 \cdots \vec{r}_N)\Psi^*(\vec{r}_1' \cdots \vec{r}_N') . \qquad (2.4a)$$

A "mixed" state is a real linear combination of pure states; i.e.,

$$\mathfrak{R}^{(N)}(\vec{r}_1 \cdots \vec{r}_N; \vec{r}_1' \cdots \vec{r}_N')$$
$$\equiv \sum_m a_m \Psi_m(\vec{r}_1 \cdots \vec{r}_N)\Psi_m^*(\vec{r}_1' \cdots \vec{r}_N') . \qquad (2.4b)$$

We will not discuss in detail the properties of the N-body density matrix. This may be found in many references.[13] From Schrödinger's equation and Eq. (2.4) one can derive an equation of motion for $\mathfrak{R}^{(N)}$

$$i\hbar \frac{\partial}{\partial t} \mathfrak{R}^{(N)}(\vec{r}_1 \cdots \vec{r}_N; \vec{r}_1' \cdots \vec{r}_N') = -\frac{\hbar^2}{2m} \sum_{i=1}^{N} (\nabla_{r_i}^2 - \nabla_{r_i'}^2)\mathfrak{R}^{(N)}(\vec{r}_1 \cdots \vec{r}_N; \vec{r}_1' \cdots \vec{r}_N')$$

$$+ \sum_{\substack{i<j \\ i=1}}^{N} [v(\vec{r}_i, \vec{r}_j) - v(\vec{r}_i', \vec{r}_j')]\mathfrak{R}^{(N)}(\vec{r}_1 \cdots \vec{r}_N; \vec{r}_1' \cdots \vec{r}_N') . \qquad (2.5)$$

We wish to consider the N-body density matrix as the quantum analog of the N-body distribution function in classical theory. This leads us to introduce the reduced density matrices $\mathfrak{R}^{(s)}(\vec{r}_1 \cdots \vec{r}_s; \vec{r}_1' \cdots \vec{r}_s')$

$$\mathfrak{R}^{(s)}(\vec{r}_1 \cdots \vec{r}_s; \vec{r}_1' \cdots \vec{r}_s') \equiv \frac{N!}{(N-s)!} \int d^3r_{s+1} \cdots d^3r_N \mathfrak{R}^{(N)}(\vec{r}_1 \cdots \vec{r}_s \vec{r}_{s+1} \cdots \vec{r}_N; \vec{r}_1' \cdots \vec{r}_s' \vec{r}_{s+1} \cdots \vec{r}_N) . \qquad (2.6)$$

Either by integrating (2.6) or using (2.2), (2.5), and (2.6), one gets the following equation of motion for $\mathfrak{R}^{(s)}$:

$$i\hbar \frac{\partial}{\partial t} \mathfrak{R}^{(s)}(\vec{r}_1 \cdots \vec{r}_s; \vec{r}_1' \cdots \vec{r}_s') = -\frac{\hbar^2}{2m} \sum_{i=1}^{s} (\nabla_{r_i}^2 - \nabla_{r_i'}^2)\mathfrak{R}^{(s)} + \sum_{\substack{i<j \\ i=1}}^{s} [v(\vec{r}_i, \vec{r}_j) - v(\vec{r}_i', \vec{r}_j')]\mathfrak{R}^{(s)}(r_1 \cdots r_s; r_1' \cdots r_s')$$

$$+ \sum_{i=1}^{s} \int d^3r_{s+1}[v(\vec{r}_i, \vec{r}_{s+1}) - v(\vec{r}_i', \vec{r}_{s+1})]\mathfrak{R}^{(s+1)}(\vec{r}_1 \cdots \vec{r}_s \vec{r}_{s+1}; \vec{r}_1' \cdots \vec{r}_s' \vec{r}_{s+1}) . \qquad (2.7)$$

The case of most practical interest is the one-body equation

$$i\hbar \frac{\partial}{\partial t} \mathfrak{R}^{(1)}(\vec{r}_1; \vec{r}_1') = -\frac{\hbar^2}{2m} (\nabla_{r_1}^2 - \nabla_{r_1'}^2)\mathfrak{R}^{(1)}(\vec{r}_1; \vec{r}_1') + \int d^3r_2[v(\vec{r}_1, \vec{r}_2) - v(\vec{r}_1', \vec{r}_2)]\mathfrak{R}^{(2)}(\vec{r}_1\vec{r}_2; \vec{r}_1'\vec{r}_2) . \qquad (2.8)$$

The set of equations represented by Eq. (2.7) relates the one-body density matrix to the two-body density matrix which is in turn related to the three-body density matrix. This hierarchy of coupled equations derived first by Bogoliubov and Gurov,[4] Born and Green;[5] and recently revived by Fröhlich,[12] Wong, Maruhn, and Welton[1] can be named the quantum kinetic equation, according to the terminology of Bogoliubov and Gurov. It forms the basis of our investigation.

III. GENERALIZED FLUID-DYNAMICAL EQUATIONS

Starting with the quantum kinetic equation, generalized fluid-dynamical equations have been de-

rived by Born and Green.[5] However, their results are written in such a form that the quantum effects are not explicitly exhibited. Their introduction of the generalized temperature is also subject to question in that it does not necessarily vanish for the ground state. We wish to rederive the same set of generalized fluid-dynamical equations using a very different method by making extensive use of the polar form of the wave function. Terms of nonclassical origin are thus explicitly exhibited. As quantum effects are of great importance for low temperature phenomena, the connection between the dynamics and these extra terms of quantum origin is therefore of great interest.

We write the many-body wave function

$\Psi(\vec{r}_1 \cdots \vec{r}_N, t)$ in terms of an amplitude $\phi(\vec{r}_1 \cdots \vec{r}_N, t)$ and phase factor $S(\vec{r}_1 \cdots \vec{r}_N, t)$:

$$\Psi(\vec{r}_1 \cdots \vec{r}_N, t) = \phi(\vec{r}_1 \cdots \vec{r}_N, t)\exp[imS(\vec{r}_1 \cdots \vec{r}_N, t)/\hbar],$$

$$(3.1)$$

where both ϕ and S are real functions. We wish to have ϕ reflect the symmetry of Ψ, and S to be symmetric with respect to exchange of particle labels and to be single valued. That this is always possible can be seen by considering the following: If Ψ is symmetric with respect to exchange of particles, then define

$$\phi \equiv |\Psi|$$

$$(3.2)$$

and

$$S = \mathcal{P}\frac{\hbar}{im}\ln\left(\frac{\Psi}{|\Phi|}\right),$$

$$(3.3)$$

where $\mathcal{P}$ stands for the principal value. If Ψ is antisymmetric, let $A(\vec{r}_1 \cdots \vec{r}_N)$ such that $A = 1$ or -1. For example, let

$$A \equiv \frac{1}{3}\sum_{i=1}^{3N}\prod_{l<m}\frac{(r_{li} - r_{mi})}{|r_{li} - r_{mi}|},$$

$$(3.4)$$

where r_{li} is the ith component of $\vec{r}_l$. Then define

$$\phi \equiv A|\Psi|$$

$$(3.5)$$

and

$$S \equiv \mathcal{P}\frac{\hbar}{im}\ln\left(\frac{A\Psi}{|\Phi|}\right).$$

$$(3.6)$$

Without loss of generality, one can always choose the spatial part of the wave function of a stationary state to be a real function of all the spatial coordinates. With such a convention, the spatial variation of phase factor S then properly describes the dynamics of the many-body system. From now on we will consider only pure states. The generalization to mixed states will be straightforward.

We first derive an equation of continuity for the s-body density defined in the space $(\vec{r}_1 \cdots \vec{r}_s)$. It is sometimes convenient to treat the configuration space $(\vec{r}_1 \cdots \vec{r}_s)$ in our problem not as s particles in a three-dimensional space but as a single particle in a $3s$-dimensional space $(x_1 \cdots x_{3s})$, particularly when we wish to simplify our notations in the gradient operator and the velocity fields. The extensive repetition with which the coordinate labels appear in our formulas calls for a need to simplify the notation of a collection of vectors in the multidimensional spaces. Accordingly, we introduce the abbreviated notation $\vec{r}^s$ to denote a collection of vectors $(\vec{r}_1 \cdots \vec{r}_s)$ in the form

$$\vec{r}^s \equiv (\vec{r}_1, \vec{r}_2 \cdots \vec{r}_s),$$

$$(3.7)$$

where the ambiguity of $\vec{r}^s$ defined above and the usual definition of the sth power of $\vec{r}$ seldom arises, and, even if it occurs, it can be easily resolved from the context. Other collections of vectors can be similarly defined. In terms of the abbreviated notation, the quantum kinetic equation can be written in the form

$$i\hbar\frac{\partial}{\partial t}\mathfrak{N}^{(s)}(\vec{r}^s; \vec{r}^{s\prime}) = \frac{\hbar^2}{2m}\sum_1^s (\nabla_{r_i}^2 - \nabla_{r_i'}^2)\mathfrak{N}^{(s)}(\vec{r}^s; \vec{r}^{s\prime}) + \sum_{\substack{i<j \\ i=1}}^s [v(\vec{r}_i, \vec{r}_j) - v(\vec{r}_i, \vec{r}_j)]\mathfrak{N}^{(s)}(\vec{r}^s; \vec{r}^{s\prime})$$

$$+ \int d^3r_{s+1}\sum_{i=1}^s [v(\vec{r}_i, \vec{r}_{s+1}) - v(\vec{r}_i, \vec{r}_{s+1})]\mathfrak{N}^{(s+1)}(\vec{r}^s \vec{r}_{s+1}; r^{s\prime}\vec{r}_{s+1}).$$

$$(3.8)$$

The equation of continuity for the s-body density can be obtained by taking the limit $\vec{r}^{s\prime} \to \vec{r}^s$ of the above equation. From Eq. (3.8) we get first

$$\frac{\partial}{\partial t}\mathfrak{N}^{(s)}(\vec{r}^s; \vec{r}^s) + \nabla_{r^s} \cdot \int d\mu_s \phi^2(\vec{r}^N)\nabla_{r^s}S(\vec{r}^N) = 0,$$

$$(3.9)$$

where we have introduced an abbreviation

$$d\mu_s = \frac{N!}{(N-s)!}d^3r_{s+1}\cdots d^3r_N.$$

$$(3.10)$$

If we define the s-body density $n^{(s)}(\vec{r}^s)$ as the diagonal matrix element of the s-body density matrix

$$n^{(s)}(\vec{r}^s) \equiv \mathfrak{N}^{(s)}(\vec{r}^s; \vec{r}^s) \equiv \int d\mu_s \phi^2(\vec{r}^N),$$

$$(3.11)$$

and the s-body average velocity $\vec{u}^{(s)}$, which has components in the $3s$-dimensional space, as

$$u_i^{(s)}(\vec{r}^s) = \int d\mu_s \phi^2(\vec{r}^N)\nabla_{x_i}S(\vec{r}^N)/n^{(s)}(\vec{r}^s),$$

$$(3.12)$$

where $i = 1, \ldots, 3s$, then we get the equation of continuity for the s-body density

$$\frac{\partial}{\partial t}n^{(s)}(\vec{r}^s) + \nabla_{r^s} \cdot [n^{(s)}(\vec{r}^s)\vec{u}^{(s)}(\vec{r}^s)] = 0.$$

$$(3.13)$$

Of particular interest is the equation of continuity for the one-body density when $s = 1$. We have

$$\frac{\partial}{\partial t}\, n(\vec{r}_1) + \vec{\nabla}\cdot[n(\vec{r}_1)\vec{u}(\vec{r}_1)] = 0 , \qquad (3.14)$$

where we have suppressed the superscript 1 and

$$n(\vec{r}_1) = \int d\mu_1\, \phi^2(\vec{r}_1\cdots\vec{r}_N) \qquad (3.15)$$

and

$$\vec{u}(\vec{r}_1) = \int d\mu_1\, \phi^2(\vec{r}_1\cdots\vec{r}_N)\nabla_{\vec{r}_1} S(\vec{r}_1\cdots\vec{r}_N)/n(\vec{r}_1). \qquad (3.16)$$

