PHYSICAL REVIEW C VOLUME 15, NUMBER 1 JANUARY 1977

Cluster spectroscopic factors for the p-shell nuclei

Yu. F. Smirnov and Yu. M. Tchuvil'sky
Institute of Nuclear Physics, Moscow State University, Moscow 117234, Union of Soviet Socialist Republics
(Received 29 April 1976)

A convenient method of calculation has been developed to determine the cluster spectroscopic factors for the ejection of excited and complex fragments (^{4}He, ^{6}Li, ^{7}Li, ^{8}Be, etc.) from the low-lying levels of the p-shell nuclei followed by transition both to the states of the lowest shell configuration of the daughter nucleus and to its hole states. The method can be used to analyze the cluster knock-out, transfer reactions induced by heavy ions, etc.

[NUCLEAR STRUCTURE p-shell nuclei; calculated cluster spectroscopic factors. Harmonic oscillator shell model.]

I. INTRODUCTION

In order to analyze various nuclear reactions involving the nucleon clusters, one should know the cluster spectroscopic factors (CSF) for the nuclear states. A shell model for calculating these quantities was proposed in Refs. 1 and most detailed description of the method was given in a monograph.[2] Subsequently, this approach was generalized to include the reactions induced by heavy ions.[3] Calculations of the d, t, and α spectroscopic factors in the intermediate coupling scheme for the p-shell nuclei have been made by many authors,[4] the results of Kurath being the most complete so far. In Ref. 5 the above method has been modified and applied to the sd-shell nuclei. Within the nucleon cluster model, the theory of CSF has been developed in Ref. 6. Almost all of these papers, however, dealt with the transfer or emission of clusters d, t, and α in their ground states.

In connection with the study of quasielastic cluster knock-out reactions at high energies,[7] there arose an urgent need for the CSF calculations for excited clusters, as well as for transitions to the highly excited hole states of the final nucleus. These quantities are required also for the theory of γ quanta and pion capture by nuclei. No less interesting is the problem of calculating CSF for complex clusters with a mass greater than 4 (both excited and unexcited). This problem is of immediate importance for the theory of reactions induced by heavy ions.[8] It should be mentioned that the formalism developed in Refs. 1–5 and based on the usual shell model is unsuitable for solving such problems, since in considering the decay of a nucleus into highly excited fragments it is of fundamental importance to eliminate from the wave functions for these fragments the "spurious" states associated with nonphysical oscillations of their centers of mass. Thus, there arises a problem of deriving such expressions for CSF characterizing the nuclear decay into highly excited fragments which, on the one hand, would correctly describe the motion of the centers of masses and, on the other hand, would retain their former mathematical structure, i.e., would include usual fractional parentage coefficients (FPC) tabulated in Refs. 12 and the various symbols from Racah's algebra. In the present work the problem stated above has been solved for all cases when the initial nuclear state with $A \leq 16$ belongs to the s^4p^n configuration. The consideration is within the framework of the translationally invariant shell model (TISM)[2,11] in which spurious states are absent. In the course of the consideration, the relation between the FPC in the TISM and the corresponding coefficients in the usual shell model is obtained.

Some aspects of the problem have already been dealt with in the literature. Thus, in Ref. 3 the transfer of a nucleon grouping from one nucleus to another has been considered in all possible states of this grouping. In Ref. 9 the CSF for the ^{4}Li transfer (the isobaric analog of excited cluster ^{4}He) have been calculated; in Ref. 10 the effective numbers for the complex clusters ^{6,7}Li and ^{8}Be in the p-shell nuclei have first been obtained. The problem of CSF for transitions to "hole" states of residual nuclei has been discussed recently in Ref. 13. These papers, however, discussed, as a rule, the sum of CSF for a set of excited states of a cluster or a final nucleus. The problem of CSF calculation for the transition resulting in the formation of two highly excited fragments in separate fixed states has not yet been solved.

II. CLUSTER SPECTROSCOPIC FACTORS IN TRANSLATIONALLY INVARIANT SHELL MODEL

In the TISM, the state of a nucleus consisting of A nucleons is described by the antisymmetric os-

cillator wave function

$$\Psi_A \equiv |AN[f](\lambda\mu)\alpha LSTJMM_T\rangle, \tag{1}$$

where N is the total number of oscillator quanta; $[f]$ is the Young scheme determining the permutational symmetry of the orbital part of the wave function; $(\lambda\mu)$ is the Elliott symbol defining the SU(3) symmetry of the state; L, S, J, and T are the orbital, spin, total angular, and isospin momenta, respectively; M and M_T are the projections of the total angular momentum J and of the isospin T; the α are the quantum numbers required for a complete labeling of the nuclear states. The spectroscopic amplitude (SA) for the transition from the state (1) to the state

$$\Psi_1 \equiv |A-bN_1[f_1](\lambda_1\mu_1)\alpha_1 L_1S_1T_1J_1M_1M_{T_1}\rangle$$

of the daughter nucleus $A-b$ with the emission of a cluster "b" in the state

$$\Psi_b \equiv |bN_2[f_2](\lambda_2\mu_2)\alpha_2 L_2S_2T_2J_2M_2M_{T_2}\rangle$$

which carries off the orbital angular momentum Λ has the form[2,11]

$$S^b_{A1} = \binom{A}{b}^{1/2}\langle\Psi_A|\Psi_1, n\Lambda, \Psi_b\rangle$$
$$= \binom{A}{b}^{1/2}\sum_{\mathfrak{L}} U(\Lambda L_2 J_0 S_2; \mathfrak{L}J_0)\begin{Bmatrix} L_1 & S_1 & J_1 \\ \mathfrak{L} & S_2 & J_0 \\ L & S & J\end{Bmatrix}[(2J_1+1)(2J_0+1)(2L+1)(2S+1)]^{1/2}$$
$$\times\langle AN[f](\lambda\mu)\alpha LST|A\text{-}bN_1[f_1](\lambda_1\mu_1)\alpha_1L_1S_1T_1;\ n\Lambda, bN_2[f_2](\lambda_2\mu_2)\alpha_2L_2S_2T_2\{\mathfrak{L}\}\rangle. \tag{2}$$

Here we assumed for simplicity that the oscillator parameters $\hbar\omega$ for the nuclei A, $A-b$, and the cluster b are the same. Therefore, the number of quanta for a degree of freedom of the relative motion of the nucleus $A-b$ and the cluster b is $n = N - N_1 - N_2$. In the expression (2) we have used the ordinary coefficients of Racah and $9j$ symbols, as well as the standard FPC in the TISM described in Refs. 2 and 11. The spectroscopic factor is the squared amplitude (2).

Thus, owing to the use of the relation (2) the problem of calculating CSF has been reduced to the calculation FPC in the TISM. In what follows, we confine our consideration to the nuclear states of the p shell ($A \leq 16$) belonging to the ground configuration s^4p^{A-4}. This implies that in the TISM we shall be considering the states of the initial nucleus (1) corresponding to the minimum number of quanta $N = N^{\min}$ permitted by the Pauli principle, which for nuclei with $A \leq 16$ is $N^{\min} = A-4$. The further consideration aims at expressing the FPC of TISM contained in (2) through the usual FPC. We note that the wave functions in the TISM and in the traditional shell model are related, according to the Elliott-Skyrme theorem,[14] as follows:

$$|s^4p^{A-4}(\lambda\mu)[f]LST\rangle = \Psi_{000}(\vec{R}_A)\,|AN^{\min}[f](\lambda\mu)LST\rangle, \tag{3}$$

where $\Psi_{000}(\vec{R}_A)$ is the wave function for zero oscillations of the center of mass of the nucleus A. For those cases when the state of the final nucleus $A-b$ also belongs to the lowest configuration s^4p^{A-b-4} (or $N_1 = N_1^{\min} = A-b-4$), from this relationship there follows the expression for the FPC in TISM[11]:

$$\langle AN^{\min}[f](\lambda\mu)LST|A-bN_1^{\min}[f_1](\lambda_1\mu_1)L_1S_1T_1;n\Lambda,bN_2[f_2](\lambda_2\mu_2)L_2S_2T_2\{\mathfrak{L}\}\rangle$$
$$= (-1)^n\left(\frac{A}{A-b}\right)^{n/2}\binom{A-4}{b}^{1/2}\binom{A}{b}^{-1/2}$$
$$\times\langle p^{A-4}(\lambda\mu)LST|\,p^{A-b-4}(\lambda_1\mu_1)L_1S_1T_1, p^b(\lambda_2\mu_2)\mathfrak{L}S_2T_2\rangle\langle p^b(\lambda_2\mu_2)\mathfrak{L}S_2T_2\,|n\Lambda, bN_2[f_2](\lambda_2\mu_2)L_2S_2T_2\rangle. \tag{4}$$

We call the last factor on the right-hand side of Eq. (4) the "cluster coefficient" (CC) for the configuration p^b; in general, the overlap integral of the shell-model wave function Ψ_L^{shell} and the wave function of TISM $|bN_2[f_2](\lambda_2\mu_2)L_2S_2T_2\rangle$ times $\psi_{n\Lambda M}(\vec{R}_b) \equiv |n\Lambda M(\vec{R}_b)\rangle$ will be called CC for the ψ_L^{shell} configuration:

$$\langle\psi_L^{\text{shell}}|n\Lambda(\vec{R}_b), bN_2[f_2](\lambda_2\mu_2)L_2S_2T_2\rangle.$$

Here we mean that the orbital angular momentum Λ of the center of mass of the cluster "b" and internal orbital angular momentum L_2 of this cluster are vector-coupled into the total orbital angular momentum L of the above shell-model configuration

$$\vec{\Lambda}+\vec{L}_2=\vec{L}. \tag{5}$$

The calculation of CC will be considered at length below and for the present we note that formula (4) is basic for all the methods of calculating CSF developed and used in Refs. 1–5. Its validity is however, limited by the transitions $N^{\min}\to N_1^{\min}$. Meanwhile, in knocking out clusters from the p-shell nuclei not only the states of the s^4p^{n-b} configuration may be excited but also the states with holes in the s shell, to give just one example. The intensity of excitation of such states in the processes of quasielastic cluster knockout is high enough. Before we extend formula (4) to such transitions, we should look at the structure of the hole states in the p-shell nuclei.

III. NUCLEAR *sp* STATES IN THE TISM

In the usual shell model the hole states of the p-shell nuclei are the states such as

$$|s^\nu p^{A-\nu}[f](\lambda\mu)LS\,T\rangle, \quad \nu<4. \tag{6}$$

Because they contain only s and p nucleons, they will be referred to as the sp states. In the general case, the sp state (6) contains not only the components with zero oscillations of the center of mass, but also the spurious excitations

$$|s^\nu p^{A-\nu}[f](\lambda\mu)LST\rangle = \Omega_A\Psi_{000}(\vec{R}_A)\,|AN=A-\nu[f](\lambda\mu)LST\rangle + \text{spurious states}. \tag{7}$$

The state $|AN=A-\nu[f](\lambda\mu)LST\rangle$ contained in the first term of the expression (7) for the hole state (6) will be referred to as the sp state in the TISM. Let us introduce the projection operator P_{000} having the properties

$$P_{000}\Psi_{NLM}(\vec{R}_A)=\delta^{000}_{NLM}\Psi_{000}(\vec{R}_A), \quad (P_{000})^2=P_{000}, \quad P^+_{000}=P_{000}, \tag{8}$$

i.e., the operator cuts off from an arbitrary wave function a component corresponding to zero oscillations of the center of masses. Acting with this projector upon the both sides of the expression (7), we obtain