We next wish to derive equations related to momentum conservation which are vector equations. To get vector equations out of the quantum kinetic equation (2.7), we operate on both sides of the equation with $(\nabla_{x_i} - \nabla_{x_i'})$ and then take the limit as $\vec{x}^{3s'} \to \vec{x}^{3s}$. For the first term of Eq. (3.8) on the right-hand side, we have (Appendix I):

$$\lim_{\vec{r}^{3s'}\to\vec{r}^{3s}} -\frac{\hbar^2}{2m}\,(\nabla_{x_i} - \nabla_{x_i'})\sum_{j=1}^{3S}(\nabla_{x_j}^2 - \nabla_{x_j'}^2)\mathfrak{X}^{(s)}(\vec{r}_1\cdots\vec{r}_i\vec{r}_j\vec{r}_s;\vec{r}_1'\cdots\vec{r}_i'\cdots\vec{r}_j'\cdots\vec{r}_s')$$

$$= 2\sum_{j=1}^{3s}\nabla_{x_j}\int d\mu_s\left(m\phi^2\nabla_{x_i}S\nabla_{x_j}S - \frac{\hbar^2}{4m}\,\nabla_{x_i}\nabla_{x_j}\,\phi^2 + \frac{\hbar^2}{m}\,\nabla_{x_i}\phi\nabla_{x_j}\,\phi\right), \qquad (3.17)$$

where and henceforth, for simplicity of notation, all the independent variables of coordinates $\vec{r}^N$ and time in ϕ and S have been understood. The above expression can be simplified. We note that

$$\int d\mu_s\,\phi^2\nabla_{x_i}S\,\nabla_{x_j}S$$

$$= n^{(s)}u^{(s)}u^{(s)} + \int d\mu_s\,\phi^2(\nabla_{x_i}S - u_i^{(s)})(\nabla_{x_j}S - u_j^{(s)}). \qquad (3.18)$$

Thus, it is appropriate to introduce an s-body "par-thermal" stress tensor $p_{ij,p}^{(s)}$ defined by

$$p_{ij,p}^{(s)}(\vec{r}^s) = m\int d\mu_s\,\phi^2(\nabla_{x_i}S - u_i^{(s)})(\nabla_{x_j}S - u_j^{(s)}). \qquad (3.19)$$

This stress tensor arises from the deviation of the s-body velocity fields from the average. We use the term par-thermal with the prefix par(io) to denote that it can give rise to (but not necessarily is) the thermal stress tensor. It becomes the thermal stress when there is local thermal equilibrium such that the s-body velocity fields deviate randomly from the average.

The expression in Eq. (3.17) suggests the usefulness of introducing also an s-body "quantum" stress tensor[14,15] defined by

$$p_{ij,q}^{(s)}(\vec{r}^s) = \int d\mu_s\left(-\frac{\hbar^2}{4m}\,\nabla_{x_i}\nabla_{x_j}\phi^2 + \frac{\hbar^2}{m}\,\nabla_{x_i}\phi\nabla_{x_j}\phi\right) \qquad (3.20a)$$

or equivalently

$$p_{ij,q}^{(s)}(\vec{r}^s) = \int d\mu_s\left(+\frac{\hbar^2}{4m}\,\nabla_{x_i}\nabla_{x_j}\phi^2 - \frac{\hbar^2}{m}\,\phi\nabla_{x_i}\nabla_{x_j}\phi\right). \qquad (3.20b)$$

The term "quantum" is introduced here to emphasize its nonclassical origin and its being operative even for static stationary states.

The separation of the stress tensor into a par-thermal part and a quantum part is a meaningful separation for a nonstationary state in our convention in which the spatial part of the wave function of a stationary state is chosen to be purely real (achieved, for example, by using real single-particle spatial basis functions for stationary states). On the other hand, if one abandons such a convention and insists on using complex spatial basis functions for stationary states, then the terms par-thermal and quantum lose their meaning. The same thing happens when one deals with a mixed state described in terms of a real linear combination of stationary states (with, for example, a Boltzmann weighting factor). In these cases, it is still possible to define a p-stress tensor and a q-stress tensor by Eqs. (3.19) and (3.20), respectively, but one should speak more appropriately of an intrinsic stress tensor as the sum of the p- and q-stress tensors, and should not associate any particular physical meaning to the subscripts and the prefixes p and q. We shall, however, adhere to our original convention so that the stress tensors have their physical origins. The usefulness of the quantum stress tensor in understanding some peculiar phenomena of many-fermion systems has been discussed recently.[14]

For the other terms in Eq. (3.8), the limit of $\vec{x}^{3s} \to \vec{x}^{3s'}$ after the operation of $\nabla_{x_i} - \nabla_{x_i'}$ can be obtained in a simple way. Putting all terms together,

we have the equation of motion for the s-body density and velocity field derived from the quantum equation:

$$\frac{\partial}{\partial t} \, mn^{(s)}u_i^{(s)} + \sum_{j=1}^{3s} \nabla_{x_j}(n^{(s)}u_i^{(s)}u_j^{(s)} + p_{ij,p}^{(s)} + p_{ij,q}^{(s)})$$

$$= -n^{(s)}\nabla_{x_i} \sum_{m \neq L}^{s} v(\vec{r}_L, \vec{r}_m) - \int d^3r_{s+1} n^{(s+1)}(\vec{r}^{s+1})\nabla_{x_i} v(\vec{r}_L, \vec{r}_{s+1}) \,, \quad (3.21)$$

where $x_i \in \vec{r}_L$. This set of equations relates the s-body current to the $(s+1)$-body density and forms a hierarchy of equations coupling the different many-body currents in an ordered manner.

Of particular interest in the case for $s = 1$ for which we have

$$\frac{\partial}{\partial t} \, mnu_i + \sum_{j=1}^{3} \nabla_{x_j}(nu_i u_j + p_{ij,p} + p_{ij,q}) = -\int d^3r_2 n^{(2)}(\vec{r}_1\vec{r}_2)\nabla_{x_i} v(\vec{r}_1, \vec{r}_2) \,, \quad (3.22)$$

where the superscript 1 has been omitted.

An energy equation for the s-body density matrix can be similarly derived. One notes that the energy density contains terms of the kind $\int d\mu_s \, \phi^2(\nabla S)^2$, $\int d\mu_s \nabla\phi \cdot \nabla\phi$, while the density matrix is of the form $\int d\mu_s \Psi\Psi^*$. So, to get an energy equation out of the quantum kinetic equation, one operates Eq. (3.8) with $\sum_{i=1}^{3s} \nabla_{x_i}\nabla_{x_i'}$ and then takes the limit as $\vec{x}^{3s'} \to \vec{x}^{3s}$. First of all, for the first term on the right-hand side of Eq. (3.8), we have

$$\lim_{\vec{r}' \to \vec{r}} \sum_{i=1}^{3s} \nabla_{x_i}\nabla_{x_i'} \, i\hbar \, \frac{\partial}{\partial t} \, \mathfrak{N}^{(s)}(\vec{r}^s; \vec{r}^{s'}) = \frac{2im}{\hbar} \, \frac{\partial}{\partial t} \, n^{(s)}(\vec{r}^s)E_k^{(s)}(\vec{r}^s) \,, \quad (3.23)$$

where $E_k^{(s)}$ is the s-body kinetic energy density defined by

$$E_k^{(s)}(\vec{r}^s) = \int d\mu_s \left[\tfrac{1}{2} m \, \phi^2(\vec{r}^N)\sum_{i=1}^{3s} (\nabla_{x_i}S(\vec{r}^N))^2 + \frac{\hbar^2}{2m} \sum_{i=1}^{3s} (\nabla_{x_i}\phi(\vec{r}^N))^2 \right] \Big/ n^{(s)}(\vec{r}^s) \,. \quad (3.24)$$

For the first term on the right-hand side, we have (Appendix II):

$$\lim_{\vec{r}^{3s'} \to \vec{r}^{3s}} -\frac{\hbar^2}{2m} \sum_{i,j=1}^{3s} \nabla_{x_i}\nabla_{x_i'}(\nabla_{x_j}^2 - \nabla_{x_j'}^2)\mathfrak{N}^{(s)}(\vec{r}^s; \vec{r}^{s'})$$

$$= -\frac{2mi}{\hbar} \sum_{i=1}^{3s} \nabla_{x_i}\left[n^{(s)}E_k^{(s)}u_i^{(s)} + \sum_{j=1}^{3} (p_{ij,p}^{(s)} + p_{ij,q}) u_j^{(s)} + F_i^{(s)} \right] \,, \quad (3.25)$$

where

$$F_i^{(s)}(\vec{r}^s) = \int d\mu_s \left[\left(\tfrac{1}{2} m\phi^2(\nabla_{x^{3s}}S)^2 + \frac{\hbar^2}{2m} (\nabla_{x^{3s}}\phi)^2 \right)(\nabla_{x_i}S - u_i) \right.$$

$$\left. + \sum_{j=1}^{3s} \left(-\frac{\hbar^2}{4m} \nabla_{x_i}\nabla_{x_j}\phi^2 + \frac{\hbar^2}{m} \nabla_{x_i}\phi\nabla_{x_j}\phi \right)(\nabla_{x_j}S - u_j) + \frac{\hbar^2}{2m} \phi\nabla_{x_i}\phi\nabla_{x^{3s}} \cdot \nabla_{x^{3s}}S \right] \,. \quad (3.26)$$

Gathering all the terms together, we have the s-body energy equation in the form

$$\frac{\partial}{\partial t} \, n^{(s)}E_k^{(s)} + \sum_{i=1}^{3s} \nabla_{x_i}\left[n^{(s)}E_k^{(s)}u_i^{(s)} + \sum_{j=1}^{3s} (p_{ij,p}^{(s)} + p_{ij,q}^{(s)})u_j + F_i^{(s)} \right]$$

$$= -n^{(s)} \sum_{i=1}^{3s} u_i^{(s)} \nabla_{x_i} \sum_{m \neq L}^{s} v(\vec{r}_L, \vec{r}_m) - \int d^3r_{s+1} n^{(s+1)}(\vec{r}^{s+1}) \sum_{i=1}^{3s} u_i^{(s+1)}(\vec{r}^{s+1})\nabla_{x_i} v(\vec{r}_L, \vec{r}_{s+1}) \,, \quad (3.27)$$

where $x_i \in \vec{r}_L$. Of particular interest in the one-body energy equation for $s = 1$. We have

$$\frac{\partial}{\partial t} \, nE_k + \sum_{i=1}^{3} \nabla_{x_i}\left[n E_k u_i + \sum_{j=1}^{3} (p_{ij,p} + p_{ij,q})u_j + F_i \right] = -\int d^3r_2 \sum_{i=1}^{3} n^{(2)}(\vec{r}_1\vec{r}_2)u_i^{(2)}(\vec{r}_1\vec{r}_2)\nabla_{x_i} v(\vec{r}_1, \vec{r}_2) \,, \quad (3.28)$$

where the variables on the left-hand side depend on the coordinate $\vec{r}_1 = (x_1, x_2, x_3)$ and the superscript (1) has been omitted for simplicity. We note in passing that in the TDHF approximation, the two-body current $n^{(2)}u^{(2)}$ can be written in terms of the one-body density matrix and we have

$$\frac{\partial}{\partial t} \, nE_k + \sum_{i=1}^{3} \nabla_{x_i} \left[nE_k u_i + \sum_{j=1}^{3} (p_{ij,p} + p_{ij,q}) u_j + F_i \right]$$

$$= - \int d^3 r_2 \sum_{i=1}^{3} \left\{ n(\vec{r}_1) n(\vec{r}_2) u_i(\vec{r}_1) - \frac{\hbar}{m} \, \mathcal{G}m[\mathfrak{N}^{(1)}(\vec{r}_2; \vec{r}_1) \nabla_{x_i} \mathfrak{N}^{(1)}(\vec{r}_1; \vec{r}_2)] \right\} \nabla_{x_i} v(\vec{r}_1, \vec{r}_2) \, , \quad (3.29)$$

where the energy flux F_i is given explicitly as

$$F_i = \sum_{\{\alpha\}} \left\{ \left[\tfrac{1}{2} m \, \phi_\alpha^2 (\nabla S_\alpha)^2 + \frac{\hbar^2}{2m} \, (\nabla \phi_\alpha)^2 \right] (\nabla_i S_\alpha - u_i) \right.$$

$$\left. + \sum_{j=1}^{3s} \left(- \frac{\hbar^2}{4m} \, \nabla_i \nabla_j \, \phi_\alpha^2 + \frac{\hbar^2}{m} \, \nabla_i \phi_\alpha \nabla_j \, \phi_\alpha \right) (\nabla_j S_\alpha - u_j) + \frac{\hbar^2}{2m} \, \phi_\alpha \nabla_i \, \phi_\alpha \nabla^2 S_\alpha \right\} . \quad (3.30)$$

The energy equation (3.29) supplements Eqs. (2.8) and (2.12) of Ref. 1 to form the set of exact macroscopic equations from the TDHF approximation.

Starting with the quantum kinetic equations, we have now obtained the generalized fluid-dynamical equation for a many-body system. It consists of the equation of continuity (3.13), the equation of motion (3.21), and the equation of energy (3.27). The set of equations of continuity, of motion, and of energy couple among each other and also with other density functions. The totality of these equations can be arranged in an ordered manner in the form of a hierarchy with the one-body quantitites coupling the two-body quantities which in turn are coupled to the three-body quantities. Such coupling continues on and terminates when the N-body quantities couple with themselves.

It might be worth noting that one can take more higher moments by operating the quantum kinetic equation with higher powers of the gradient ∇_{x_i} $- \nabla_{x'_i}$ and taking the limits of $\vec{x}^{3s'} \to \vec{x}^{3s}$ afterwards. In this context, one recognizes that for each value of s, the continuity, momentum, and energy equations are just the first three moments of the sth quantum kinetic equation.

The set of equations we have written down is the same set obtained by Born and Green.[5] We have, however, derived it with a different method making extensive use of the polar form of the wave function which allows one to separate the stress tensor into two components. Thus, for the static ground state for which all the velocity fields (both microscopic and macroscopic) can be chosen to vanish, we can see clearly the importance of the quantum stress tensor in providing the necessary force leading to static equilibrium. The importance of the quantum stress tensor in many problems of physical interest cannot be overstated. For many-fermion systems, the peculiar dependence of the quantum stress tensor on the second spatial derivatives of the density leads to nonuniform volume-type density oscillations in their ground states.[14] For finite nuclei, such density oscilla-

tions, proposed by Friedel, Thorpe, and Thouless[16] using very different arguments, appears to have recently been observed.[17] In a finite nucleus, detailed investigation on the mechanism leading to static equilibrium in terms of the quantum stress tensor has not yet been carried out. However, as the two-body nuclear interaction is short-ranged, one expects that the nonuniformity and the nonisotropy of the quantum stress tensor are responsible for the permanent ground state deformations of many nuclei. For the energy equation, we use the kinetic energy density as our dynamical variable instead of the generalized temperature of Born and Green[5] which has the disadvantage that it does not vanish for the ground state and that its correspondence with the temperature defined in the usual way is rather obscure. In addition, we have also written the energy flux explicitly in terms of the many-body wave function.

IV. QUANTUM KINETIC EQUATION IN PHASE SPACE IN TERMS OF THE REDUCED WIGNER FUNCTIONS

An alternative way of deriving the same fluid-dynamical equations which appear in Sec. III makes use of the Wigner function by going to the phase space. Being a function of both position and momentum coordinates, its correspondence with the classical distribution function was immediately realized when it was first introduced by Wigner,[9] as the classical Liouville equation could be obtained from the equation of motion for the Wigner function in the limit of $\hbar \to 0$. The Wigner function cannot, however, be really interpreted as the simultaneous probability for coordinates and momenta, as it is clear that it may take on negative values. On the other hand, the Wigner function obeys many relations which are expected of such an interpretation. It is therefore useful to treat it as an auxiliary function, analogous with, but not identical to, the classical distribution function.

One knows that the classical Liouville equation

leads to the hierarchy of BBGKY equations and the BBGKY equations in turn lead to the equation of classical fluid dynamics.[4-8] One may wish to adopt a similar approach here. One can start with the equation of motion for the Wigner function and obtain the quantum analog of the BBGKY equation involving reduced Wigner functions. This is just the quantum kinetic equation (2.7) written in phase space first obtained by Irving and Zwanzig.[11] The generalized fluid-dynamical equation can then be obtained by taking the various moments of this hierarchy of equations.

The results of Irving and Zwanzig[11] can be sum-

marized, with a slight change of notations, as follows:

We first introduce the coordinate transformations from $\vec{r}_i, \vec{r}_i'$ to $\vec{R}_i$ and $\vec{y}_i$:

$$\vec{r}_i = \vec{R}_i - \tfrac{1}{2}\vec{y}_i \tag{4.1a}$$

and

$$\vec{r}_i' = \vec{R}_i + \tfrac{1}{2}\vec{y}_i . \tag{4.1b}$$

With this transformation, the N-body Wigner function $f^{(N)}$ for a system of N identical particles is defined in terms of the density matrix $\mathfrak{K}^{(N)}$ by

$$f^{(N)}(\vec{R}_1\cdots\vec{R}_N; \vec{p}_1\cdots\vec{p}_N) \equiv \int d^3y_1\cdots d^3y_N \exp\left(i\sum_{i=1}^{N}\vec{p}_i\cdot\vec{y}_i/\hbar\right)\mathfrak{K}^{(N)}(\vec{R}_1 - \tfrac{1}{2}\vec{y}_1\cdots R_N - \tfrac{1}{2}\vec{y}_N; \vec{R}_1 + \tfrac{1}{2}\vec{y}_1\cdots\vec{R}_N + \tfrac{1}{2}\vec{y}_N), \tag{4.2}$$

while the reduced s-body Wigner function is defined in two equivalent ways:

$$f^{(s)}(\vec{R}_1\cdots\vec{R}_s; \vec{p}_1\cdots\vec{p}_s) \equiv \frac{N!}{(N-s)!}\int d^3R_{s+1}\cdots d^3R_N \frac{d^3p_{s+1}\cdots d^3p_N}{(2\pi\hbar)^{N-s}} f^{(N)}(\vec{R}_1\cdots\vec{R}_N; \vec{p}_1\cdots\vec{p}_N) \tag{4.3}$$

$$\equiv \int d^3y_1\cdots d^3y_s \exp\left(i\sum_{i=1}^{3}\vec{p}_i\cdot\vec{y}_i/\hbar\right)\mathfrak{K}^{(s)}(\vec{R}_1 - \tfrac{1}{2}\vec{y}_1\cdots\vec{R}_s - \tfrac{1}{2}\vec{y}_s; \vec{R}_1 + \tfrac{1}{2}\vec{y}_1\cdots\vec{R}_s + \tfrac{1}{2}\vec{y}_s). \tag{4.4}$$

For simplicity, we use the abbreviated notation introduced before

$$\vec{R}^s = (\vec{R}_1\cdots\vec{R}_s), \tag{4.5a}$$

$$\vec{p}^s = (\vec{p}_1\cdots\vec{p}_s), \tag{4.5b}$$

$$\vec{y}^s = (\vec{y}_1\cdots\vec{y}^s), \tag{4.5c}$$

and $d^{3s}R = d^3R_1\cdots d^3R_s$, etc.

To write down the quantum kinetic equation (2.7) in phase space, we integrate Eq. (2.7) with $\int d^{3s}y$ $\times \exp\{i\vec{p}^s\cdot\vec{y}^s/\hbar\}$ and we obtain the quantum kinetic equation in phase space[11]

$$\frac{\partial}{\partial t}f^{(s)}(\vec{R}^s, \vec{p}^s) + \sum_{i}^{3s}\frac{\vec{p}_i}{m}\cdot\nabla_{R_i}f^{(s)}(\vec{R}^s, \vec{p}^s) = A^{(s)}\cdot f^{(s)}(\vec{R}^s, \vec{p}^s) + \int d^3R_{s+1}\frac{d^3p_{s+1}}{(2\pi\hbar)^3}B^{(s+1)}\cdot f^{(s+1)}(\vec{R}^{s+1}, \vec{p}^{s+1}), \tag{4.6}$$

where the operators $A^{(s)}$ and $B^{(s+1)}$ are given either in an integral of differential form. In the differential form, the operators $A^{(s)}$ and $B^{(s+1)}$ are

$$A^{(s)}\cdot f^{(s)}(\vec{R}^s, \vec{p}^s) = \frac{2}{\hbar}\sum_{\substack{i<j \\ i=1}}^{s}\{\sin[\tfrac{1}{2}\hbar(\nabla_{p_i}^f\cdot\nabla_{R_i}^v + \nabla_{p_j}^f\cdot\nabla_{R_j}^v)]\}\,v(\vec{R}_i, \vec{R}_j)f^{(s)}(\vec{R}^s, \vec{p}^s) \tag{4.7}$$

and

$$B^{(s+1)}\cdot f^{(s+1)}(\vec{R}^{s+1}, \vec{p}^{s+1}) = \frac{2}{\hbar}\sum_{i=1}^{s}\{\sin[\tfrac{1}{2}\hbar(\nabla_{p_i}^f\cdot\nabla_{R_i}^v)]\}\,v(\vec{R}_i, \vec{R}_{s+1})f^{(s+1)}(\vec{R}^{s+1}, \vec{p}^{s+1}), \tag{4.8}$$

where the operator $\nabla_{R_i}^v$ and $\nabla_{R_j}^v$ inside the sine function operate only on v; and $\nabla_{p_i}^f$ and $\nabla_{p_j}^f$ operate only on $f^{(s)}$ and $f^{(s+1)}$.

In the integral form, the operators $A^{(s)}$ and $B^{(s+1)}$ are

$$A^{(s)}\cdot f^{(s)}(\vec{R}^s, \vec{p}^s) = \int\frac{d^{3s}p'}{(2\pi\hbar)^{3s}}K^{(s)}(\vec{R}^s, \vec{p}^s - \vec{p}^{s'})f^{(s)}(\vec{R}^s, \vec{p}^{s'}) \tag{4.9}$$

and

$$B^{(s+1)} \cdot f^{(s+1)}(\vec{R}^{s+1}, \vec{p}^{s+1}) = \int \frac{d^{3s}p'}{(2\pi\hbar)^{3s}} \; L(\vec{R}^{s+1}, \vec{p}^s - \vec{p}^{s'}) f^{(s+1)}(\vec{R}^{s+1}, \vec{p}^{s'}\vec{p}_{s+1}) , \tag{4.10}$$

where

$$i\hbar K^{(s)}(\vec{R}^s, \vec{p}^s - \vec{p}^{s'}) = \int d^{3s}y \, \exp[i\,(\vec{p}^s - \vec{p}^{s'}) \cdot \vec{y}^s/\hbar] \sum_{\substack{i<j \\ i=1}}^{s} [v(\vec{R}_i - \tfrac{1}{2}\vec{y}_i, \vec{R}_j - \tfrac{1}{2}\vec{y}_j) - v(\vec{R}_i + \tfrac{1}{2}\vec{y}_i, \vec{R}_j + \tfrac{1}{2}\vec{y}_j)] \tag{4.11}$$

and

$$i\hbar L(\vec{R}^{s+1}, \vec{p}^s - \vec{p}^{s'}) = \int d^{3s}y \, \exp[i\,(\vec{p}^s - \vec{p}^{s'}) \cdot \vec{y}^s/\hbar] \sum_{i=1}^{s} [v(\vec{R}_i - \tfrac{1}{2}\vec{y}_i, \vec{R}_{s+1}) - v(\vec{R}_i + \tfrac{1}{2}\vec{y}_i, \vec{R}_{s+1})] . \tag{4.12}$$

In the particular case when $s=N$, we have $B^{(s+1)} = 0$ and we obtain the equation of motion for the Wigner function $f^{(N)}$:

$$\frac{\partial}{\partial t} f^{(N)}(\vec{R}^N, \vec{p}^N) + \sum_i \frac{p_i}{m} \cdot \nabla_{R_i} f^{(N)}(\vec{R}^N, \vec{p}^N) = \frac{2}{\hbar} \sum_{\substack{i<j \\ i=1}}^{s} \{\sin[\tfrac{1}{2}\hbar (\nabla_{p_i}^f \cdot \nabla_{R_i}^v + \nabla_{p_j}^f \cdot \nabla_{R_j}^v)]\} v(\vec{R}_i, \vec{R}_j) f^{(N)}(\vec{R}^N, \vec{p}^N) . \tag{4.13}$$

Another case of interest is when $s = 1$ for which $A^{(1)} = 0$, and we have

$$\frac{\partial}{\partial t} f^{(1)}(\vec{R}_1, \vec{p}_1) + \frac{\vec{p}}{m} \cdot \nabla_{R_1} f^{(1)}(\vec{R}_1, \vec{p}_1) = \int d^3R_2 \frac{d^3p_2}{(2\pi\hbar)^3} \frac{2}{\hbar} \{\sin[\tfrac{1}{2}\hbar(\nabla_{p_1}^f \cdot \nabla_{R_1}^v)]\} v(\vec{R}_1, \vec{R}_2) f^{(2)}(\vec{R}_1\vec{R}_2, \vec{p}_1\vec{p}_2) . \tag{4.14}$$

The quantum kinetic Eq. (4.6) is an equation of motion for the Wigner function. It couples the s-body Wigner function with the $(s + 1)$-body Wigner function and is the quantum analog of the BBGKY equation.

So far, the quantum kinetic equation in the phase state given by Irving and Zwanzig[11] is based on a two-body momentum-independent potential. To anticipate the application of the quantum kinetic equation to the Fermi-liquid theory of Landau,[18] we wish to write down the quantum kinetic equation in the case of a momentum-dependent two-body interaction $v(\vec{r}_i\vec{p}_i, \vec{r}_j'\vec{p}_j')$. It is easy to prove that we obtain again the quantum kinetic equation (4.6) with a modified definition of the operators $A^{(s)}$ and $B^{(s+1)}$. In differential forms, the operators $A^{(s)}$ and $B^{(s+1)}$ become

$$A^{(s)} \cdot f^{(s)}(\vec{R}^s, \vec{p}^s)$$

$$= \frac{2}{\hbar} \sum_{\substack{i<j \\ i=1}}^{s} \sin[\tfrac{1}{2}\hbar(\nabla_{p_i}^f \cdot \nabla_{R_i}^v + \nabla_{p_i}^f \cdot \nabla_{R_j}^v)]$$

$$\times v(\vec{R}_i\vec{p}_i, \vec{R}_j\vec{p}_j) f^{(s)}(\vec{R}^s, \vec{p}^s) \tag{4.15}$$

and

$$B^{(s+1)} \cdot f^{(s+1)}(\vec{R}^{s+1}, \vec{p}^{s+1})$$

$$= \frac{2}{\hbar} \sum_{i=1}^{s} \sin[\tfrac{1}{2}(\nabla_{p_i}^f \cdot \nabla_{R_i}^v)] v(\vec{R}_i\vec{p}_i, \vec{R}_{s+1}\vec{p}_{s+1})$$

$$\times f^{(s+1)}(\vec{R}^{s+1}, \vec{p}^{s+1}) , \tag{4.16}$$

where the operators $\nabla_{R_i}^v$ and $\nabla_{R_j}^v$ operate only on the two-body potential v and the operators $\nabla_{p_i}^f$ and

$\nabla_{p_j}^f$ operate only on the reduced Wigner function $f^{(s)}(\vec{R}^s, \vec{p}^s)$ and $f^{(s+1)}(\vec{R}^{s+1}, \vec{p}^{s+1})$.

Starting with the classical BBGKY equation, Born and Green showed how it was possible to obtain the generalized fluid-dynamical equation referring to a cluster of s particles.[5] The procedure involves taking the various moments of the BBGKY equation. Specializing in the TDHF and the corresponding on one-body Wigner function, Koonin and Kerman[19] followed similar procedures and obtained the one-body fluid-dynamical equations from the TDHF approximation.

With the knowledge of the quantum kinetic equation in phase space, we can carry out similar procedures to obtain the same set of generalized fluid-dynamical equations (3.13), (3.21), and (3.27). These procedures of taking the various moments of the *full* quantum kinetic equation are lengthy but straightforward. They will not be reproduced here but will be supplied upon request.

V. MACROSCOPIC DESCRIPTION: ELASTIC RESPONSE

All the results in the previous sections have been obtained in a rather general manner without approximations. The resultant generalized fluid-dynamical equations are exact relations from the many-body Schrödinger equation. They are similar in form to those one encounters in the classical kinetic theory of fluid, with the exception of additional quantum stress terms proportional to $\hbar^2$.

The generalized s-body fluid-dynamical equation

is useful in the discussion of the dynamics of a cluster of s particles relative to the other particles and may find its application in some quasi-elastic scattering processes in heavy-ion reactions. For many processes, it suffices to study the density of a single particle and its time evolution. We shall limit our attention to such a consideration.

We would like to write out explicitly the one-body equation of motion

$$\frac{\partial}{\partial t}\ mn(\vec{r}_1)u_i(\vec{r}_1)+\sum_{j=1}^{3}\frac{\partial}{\partial r_{1j}}\ [n(\vec{r}_1)u_j(\vec{r}_1)u_i(\vec{r}_1)+p_{ji,p}(\vec{r}_1)+p_{ji,q}(\vec{r}_1)] = -\int d^3r_2 n^{(2)}(\vec{r}_1,\vec{r}_2)\frac{\partial}{\partial r_{1i}}\ v(\vec{r}_1,\vec{r}_2) \tag{5.1}$$

which is written in such a way as to exhibit the spatial dependence on the radius vector $\vec{r}_1$ (with components r_{1i}) and $\vec{r}_2$, the superscript (1) being omitted for simplicity. In order to show the importance of the interaction v in contributing to a pressure stress tensor, we can write Eq. (5.1) in a slightly different way. We separate the two-body interaction in terms of a short-range part v_s and a long-range part. We further assume that the short-range part of the interaction depends only on the magnitude of the relative coordinate $\vec{r}=\vec{r}_2-\vec{r}_1$:

$$v(\vec{r}_1,\vec{r}_2)=v_s(r)+v_L(\vec{r}_1,\vec{r}_2)\ . \tag{5.2}$$

For the short-range force, one can adequately expand the two-body density matrix in powers of $\vec{r}$ and obtain[5,20]

$$-\int d^3r_2 n^{(2)}(\vec{r}_1,\vec{r}_2)\ \frac{\partial}{\partial r_{1i}}\ v_s(\vec{r}_1,\vec{r}_2)=\sum_{j=1}^{3}\frac{\partial}{\partial r_{1j}}\ \frac{1}{2}\int d^3r\frac{x_i x_j}{r^2}\ \frac{dv_s(r)}{dr}\ n^{(2)}(\vec{r}_1-\tfrac{1}{2}\vec{r},\mathrm{r}_1+\tfrac{1}{2}\vec{r})\ , \tag{5.3}$$

where x_i and x_j are components of $\vec{r}$. It is thus possible to write the equation of motion in the form

$$\frac{\partial}{\partial t}\ mn(\vec{r}_1)u_i(\vec{r}_1)+\sum_{j=1}^{3}\frac{\partial}{\partial r_{1j}}\ [n(\vec{r}_1)u_i(\vec{r}_1)u_j(\vec{r}_1)+p_{ij,\delta}(\vec{r}_1)+p_{ij,p}(\vec{r}_1)+p_{ij,q}(\vec{r}_1)] = -\int d^3r\, n^{(2)}(\vec{r}_1,\vec{r}_2)\ \frac{\partial}{\partial r_{1i}}\ v_L(\vec{r}_1,\vec{r}_2)\ , \tag{5.4}$$

where the stress tensor $p_{ij,\delta}(\vec{r}_1)$ is given by

$$p_{ij,\delta}(\vec{r}_1)=-\frac{1}{2}\int d^3r\ \frac{x_i x_j}{r^2}\ \frac{dv_s(r)}{dr}\ n^{(2)}(\vec{r}_1-\tfrac{1}{2}\ \vec{r},\vec{r}_1+\tfrac{1}{2}\vec{r})\ , \tag{5.5}$$

and the subscript δ denotes that this is a stress tensor arising from the short-range interaction between particles. It is clear that for a strongly interacting dense system, the pressure stress tensor arising from the interaction $p_{ij,\delta}$ can be substantially greater than that arising from the thermal-like motion of the particles $p_{ij,p}$.

The formal analogy between the one-body Eqs. (3.14), (5.4), and (3.28) and the equations of classical fluid dynamics does not immediately imply the validity of a *completely* macroscopic description of a quantum fluid. Additional conditions need to be satisfied. Of all the variables we introduce in the fluid-dynamical equation, we can take $n(\vec{r}_1,t)$, $u(\vec{r}_1,t)$, and $E_k(\vec{r}_1,t)$ as our basic macroscopic variables and consider the equations of continuity, of motion, and of energy as the necessary equations to propagate these quantities in time. The equation of continuity involves only macroscopic variables; however, the equations of motion and of energy contain subsidiary variables $p_{ij}(\vec{r}_1)$, $F_i(\vec{r}_1)$, $n^{(2)}(\vec{r}_1,\vec{r}_2)$, and $u^{(2)}(\vec{r}_1,\vec{r}_2)$ which are given in terms of the microscopic many-body wave functions. The

equations (3.14), (5.4), and (3.28) can therefore be viewed as equations coupling the microscopic motion with the macroscopic quantities n, u, and E_k. Such a view is a useful one in some problems where the coupling gives rise to interesting effects. For example, in the consideration of nuclear ground states, the possibility of an anisotropic and non-uniform $p_{ij,q}(r_1)$ arising from the microscopic wave functions is very likely the origin of the permanent deformation of a nonspherical nucleus.

In many problems it is desirable to seek a *completely* macroscopic description of the dynamics. It is then required that the subsidiary variables can either be expressible in terms of the basic macroscopic variables n, u, and E_k or be evaluated by other means. Even when a macroscopic description is possible, the pattern of behavior of the fluid is not necessarily that of hydrodynamics. In spite of the close analogy between Eq. (5.4) and the Euler equation, the pattern of behavior of a quantum fluid is governed in an important way by the quantum stress tensor, the extra term of quantum origin. In some dynamical motion of the system such as

that described in a special case of the random phase approximation, one obtains the quantum stress tensor proportional to the first spatial derivatives of the displacement.[21] The dynamic equation obtained from Eq. (5.4) is the Lamé equation governing the propagation of elastic waves. As was first observed by Bertsch,[22] nuclear giant multipole resonances can be properly described by such an elastic response of the nuclear fluid. Therefore, nuclear elasticity is of physical interest.

We shall discuss briefly the elastic response of a Fermi fluid as a proper limit of Eq. (5.4). Detailed discussions will be found elsewhere.[21] We can imagine that initially we have a Fermion system in its ground state properly described by a single determinant of product wave functions. The single-particle spatial wave function $\psi_\alpha^{(0)}(x, 0)$ can be chosen to be real functions of spatial coordinates. Then, in special cases of the random phase approximation (RPA) when the strength for collective motion is concentrated in one particular state, the time dependence of the spatial wave function can be properly described in terms of a displacement $\vec{D}$ and a phase factor S by[22]

$$\psi_\alpha(\vec{r}, t) = \frac{\psi_\alpha^{(0)}(\vec{r} + \vec{D}, 0) \exp[imS(\vec{r}, t)/\hbar + i\epsilon_\alpha(t)t/\hbar]}{(1 - \nabla \cdot \vec{D})^{1/2}} \,. \tag{5.6}$$

Note that in Eq. (5.6), all the different single-particle states are governed by the same displacement and phase factor S. Knowing the wave function, it is a simple matter to evaluate the par-thermal stress tensor and the quantum stress tensor. One finds

$$p_{ij, p} = 0 \tag{5.7}$$

and for the quantum stress tensor

$$p_{ij, q} = p_{ij, q}^{(0)} + 2\tau_{ij}^{(0)} \nabla \cdot \vec{D}$$

$$+ \sum_\gamma [\nabla_i(p_{j\gamma, q}^{(0)} D_\gamma) + \nabla_j(p_{i\gamma, q}^{(0)} D_\gamma)] + O(\nabla^3 D_\gamma) \,, \tag{5.8}$$

where the superscript (0) denotes equilibrium quantities and $\tau_{ij}^{(0)}$ is the equilibrium kinetic energy density. One notes that for a Fermi fluid which is initially spatially homogeneous, the quantum stress tensor is proportional to the first spatial derivative of the displacement. Furthermore, from the equation of continuity, we obtain

$$\vec{u} = -\frac{\partial \vec{D}}{\partial t} \tag{5.9}$$

and thus Eq. (5.4) becomes the Lamé equation which describes the propagation of elastic waves.

VI. MACROSCOPIC DESCRIPTION: HYDRODYNAMICS

The dynamics of the quantum fluid can take on a different form in the presence of a different type of perturbation. In problems where the time scale for the macroscopic motion is long compared with the microscopic relaxation time such that local equilibrium can be maintained throughout, a plausible approximation to the exact many-body problem is to introduce a local specific entropy σ to identify the par-thermal stress tensors as the thermal stress tensor to be written in terms of the local basic macroscopic variables. The approximation of the quantum pressure as a function of local macroscopic variables imposes a lower limit on the dimension of the system. By such an approximation, one also foregoes the hope of describing irregularities of single-particle origin. Likewise, the replacement of $p_{ij, \theta}$ by local macroscopic variables requires that the dimension of the system under consideration is large compared with the range of the short-range interaction. Indeed, in large enough systems such as in classical or in the ^{3}He and ^{4}He fluid, these conditions are satisfied for many phenomena and a completely hydrodynamical description has been found useful.[23] The presence of off-diagonal long-range order[24] in some quantum fluid may give rise to peculiar dynamical properties which can still be included in the hydrodynamical description in terms of a new degree of freedom.[25]

For a Fermi liquid, we can make use of the Thomas-Fermi approximation for the quantum and thermal stress tensor to write

$$p_{ij, p} + p_{ij, q} \rightarrow \left[\frac{1}{5} \frac{\hbar^2}{m} (1.5\pi^2)^{2/3} n^{5/3}\right.$$

$$\left. + \frac{1}{3} \frac{\hbar^2}{m} \left(\frac{6}{4\pi}\right)^{2/3} \sigma^2 n^{5/3}\right] \delta_{ij}. \tag{6.1}$$

We hope to give in the near future an estimate on the time scale and the length scale for which the above approximation leading to a completely hydrodynamical description can be a meaningful concept.

Assuming that the necessary conditions are satisfied, we can write down, at least formally, the Navier-Stokes equation following the general expansion procedure of Born and Green.[5,20] Such an expansion was used by Born and Green[5,20] and Bogoliubov[26] for classical fluids and by Bogoliubov[26] and Fröhlich[12] for quantum fluid.

We consider first the equilibrium case for which the two-body density function $\bar{n}^{(2)}(\vec{r}_1, \vec{r}_2)$ is obtained. In the presence of a velocity gradient, the two-body density gradient will be distorted. To the first order in the velocity gradient, we expand the two-body density $n^{(2)}(\vec{r}_1, \vec{r}_2)$ as

$$n^{(2)}(\vec{r}_1, \vec{r}_2) = \bar{n}^{(2)}(\vec{r}_1, \vec{r}_2) + \sum_{i,j} \xi_{ij} \frac{x_i x_j}{r^2} \left(\frac{\partial u_i}{\partial R_j} + \frac{\partial u_j}{\partial R_i} - \frac{2}{3} \sum_{l=1}^{3} \frac{\partial u_l}{\partial R_l} \delta_{ij} \right) + \sum_{i=1}^{3} \xi_{ii} \frac{x_i x_j}{r^2} \frac{\partial u_i}{\partial R_i} , \tag{6.2}$$

where the functions ξ_{ij} may, of course, depend on $\vec{r} = \vec{r}_2 - \vec{r}_1 = (x_1, x_2, x_3)$ and $\vec{R} = \frac{1}{2}(\vec{r}_1 + \vec{r}_2)$. Not only is the two-body density distorted by the presence of the velocity gradient, so is the one-body Wigner function which can be expanded about the one-body Wigner function for the equilibrium case $\bar{f}^{(1)}(\vec{R}_1, \vec{p}_1)$ by:

$$f^{(1)}(\vec{R}_1, \vec{p}_1) = \bar{f}^{(1)}(\vec{R}_1, \vec{p}_1) + \sum_{i,j} \omega_{ij} \frac{(p_{1i} - mu_i)(p_{1j} - mu_j)}{(\vec{p} - m\vec{u}_1)^2} \left(\frac{\partial u_i}{\partial R_{1j}} + \frac{\partial u_j}{\partial R_{1i}} - \frac{2}{3} \sum_{l} \frac{\partial u_l}{\partial R_{1l}} \delta_{ij} \right) , \tag{6.3}$$

where ω_{ij} depends on $\vec{R}_1$ and $\vec{p}_1$ and only $i \neq j$ need be considered because of the normalization requirement

$$\int f^{(1)}(\vec{R}_1, \vec{p}_1) \frac{d^3 p_1}{(2\pi\hbar)^{3/2}} = n(\vec{R}_1) . \tag{6.4}$$

As a result of the expansion of $n^{(2)}$ and $f^{(1)}$, we have

$$p_{ij,p}(\vec{r}_1) + p_{ij,6}(\vec{r}_1) = \overline{p_{ij,p}(\vec{r}_1)} + \overline{p_{ij,6}(\vec{r}_1)} + \sum_{i,j}^{3} \eta_{ij} \left(\frac{\partial u_i}{\partial r_{1j}} + \frac{\partial u_j}{\partial r_{1i}} - \frac{2}{3} \sum_{l=1}^{3} \frac{\partial u_l}{\partial r_{1l}} \right) + \sum_{i=1}^{3} \zeta_i \frac{\partial u_i}{\partial r_{1i}} , \tag{6.5}$$

where $\overline{p_{ij,p}(\vec{r}_1)}$ and $\overline{p_{ij,6}(\vec{r}_1)}$ are evaluated with the equilibrium one-body Wigner function and two-body density matrix. The coefficients of shear viscosity are, for $i \neq j$

$$\eta_{ij} = \int \frac{d^3 p_1}{(2\pi\hbar)^{3/2}} \omega_{ij} \frac{(p_{1i} - mu_i)^2 (p_{1j} - mu_j)^2}{(\vec{p}_1 - m\vec{u})^2} - \frac{1}{2} \int d^3 r \, \xi_{ij} \frac{x_i x_j}{r^2} \frac{dv_s(r)}{dr} , \tag{6.6}$$

and the coefficients of compressional viscosity are

$$\zeta_i = -\frac{1}{2} \int d^3 r \, \xi_{ii} \frac{x_i x_i}{r^2} \frac{dv_s(r)}{dr} . \tag{6.7}$$

With the extra terms linear in the velocity gradient, Eq. (5.4) becomes now the Navier-Stokes equation.

Although it is possible to obtain a formal definition for the viscosity coefficients, it is still necessary to calculate the distortion of $f^{(1)}$ and $n^{(2)}$ in the presence of a velocity gradient. This may seem a difficult task, but its usefulness lies in requiring only very specific information from a microscopic calculation for its evaluation and not of the complete microscopic solution.

VII. QUANTUM KINETIC EQUATION AND LANDAU'S FERMI-LIQUID THEORY

As we have emphasized, the quantum kinetic equations are general and exact equations. They may be the starting point for many simplifying approximations for the investigation of the dynamics of the quantum many-body systems. We shall illustrate another application of the quantum kinetic equation to the discussion of transport phenomena of the Fermi liquid at very low temperatures and shall establish the connection between the quantum kinetic equation of the Landau's transport equation for simple cases. Such a connection is a simple observation as the Boltzmann equation is the

starting point of Landau's transport equation for quasiparticles. Our attempt to go into some detail to establish such a connection is to indicate the tacit assumptions entering into the derivation so that correction terms or alternative assumptions can be systematically introduced if one so desires.

First of all, we identify the one-body Wigner function $f^{(1)}(\vec{R}, \vec{p})$ as the distribution function for the "quasiparticles." To make our problem simple, we shall consider a temperature near absolute zero so that collisions between quasiparticles can be neglected. Because of having no collisions between quasiparticles, the two-body Wigner function can be approximated as a simple product of one-body Wigner functions:

$$f^{(2)}(\vec{R}_1 \vec{R}_2; \vec{p}_1 \vec{p}_2) = f^{(1)}(\vec{R}_1, \vec{p}_1) f^{(1)}(\vec{R}_2, \vec{p}_2) . \tag{7.1}$$

We consider now small deviations from equilibrium:

$$f^{(1)}(\vec{R}, \vec{p}) = \overline{f^{(1)}(\vec{R}, \vec{p})} + \delta f(\vec{R}, \vec{p}), \tag{7.2}$$

where the equilibrium distribution function $\overline{f^{(1)}}(\vec{R}, \vec{p})$ can be chosen to be time independent by a proper choice of our coodinate system. From Eq. (4.14) it then satisfies the equation

$$\frac{\vec{p}}{m} \cdot \nabla_{\vec{R}} \ \overline{f^{(1)}(\vec{R},\vec{p})} = \int \frac{d^3R_2 d^3p_2}{(2\pi\hbar)^3} \cdot \frac{2}{\hbar} \left(\sin \frac{2}{\hbar} \ \nabla_p^f \cdot \nabla_R^v \right) v(\vec{R}\vec{p}; \vec{R}_2\vec{p}_2) \overline{f^{(1)}(\vec{R},\vec{p})} \ \overline{f^{(1)}(\vec{R}_2,\vec{p}_2)}$$ (7.3)

while the deviation from equilibrium satisfies, up to the first order in $\delta f^{(1)}$

$$\frac{\partial}{\partial t} \ \delta f^{(1)}(R,p) + \frac{\vec{p}}{m} \cdot \nabla_{\vec{R}} \ \delta f^{(1)}(\vec{R},\vec{p})$$

$$= \int \frac{d^3R_2 d^3p_2}{(2\pi\hbar)^3} \cdot \frac{2}{\hbar} \left(\sin \frac{\hbar}{2} \ \nabla_p^f \ \nabla_R^v \right) v(\vec{R}\vec{p}; \vec{R}_2\vec{p}_2)[\ \overline{f^{(1)}(\vec{R},\vec{p})} \delta f^{(1)}(\vec{R}_2,\vec{p}_2) + \overline{f^{(1)}(\vec{R}_2,\vec{p}_2)} \delta f^{(1)}(\vec{R},\vec{p})] \ .$$ (7.4)

Upon making the expansion of the sine function in Eq. (7.4) and neglecting terms of order $\hbar^2$ and higher, and assuming further that

$$v(\vec{R}_1\vec{p}_1; \vec{R}_2\vec{p}_2) = V(\vec{p}_1\vec{p}_2)\delta(\vec{R}_1 - \vec{R}_2) \ ,$$ (7.5)

we have the following transport equation for the deviation of the quasiparticle distribution function from equilibrium:

$$\frac{\partial}{\partial t} \ \delta f^{(1)}(\vec{R},\vec{p}) + \frac{\vec{p}}{m} \cdot \nabla_{\vec{R}} \delta f^{(1)} (\vec{R},\vec{p}) - \nabla_p \overline{f^{(1)}(\vec{R},\vec{p})} \int \frac{d^3p_2}{(2\pi\hbar)^3} \ V(\vec{p} \ \vec{p}_2)\nabla_R \delta f^{(1)}(\vec{R}_1,\vec{p}_2)$$

$$- \nabla_p \ \delta f^{(1)}(\vec{R},\vec{p}) \int \frac{d^3p_2}{(2\pi\hbar)^3} \ V(\vec{p} \ \vec{p}_2)\nabla_R \overline{f^{(1)}(\vec{R},\vec{p}_2)} = 0 \ .$$ (7.6)

In the case of an infinite medium for which the equilibrium distribution function $f^{(1)}$ is independent of positions, the last term vanishes and Eq. (7.6) is identical to Landau's collisionless transport equation (which is also known as the Landau-Vlasov equation[27]). The deviation $\delta f^{(1)}$ is now coupled to the equilibrium distribution function $f^{(1)}$ through $\nabla_p f^{(1)}$. In Landau's theory the equilibrium distribution $f^{(1)}$ is, however, not evaluated with Eq. (7.3) but taken to be a Fermi-Dirac distribution (as a definition of his "normal" Fermi liquid), justification of which is discussed in great detail by Noziéres.[28] The study of the dynamics of the system can thus be greatly simplified.

Our simple result outlined above gives an indication of how generalizations and corrections can be introduced. First of all, in situations where the equilibrium distribution $f^{(1)}$ is spatially inhomogeneous such as in a finite nucleus, there is the additional last term in Eq. (7.6) which must be included in the dynamics. Secondly, the expansion of the sine function leads to quantum corrections of order $\hbar^2, \hbar^4, \dots$ and higher-order derivatives of the potential and of the equilibrium distribution function. Finally, in situations where the collisions between quasiparticles cannot be neglected, Eq. (7.1) needs to be modified to include the effect of correlation in the two-body Wigner function. One can follow, for example, the arguments of Ross and Kirkwood[29] to obtain a transport equation with the proper collision term.

VIII. TDHF AND MDTDHF AS SPECIAL APPROXIMATIONS OF THE QUANTUM KINETIC EQUATIONS

It is clear that in a microscopic theory it is difficult to solve the set of quantum kinetic equations in its entirety. One usually resorts to simple truncations so that the hierarchy of equations terminate. In this section, we shall restrict ourselves to the considerations of Fermion systems for which the microscopic time-dependent Hartree-Fock approximation has been used to study the dynamics of many nuclear systems.[30-32] In Ref. 1 we discuss how TDHF can be viewed as a special case in which the quantum kinetic equation can be truncated at the lowest level. Specifically, with the assumption of a single Slater determinant consisting of single-particle wave functions $\psi_\alpha(r)$ for a Fermion system, the one-body and two-body density matrices are

$$\mathfrak{N}^{(1)}(\vec{r}; \vec{r}') = \sum_\alpha \psi_\alpha(\vec{r})\psi_\alpha^*(\vec{r}') \ ,$$ (8.1)

where the sum is carried over the occupied states, and

$$\mathfrak{N}^{(2)}(\vec{r}_1\vec{r}_2; \vec{r}_1'\vec{r}_2') = \mathfrak{N}^{(1)}(\vec{r}_1; \vec{r}_1')\mathfrak{N}^{(1)}(\vec{r}_2; \vec{r}_2')$$

$$- \mathfrak{N}^{(2)}(\vec{r}_1; \vec{r}_2')\mathfrak{N}^{(1)}(\vec{r}_2; \vec{r}_1') \ ,$$ (8.2)

As the two-body density matrix is now written in terms of the one-body density matrix, the hierarchy of the quantum kinetic equation terminates at the lowest level and one obtains from Eq. (2.8)

the TDHF equation

$$i\hbar \frac{\partial}{\partial t} \mathfrak{N}^{(1)}(\vec{r}_1;\vec{r}_1') = -\frac{\hbar^2}{2m}(\nabla_{r_1}^2 - \nabla_{r_1'}^2)\mathfrak{N}^{(1)}(\vec{r}_1;\vec{r}_1')$$

$$+ \int d^3r_2[v(\vec{r}_1\vec{r}_2) - v(\vec{r}_1'\vec{r}_2)][\mathfrak{N}^{(1)}(\vec{r}_1;\vec{r}_1')\mathfrak{N}^{(1)}(\vec{r}_2;\vec{r}_2) - \mathfrak{N}^{(1)}(\vec{r}_1;\vec{r}_2)\mathfrak{N}^{(1)}(\vec{r}_2;\vec{r}_1')] . \tag{8.3}$$

The transition from the microscopic description of TDHF to a macroscopic description in terms of the total density field and velocity field was discussed in Ref. 1. We shall briefly summarize the results. One introduces an amplitude factor $\phi_\alpha(\vec{r},t)$ and phase factor $S_\alpha(\vec{r},t)$ to write the wave function of an occupied single-particle state ψ_α in the form

$$\psi_\alpha(\vec{r},t) = \phi_\alpha(\vec{r},t)\exp[imS_\alpha(\vec{r},t)/\hbar - i\Omega_\alpha(t)], \tag{8.4}$$

where $\Omega_\alpha(t)$ is a factor which depends only on time. One obtains from TDHF the equation of continuity

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\vec{u}) = 0 \tag{8.5}$$

and the equation of motion

$$\frac{\partial}{\partial t}nu_i + \sum_{j=1}^{3}\frac{\partial}{\partial x_j}nu_iu_j$$

$$= -\frac{1}{m}\sum_{j=1}^{3}\frac{\partial}{\partial x_j}(p_{ij,p}+p_{ij,q})$$

$$-\frac{n}{m}\int d^3r_2 n(\vec{r}_2)\frac{\partial}{\partial x_i}v(\vec{r},r_2)$$

$$+\frac{1}{m}\int d^3r_2 |\mathfrak{N}^{(1)}(\vec{r};\vec{r}_2)|^2 \frac{\partial}{\partial x_i}v(\vec{r},\vec{r}_2), \tag{8.6}$$

where

$$n = \sum_\alpha \phi_\alpha^2(\vec{r},t), \tag{8.7}$$

$$\vec{u}(\vec{r},t) = \sum_\alpha \phi_\alpha^2\nabla_r S_\alpha/n , \tag{8.8}$$

and

$$\vec{r} = (x_1,x_2,x_3). \tag{8.9}$$

The par-thermal stress tensor $p_{ij,p}$ is

$$p_{ij,p} = m\sum_\alpha \phi_\alpha^2 \left(\frac{\partial S_\alpha}{\partial x_i} - u_i\right)\left(\frac{\partial S_\alpha}{\partial x_j} - u_j\right) \tag{8.10}$$

and the quantum stress tensor is

$$p_{ij,q} = -\frac{\hbar^2}{4m}\frac{\partial^2}{\partial x_i \partial x_j}n + \frac{\hbar^2}{m}\sum_\alpha\left(\frac{\partial}{\partial x_i}\phi_\alpha\right)\left(\frac{\partial}{\partial x_j}\phi_\alpha\right). \tag{8.11}$$

An independent approach undertaken by Koonin and Kerman[31] by starting with the TDHF equation

and using the one-body Wigner function led to similar results.

It is clear that a single Slater determinant cannot be the complete solution of the many-body problem. It is necessary to include the possibility of transition from one Slater determinant into another. We can generalize the TDHF formulation of the many-body problem to include many determinants and obtain the multideterminant time-dependent Hartree-Fock (MDTDHF) formulation of the many-body problem.

We envisage that we can construct a set of TDHF solutions Ψ_k such that at time $t = 0$, the expectation values of the many-body Hamiltonian, the total energy momentum, and the angular momentum are all the same for the different determinants Ψ_k:

$$(\Psi_k(\vec{r}_1\cdots\vec{r}_N,t=0),H\Psi_k(\vec{r}_1\cdots\vec{r}_N,t=0)) = E , \tag{8.12}$$

$$\left(\Psi_k(\vec{r}_1\cdots\vec{r}_N,t=0), \frac{\hbar}{i}\sum_i\nabla_{r_i}\Psi_k(\vec{r}_1\cdots\vec{r}_N,t=0)\right) = \vec{P} , \tag{8.13}$$

$$\left(\Psi_k(\vec{r}_1\cdots\vec{r}_N,t=0),m\sum_i\vec{r}_{ix}\nabla_{r_i}\Psi_k(\vec{r}_1\cdots\vec{r}_N,t=0)\right) = \vec{L}. \tag{8.14}$$

In addition, we require that at $t = 0$, the first and the second moments of distances are also the same for the different determinants:

$$\left(\Psi_k(\vec{r}_1\cdots\vec{r}_N,t=0),\sum_i\vec{r}_i\Psi_k(\vec{r}_1\cdots\vec{r}_N;t=0)\right) = \vec{R} , \tag{8.15}$$

$$\left(\Psi_k(\vec{r}_1\cdots\vec{r}_N,t=0),\sum_i x_i^2\Psi_k(\vec{r}_1\cdots\vec{r}_N;t=0)\right) = X^2 , \tag{8.16}$$

$$\left(\Psi_k(\vec{r}_1\cdots\vec{r}_N,t=0),\sum_i y_i^2\Psi_k(\vec{r}_1\cdots\vec{r}_N;t=0)\right) = Y^2 , \tag{8.17}$$

and

$$\left(\Psi_k(\vec{r}_1\cdots\vec{r}_N,t=0),\sum_i z_i^2\Psi_k(\vec{r}_1\cdots\vec{r}_N;t=0)\right) = Z^2 . \tag{8.18}$$

These conditions insure that all these different determinants Ψ_k have the same relevant initial con-

ditions. The determinants differ from each other
by the different ways energy and/or mass can be
partitioned. For example, consider two identical
nuclei approaching each other at time $t = 0$. We
can choose $\Psi_0(\vec{r}_1 \cdots \vec{r}_N; t = 0)$ as the single Slater
determinant for which the lowest single-particle
states up to a certain Fermi level are occupied
and the two nuclei move toward each other with a
velocity v_0 which enters into the phase factor for
the single-particle states. We can construct Ψ_1
as the self-consistent Slater determinant for which
one of the single particles is promoted to occupy
a higher single-particle state and the two nuclei
move toward each other with a slightly smaller
velocity v_1. Other determinants Ψ_k can be sim-
ilarly constructed to partition the energy E into
intrinsic and relative energies and/or the mass
number as shown in Fig. 1.

For each of the determinants $\Psi_k(t = 0)$, one, can,
in principle, follow the time evolution with the
TDHF equations. One then obtains the set of Ψ_k's
which are now known functions of time and space.

The general solution can be written using the
Ψ_k's as the basis wave function:

$$\Psi(\vec{r}_1\vec{r}_2 \cdots \vec{r}_N; t) = \sum_k a_k(t)\Psi_k(\vec{r}_1\vec{r}_2 \cdots \vec{r}_N; t) \quad (8.19)$$
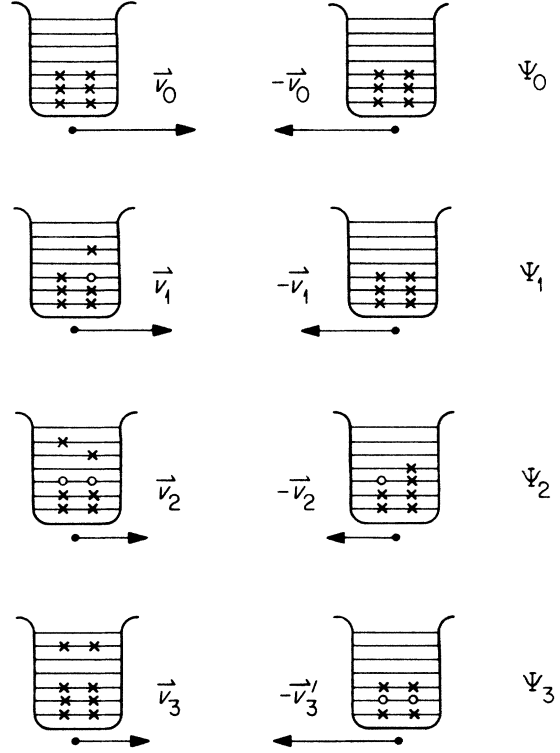
which satisfies the many-body Hamiltonian:

$$i\hbar \frac{\partial}{\partial t} \Psi(\vec{r}_1\vec{r}_2 \cdots \vec{r}_N; t) = H\Psi(\vec{r}_1\vec{r}_2 \cdots \vec{r}_N; t). \quad (8.20)$$

Therefore, we have

$$i\hbar \sum_k \dot{a}_k \Psi_k = \sum_k a_k \left(H - i\hbar \frac{\partial}{\partial t} \right) \Psi_k. \quad (8.21)$$

Upon taking the scalar product with $\Psi_{k'}$, we have

$$i\hbar \sum_k \dot{a}_k (\Psi_{k'}, \Psi_k) = \sum_k a_k \left(\Psi_{k'}, \left(H - i\hbar \frac{\partial}{\partial t} \right) \Psi_k \right). \quad (8.22)$$

AND SO ON . . .

FIG. 1. Schematic representation for the set of Ψ_k's
used as basis wave functions in describing the many-
body wave function for the collision of two identical nuc-
lei.

In Eq. (8.22), the scalar products can, in princi-
ple, be calculated as all the Ψ_k's are known func-
tions of space and time. It therefore allows the
evaluation of $a_k(t)$ once the initial amplitudes $a_k(0)$
are known. With the knowledge of $a_k(t)$, one can
construct the one-body density matrix

$$\mathfrak{N}^{(1)}(\vec{r}_1, \vec{r}_1') = \sum_k |a_k|^2 \mathfrak{N}_k^{(1)}(\vec{r}_1, \vec{r}_1')$$

$$+ \sum_{\substack{k, k' \\ k \neq k'}} a_k a_{k'}^* N \int d^3r_2 \cdots d^3r_N \, \Psi_k(\vec{r}_1\vec{r}_2 \cdots \vec{r}_N)\Psi_{k'}^*(\vec{r}_1'\vec{r}_2 \cdots \vec{r}_N) \quad (8.23)$$

and the two-body density

$$\mathfrak{N}^{(2)}(\vec{r}_1\vec{r}_2; \vec{r}_1'\vec{r}_2) = \sum_k |a_k|^2 \mathfrak{N}_k^{(2)}(\vec{r}_1\vec{r}_2; \vec{r}_1'\vec{r}_2)$$

$$+ \sum_{\substack{k, k' \\ k \neq k'}} a_k a_{k'}^* N(N-1) \int d^3r_3 \cdots d^3r_N \, \Psi_k(\vec{r}_1\vec{r}_2 \cdots \vec{r}_N)\Psi_{k'}^*(\vec{r}_1'\vec{r}_2 \cdots \vec{r}_N). \quad (8.24)$$

These density matrices describe the one-body and two-body density correlations exactly.

It is clear that with a many-body system, it is virtually impossible to make all the measurements on all the variables whose operators commute with H. This being the case, one resorts to a statistical description of the system and gives up hope of ever measuring the phase relationship between the different basis wave functions. Therefore, even though one may begin by having the system described as a pure state, in subsequent times, the system becomes a mixed state where one knows only the probability for the occurrence of a certain Slater determinant Ψ_k and not the phase relationship between them. In mathematical terms, a statistical description of the system amounts to writing the density matrix in the form

$$\mathfrak{N}^{(1)}(\vec{r}_1; \vec{r}_1') = \sum_k w_k \mathfrak{N}_k^{(1)}(\vec{r}_1; \vec{r}_1') \qquad (8.25)$$

and

$$\mathfrak{N}^{(2)}(\vec{r}_1\vec{r}_2; \vec{r}_1'\vec{r}_2') = \sum_k w_k \mathfrak{N}_k^{(2)}(\vec{r}_1\vec{r}_2; \vec{r}_1'\vec{r}_2') . \qquad (8.26)$$

Comparing Eqs. (8.25) and (8.26) with Eqs. (8.23) and (8.24), one recognizes that the passage from the microscopic to a quantum statistical description is completed by making the approximations (8.25) and (8.26) so that the cross terms in Eqs. (8.23) and (8.24) do not appear. It is obvious that because of the loss of the phase relationship between the a_k's, the coherence of the different Ψ_k's is lost, and one cannot construct by passage backward in time the original wave packet. In other words, the microscopic time reversibility is violated by making the approximations.

With the knowledge of only the density matrices $\mathfrak{N}^{(1)}$ and $\mathfrak{N}^{(2)}$, we forego also any hope of obtaining higher correlations between three and larger numbers of particles. The set of many-body equations can be truncated at the lowest level. The equation of motion is then as given by Eq. (2.8).

To obtain the macroscopic equation of motion, we again introduce an amplitude factor and a phase factor for each of the single-particle states α_k belonging to the kth determinant

$$\psi_{\alpha k}(\vec{r}, t) = \phi_{\alpha k}(\vec{r}, t) \exp[imS_{\alpha k}(\vec{r}, t)/\hbar - i\Omega_{\alpha k}(t)] . \qquad (8.27)$$

After substituting Eq. (8.27) into (8.25) and (8.26) and making use of Eq. (8.8), one obtains, after much simplification,

$$\frac{\partial n}{\partial t} + \nabla \cdot n\vec{u} = 0 , \qquad (8.28)$$

$$m\frac{\partial nu_i}{\partial t} + \sum_{j=1}^{3} \frac{\partial}{\partial x_j}(mnu_iu_j + p_{ij,p} + p_{ij,q}) = -n\int d^3r_2\, n(\vec{r}_2)\frac{\partial}{\partial x_i}v(\vec{r}, \vec{r}_2) + \int d^3r_2 |\mathfrak{N}^{(1)}(\vec{r}, \vec{r}_2)|^2 \frac{\partial}{\partial x_i}v(\vec{r}, \vec{r}_2)$$

$$-\int d^3r_2 \sum_k w_k[n_k(\vec{r}) - n(\vec{r})][n_k(\vec{r}_2) - n(\vec{r}_2)]\frac{\partial}{\partial x_i}v(\vec{r}, \vec{r}_2)$$

$$+\int d^3r_2 \sum_k w_k |\mathfrak{N}_k^{(1)}(\vec{r}, \vec{r}_2) - \mathfrak{N}^{(1)}(\vec{r}, \vec{r}_2)|^2 \frac{\partial}{\partial x_i}v(\vec{r}, \vec{r}_2) , \qquad (8.29)$$

where the density is now

$$n(\vec{r}) = \sum_k w_k\, n_k(\vec{r}) = \sum_k w_k \sum_{\alpha_k} \phi_{\alpha k}{}^2(\vec{r}) , \qquad (8.30)$$

the average velocity is

$$u(\vec{r}) = \sum_k w_k \sum_{\alpha_k} \phi_{\alpha k}{}^2(\vec{r})\nabla S_{\alpha k}(\vec{r})/n(\vec{r}) , \qquad (8.31)$$

$$p_{ij,p} = m\sum_k w_k \sum_{\alpha_k} \phi_{\alpha k}{}^2\left(\frac{\partial}{\partial x_i}S_{\alpha k} - u_i\right)\left(\frac{\partial}{\partial x_j}S_{\alpha k} - u_j\right), \qquad (8.23)$$

and

$$p_{ij,q} = -\frac{\hbar^2}{4m}\sum_k w_k \frac{\partial^2}{\partial x_i\,\partial x_j}n_k + \frac{\hbar^2}{m}\sum_k w_k \sum_{\alpha_k}\left(\frac{\partial}{\partial x_i}\phi_{\alpha k}\right)\left(\frac{\partial}{\partial x_j}\phi_{\alpha k}\right) . \qquad (8.33)$$

In Eq. (8.29), all the terms but the last two have a correspondence with the terms in Eq. (8.6) obtained for TDHF. The last two terms are new, and they represent viscous forces arising from the dispersion of the original state into other states.

To close the set of dynamical equations, it is necessary to know the behavior of w_k on time. This can perhaps be done by considering the Pauli master equation for w_k and looking for its solutions. On physical grounds, one knows that for a closed system where there is no current dispersing into infinity, the probability of occupation w_k behaves as shown in Fig. 2. Suppose at time $t = 0$ we have a pure state such that $w_0 = 1$. As time proceeds, the weight w_0 goes nearly to zero, but after a while will increase again to unity. The recurrence time is of the order $\hbar/\delta$ where δ is the level spacing. Of course, when there are currents radiating toward infinity, the recurrence time is infinite. In a heavy-ion collision, one expects that the collision time involved is much smaller than the recurrence time, so that within the time interval of interest, w_k is essentially linear in t. Because of such linear dependence, the operation of time reversal onto Eq. (8.29) will alter the equation and hence violates time reversibility.

What we have presented is a framework by which dissipative mechanisms can be studied. What remains to be investigated is how $p_{ij,p}$, the kinetic energy term, and the extra new terms in Eq. (8.29) depend on macroscopic quantities. In this connection, the recent work of Bonche, Koonin, and Negele[30] on a direct solution of the TDHF will be of great help in determining such dependence.

IX. DISCUSSIONS AND CONCLUSIONS

On the basis of the quantum kinetic equation derived from the many-body Schrödinger equation, we investigated the equations of motion involving macroscopic
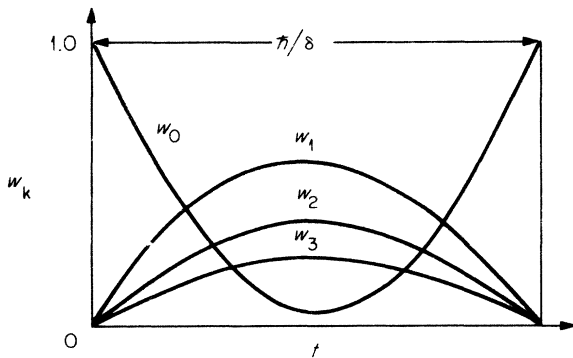

FIG. 2. Schematic representation of the occupation probabilities for various determinants Ψ_k as a function of time.

variables such as the s-body density, current, and kinetic energy. We obtain a hierarchy of generalized fluid dynamical equations involving these variables. Some of the results we presented are given in a general and exact form. Their usefulness lies in being the starting point of many simplifying approximations for the investigation of the dynamics. We mentioned TDHF and MDTDHF as two approximations to the quantum kinetic equation. One can also consider the elastic response and hydrodynamics even for quantum fluids. There are also other kinds of approximations one may make in connection with off-diagonal long-range order correlations where one writes down the appropriate form of the density matrix and proceeds to study its dynamics.[12] The possibility of having mixed states which depend on temperature makes the use of the density matrix even more attractive.

Another important application of the quantum kinetic equation is the possibility of a systematic elucidation of the connections between the Fermi-liquid theory of Landau[18] and the microscopic many-body problem. One knows that starting with the classical BBGKY equation, one can derive the Boltzmann equation for the one-body distribution function. On the other hand, the Boltzmann equation has been taken as the starting point for the transport of "quasiparticles" in the Fermi-liquid theory of Landau. One expects that the transport equation in the Fermi-liquid theory of Landau is intimately connected with the one-body quantum kinetic equation in phase space. We outline the assumptions involved in connecting the one-body kinetic equation and the Landau transport equation for the simple case when there is no collision. It will be of interest to investigate more complicated cases and the corrections to the Landau Fermi-liquid theory.

As with the classical BBGKY equation, the necessary and sufficient conditions for a truncation and approximation of the hierarchy of the quantum kinetic equations are difficult to spell out for the general case. One is usually guided in a qualitative way by the importance of the two-body collisions or three-body collisions, denseness or diluteness of the system, time scales and length scales, and whatever physical intuition there is to help select a certain type of truncation or approximation. Our lack of a systematic guideline is a shortcoming in working with the quantum kinetic equation (and its related macroscopic equations). In consequence, the ultimate confrontation between experimental results and theoretical results obtained from different truncations and approximations is of singular importance in determining the correct approach.

With regard to the nuclear fluid, there is the

additional problem of the nuclear hard-core inter-
action which gives rise to special short-range cor-
relation between nucleons. In the Brueckner theo-
ry, this is studied with the defect wave function
and the independent pair approximation. How such
a correlation can be studied within the framework
of the quantum kinetic equation needs to be

spelled out in detail. The recent work of Kümmel
and Zabolitzky[33] for the ground states of many
finite nuclei contains a hierarchy of equations
which is similar to the quantum kinetic equation in
nature. A simple extension of their results to non-
equilibrium cases and a careful comparison with
the quantum kinetic equation will be of interest.

APPENDIX I.

We wish to evaluate

$$\lim_{\vec{r}^{3s'} \to \vec{r}^{3s}} -\frac{\hbar^2}{2m}(\nabla_{x_i} - \nabla_{x_i'}) \sum_{j=1}^{3s} (\nabla_{x_j}^2 - \nabla_{x_j'}^2)\mathfrak{N}^{(s)}(\vec{r}^s; \vec{r}^{s'}) \, , \tag{I1}$$

where $x_i, x_j \in \vec{r}^s$ and $x_i', x_j' \in \vec{r}^{s'}$. We consider first the limit

$$\lim_{\vec{r}^{3N'} \to \vec{r}^{3N}} (\nabla_i - \nabla_i')(\nabla_j^2 - \nabla_j'^2)\Psi(\vec{r}^N)\Psi^*(\vec{r}^{N'}) = 2\,\mathrm{Re}\{[\nabla_i\nabla_j\nabla_j\Psi(\vec{r}^N)]\Psi^*(\vec{r}^N) - \nabla_i\Psi(\vec{r}^N)\nabla_j^2\Psi^*(\vec{r}_N)\} \, , \tag{I2}$$

where for convenience we use the simplified notation ∇_i and ∇_i' to stand for ∇_{x_i} and $\nabla_{x_i'}$, respectively.
Upon using the polar form of (3.1) to represent the wave function Ψ, we have

$$\nabla_i\Psi = e^{imS/\hbar}\left(\nabla_i\phi + \frac{im}{\hbar}\phi\nabla_i S\right) \, , \tag{I3}$$

$$\nabla_j\nabla_i\Psi = e^{imS/\hbar}\left\{\nabla_i\nabla_j\phi - \frac{m^2}{\hbar^2}\phi\nabla_i S\nabla_j S + \frac{im}{\hbar}[\nabla_i(\phi\nabla_j S) + \nabla_j\phi\nabla_i S]\right\} \, , \tag{I4}$$

$$\nabla_j^2\Psi^* = e^{-imS/\hbar}\left\{\nabla_j^2\phi - \frac{m^2}{\hbar^2}\phi\nabla_j S\nabla_j S - \frac{im}{\hbar}[\nabla_j(\phi\nabla_j S) + \nabla_j\phi\nabla_j S]\right\} \, , \tag{I5}$$

and

$$\nabla_j(\nabla_j\nabla_i\Psi) = e^{imS/\hbar}\left(\nabla_j + \frac{im}{\hbar}\nabla_j S\right)\left[\nabla_i\nabla_j\phi - \frac{m^2}{\hbar^2}\phi\nabla_i S\nabla_j S + \frac{im}{\hbar}(\nabla_i(\phi\nabla_j S) + \nabla_j\phi\nabla_i S)\right] \, . \tag{I6}$$

Substituting (I3)–(I6) into (I2), we obtain

$$\lim_{\vec{r}^{3N'} \to \vec{r}^{3N}} (\nabla_i - \nabla_i')(\nabla_j^2 - \nabla_j'^2)\Psi(\vec{r}^N)\Psi^*(\vec{r}^{N'}) = 2\left[-\frac{2m^2}{\hbar^2}\nabla_j(\phi^2\nabla_i S\nabla_j S) - 2\nabla_j(\nabla_i\phi\nabla_j\phi) + \nabla_i\nabla_j^2(\tfrac{1}{2}\phi^2)\right] \, . \tag{I7}$$

Thus, we have

$$\lim_{x^{3N'} \to x^{3N}} -\frac{\hbar^2}{2m}(\nabla_i - \nabla_i') \sum_{j=1}^{3s} (\nabla_j^2 - \nabla_j'^2)\Psi(\vec{r}^N)\Psi^*(\vec{r}^{N'}) = 2\sum_{j=1}^{3s} \nabla_j\left(m\phi^2\nabla_i S\nabla_j S - \frac{\hbar^2}{4m}\nabla_i \cdot \nabla_j\phi^2 + \frac{\hbar^2}{m}\nabla_i\phi\nabla_j\phi\right) \, . \tag{I8}$$

We integrate both sides of the above equation by $\int d\mu_s$. Because ∇_i and ∇_j commute with $\int d\mu_s$, we obtain

$$\lim_{x^{3s'} \to x^{3s}} -\frac{\hbar^2}{2m}(\nabla_i - \nabla_i') \sum_{j=1}^{3s} (\nabla_j^2 - \nabla_j'^2)\mathfrak{N}^{(s)}(\vec{r}^s; \vec{r}^{s'}) = 2\sum_{j=1}^{3s} \nabla_j \int d\mu_s\left(m\phi^2\nabla_i S\nabla_j S - \frac{\hbar^2}{4m}\nabla_{x_i} \cdot \nabla_{x_j}\phi^2 + \frac{\hbar^2}{m}\nabla_i\phi\nabla_j\phi\right) \tag{I9}$$

which is Eq. (3.17).

APPENDIX II.

We wish to evaluate

$$\lim_{x^{3s'} \to x^{3s}} -\frac{\hbar^2}{2m} \sum_{i=1}^{3s} \nabla_{x_i}\nabla_{x_i'} \sum_{j=1}^{3s} (\nabla_{x_j}^2 - \nabla_{x_j'}^2)\mathfrak{N}^{(s)}(\vec{r}^s; \vec{r}^{s'}) \, , \tag{II1}$$

where $x_i, x_j \in \vec{r}^s$ and $x_i', x_j' \in \vec{r}^{s'}$. We consider first the limit

$$\lim_{x^{3N'} \to x^{3N}} (\nabla_i - \nabla_{i'})(\nabla_j^2 - \nabla_{j'}^2)\Psi(\vec{r}^N)\Psi^*(\vec{r}^{N'}) = 2i\mathcal{I}m\{[\nabla_i\nabla_j^2\Psi(\vec{r}^N)]\nabla_i\Psi^*(\vec{r}^N)\}, \tag{II2}$$

where, for convenience, as in Appendix I, we used the simplified notation of ∇_i to stand for ∇_{x_i}. Using the polar form of Eq. (3.1) to represent the wave function Ψ, we have

$$\nabla_i\Psi^* = e^{-imS/\hbar}\left(\nabla_i\phi - \frac{im}{\hbar}\phi\nabla_iS\right) \tag{II3}$$

and

$$\nabla_i\nabla_j^2\Psi = e^{imS/\hbar}\left(\nabla_i + \frac{im}{\hbar}\nabla_iS\right)\left(\nabla_j^2\phi - \frac{m^2}{\hbar^2}\phi(\nabla_jS)^2 + \frac{im}{\hbar}[\nabla_j(\phi\nabla_jS) + \nabla_j\phi\nabla_jS]\right). \tag{II4}$$

Substituting (II3) and (II4) into Eq. (II2), we obtain

$$\lim_{x^{3N'} \to x^{3N}} -\frac{\hbar^2}{2m}\nabla_i\nabla_{i'}(\nabla_j^2 - \nabla_{j'}^2)\Psi(\vec{r}^N)\Psi^*(\vec{r}^{N'})$$

$$= -\frac{2mi}{\hbar}\nabla_i\left[\frac{1}{2}m\phi^2(\nabla_jS)^2\nabla_iS + \frac{\hbar^2}{2m}(\nabla_j\phi)^2\nabla_iS + \left(-\frac{\hbar^2}{4m}\nabla_j^2\phi^2\nabla_iS + \frac{\hbar^2}{m}\nabla_i\phi\nabla_jS\right) + \frac{\hbar^2}{2m}\phi\nabla_i\phi\nabla_j^2S\right].$$

Hence, we have

$$\lim_{x^{3N'} \to x^{3N}} -\frac{\hbar^2}{2m}\sum_{i,j=1}^{3s}\nabla_i\nabla_i'(\nabla_j^2 - \nabla_j'^2)\Psi(\vec{r}^N)\Psi^*(\vec{r}^{N'})$$

$$= -\frac{2mi}{\hbar}\sum_{i=1}^{3s}\nabla_i\left[\frac{1}{2}m\phi^2(\nabla_{x3s}S)^2\nabla_iS + \frac{\hbar^2}{2m}(\nabla_{x3s}\phi)^2\nabla_iS + \sum_{j=1}^{3s}\left(-\frac{\hbar^2}{4m}\nabla_i\nabla_j\phi^2 + \frac{\hbar^2}{m}\nabla_i\phi\nabla_j\phi\right)\nabla_jS + \frac{\hbar^2}{2m}\phi\nabla_i\phi\nabla_{x3s} \cdot \nabla_{x3s}S\right].$$

Upon integrating both sides by $\int d\mu_s$, we obtain

$$\lim_{x^{3s'} \to x^{3s}} -\frac{\hbar^2}{2m}\sum_{i,j=1}^{3s}\nabla_i\nabla_i'(\nabla_j^2 - \nabla_j'^2)\mathfrak{N}^{(s)}(\vec{r}^s, \vec{r}^{s'})$$

$$= -\frac{2mi}{\hbar}\sum_{i=1}^{3s}\nabla_i\int d\mu_s\left[\frac{1}{2}m\phi^2(\nabla_{x3s}S)^2\nabla_iS + \frac{\hbar^2}{2m}(\nabla_{x3s}\phi)^2\nabla_iS\right.$$

$$\left. + \sum_{j=1}^{3s}\left(-\frac{\hbar^2}{4m}\nabla_i\nabla_j\phi^2 + \frac{\hbar^2}{m}\nabla_i\phi\nabla_j\phi\right)\nabla_jS + \frac{\hbar^2}{2m}\phi\nabla_i\phi\nabla_{x3s} \cdot \nabla_{x3s}S\right],$$

which leads to Eq. (3.25).

---

[1] C. Y. Wong, J. A. Maruhn, and T. A. Welton, Nucl.
Phys. A256, 469 (1975).

[2] C. Y. Wong, T. A. Welton, and J. A. Maruhn, Phys.
Rev. C 15, 1558 (1977).

[3] N. N. Bogoliubov, Lectures in Quantum Statistics
(Gordon and Breach, New York, 1967) Vol. I.

[4] N. N. Bogoliubov and K. P. Gurov, Zh. Eksp. Teor.
Fiz. 17, 614 (1947)].

[5] M. Born and H. S. Green, A General Kinetic Theory
of Liquids (Cambridge U.P., London, 1949), p. 48.

[6] N. N. Bogoliubov, J. Phys. (USSR) 10, 256, 265 (1946).

[7] J. G. Kirkwood, J. Chem. Phys. 15, 72 (1947).

[8] J. Yvon, La Theorie Statistique des Fluides et l' Equa-
tion d' Etat (Hermann, Paris, 1935).

[9] E. Wigner, Phys. Rev. 40, 749 (1932).

[10] For a review of the Wigner functions, see e.g., H. Mori,
I. Oppenheimer, and J. Ross, in Studies in Statistical
Mechanics, edited by J. de Boer and G. E. Uhlenbeck
(North-Holland, Amsterdam, 1962), Vol. I and also
E. A. Remler, Ann. Phys. (N.Y.) 95, 455 (1975).

[11] J. H. Irving and R. W. Zwanzig, J. Chem. Phys. 19,
1173 (1951).

[12] H. Fröhlich, Riv. Nuovo Cimento 3, 490 (1973); H. Fröh-
lich, Collect. Phenom. 1, 173 (1974).

[13] U. Fano, Rev. Mod. Phys. 28, 74 (1957); F. D. Peat,
in Physical Chemistry, an Advanced Treatise, edited
by D. Henderson (Academic, New York, 1975), Vol.

IIA.

[14]C. Y. Wong, J. Math. Phys. 17, 1008 (1976); ORNL report (unpublished).

[15]Earlier discussions of the "quantum pressure" were given by E. P. Gross, J. Math. Phys. 4, 195 (1964). However, the quantum pressure defined there has the dimension and the function of a potential and is properly called the quantum potential by many other workers [see, e.g., J. O. Hirschfelder and K. T. Tang, J. Chem. Phys. 64, 760 (1976); J. O. Hirschfelder and A. C. Christoph, J. Chem. Phys. 61, 5435 (1974); L. E. Belinfante, Survey of Hidden Variables Theory, Natural Philosophy Monographs (Pergamon, New York, 1973), Vol. 55]. Our quantum stress tensor has the dimension and the function of a stress tensor and is different from the "quantum pressure" of Gross.

[16]J. Friedel, Nuovo Cimento Suppl. 7, 287 (1958); M. A. Thorpe and D. J. Thouless, Nucl. Phys. A156, 225 (1970).

[17]J. L. Friar and J. W. Negele, Nucl. Phys. A212, 93 (1973).

[18]L. D. Landau, Zh. Eksp. Teor. Fiz. 30, 1058 (1956) [Sov. Phys.-JETP 3, 920 (1956)]; see also D. Pines and P. Nozières, The Theory of Quantum Liquids (Benjamin, New York, 1966), Vol. I.

[19]S. E. Koonin and A. K. Kerman (unpublished); also S. E. Koonin, Ph.D. thesis, MIT, June, 1975 (unpublished).

[20]H. S. Green, The Molecular Theory of Fluids (North-Holland, Amsterdam, 1952).

[21]C. Y. Wong (unpublished).

[22]G. F. Bertsch, Ann. Phys. (N.Y.) 86, 138 (1974); Nucl. Phys. A249, 253 (1975).

[23]W. E. Keller, Helium-3 and Helium-4 (Plenum, New York, 1969).

[24]C. N. Yang, Rev. Mod. Phys. 34, 694 (1962).

[25]I. M. Khalatinikov, An Introduction to the Theory of Superfluidity (Benjamin, New York, 1965).

[26]N. N. Bogoliubov, in Studies in Statistical Mechanics (see Ref. 10); Lectures in Quantum Statistics (Gordon and Breach, New York, 1970), Vol. II.

[27]See, e.g., L. P. Kadanoff and G. Baym, Quantum Statistical Mechanics (Benjamin, New York, 1976), p. 54.

[28]P. Nozières, The Theory of Interacting Fermi Systems (Benjamin, New York, 1964).

[29]J. Ross and K. Kirkwood, J. Chem. Phys. 22, 1094 (1954).

[30]P. Bonche, S. E. Koonin, and J. W. Negele, Phys. Rev. C 13, 1226 (1976).

[31]S. E. Koonin, talk presented at the International Workshop IV on Gross Properties of Nuclei and Nuclear Excitations, Hirschegg, Austria, 1976 (unpublished).

[32]R. Y. Cusson, R. K. Smith, and J. A. Maruhn, Phys. Rev. Lett. 36, 1166 (1976).

[33]H. Kümmel and J. G. Zabolitzky, Phys. Rev. C 7, 547 (1973).