$$\Psi_{000}(\vec{R}_A)\,|AN=A-\nu[f](\lambda\mu)LST\rangle = \frac{1}{\Omega_A}P_{000}\,|s^\nu p^{A-\nu}[f](\lambda\mu)LST\rangle, \tag{9}$$

where

$$\Omega_A^{\,2}=\langle s^\nu p^{A-\nu}[f](\lambda\mu)LST\,|P_{000}|\,s^\nu p^{A-\nu}[f](\lambda\mu)LST\rangle. \tag{10}$$

By Ref. 15

$$P_{000}=:\exp(-\underline{A}^\dagger\underline{A}):. \tag{11}$$

Here $\underline{A}^\dagger=(1/\sqrt{A})\sum_{i=1}^{A}\underline{a}_i^\dagger$, where $\underline{a}_i^\dagger$ and $\underline{a}_i$ are the usual operators for the creation and annihilation of an oscillator quantum in the single-particle variable r_i for the ith nucleon; the symbol of the normal product ": :" means that in the expansion of the exponent (11) into a power series all the operators $\underline{A}^\dagger$ should be placed on the left, while the operators $\underline{A}$ are on the right. Substituting (11) in (10), we can calculate the normalization Ω_A by using the usual shell-model methods. This will be done in Sec. IV. Now it is important that using expressions (9–11) we may write explicitly the structure of the hole state in the TISM. We expand the function (9) in the shell states with the total number of quanta $N=A-\nu$:

$$\Psi_{000}(\vec{R}_A)\,|AN=A-\nu[f](\lambda\mu)LST\rangle=\Omega_A\,|s^\nu p^{A-\nu}[f](\lambda\mu)LST\rangle+C_1\,|s^{\nu+1}p^{A-\nu-2}(2s-2d):[f](\lambda\mu)LST\rangle+\cdots, \tag{12}$$

where dots stand for the terms containing two or more of the $(2s-2d)$ nucleons or nucleons from higher-lying shells. The coefficients of the expansion C_1 and the other ones are calculated as Eq. (10). For instance, for C_1 we have

$$C_1=\frac{1}{\Omega_A}\langle s^{\nu+1}p^{A-\nu-2}(2s-2d):[f](\lambda\mu)LST\,|P_{000}|\,s^\nu p^{A-\nu}[f](\lambda\mu)LST\rangle. \tag{13}$$

Having calculated the coefficients $\Omega_A, C_1, \ldots$, we obtain the explicit form of the sp state in the TISM in the shell-model representation (12). It should be mentioned that the sp states, with $N=A-\nu$ of course, do not exhaust all the states in the TISM for the nucleons A with the number of quanta $N=A-\nu$. These are only a part of the complete set of the states. It is important, however, that the remaining states (we call them C states) do not contain the component $|s^\nu p^{A-\nu}\rangle$ ($\Omega_A=0$):

$$\Psi_{000}(\vec{R}_A)\,|AN=A-\nu[f](\lambda\mu)LST\rangle_C = C_1\,|s^{\nu+1}p^{A-\nu-2}(2s-2d):[f](\lambda\mu)LST\rangle+\cdots \tag{14}$$

in their shell-model representation (12). Because of this, the C states are absent from the fractional parentage development of states with $N=N^{\min}$ of the p-shell nuclei.

Indeed, let us separate "b" nucleons in the shell-model wave function (3) (partly from the s shell and partly from the p shell). Then the remaining nucleons correspond to the configuration $s^{\nu_1}p^{n_1}(\nu_1+n_1=A-b)$. According to (7), this fact, expressed in terms of TISM, means that the fractional parentage development of the wave function in TISM for $A\leq 16$,

$$|AN^{\min}[f](\lambda\mu)LST\rangle = \sum \langle AN^{\min}[f](\lambda\mu)LST\,|A-bN_1[f_1](\lambda_1\mu_1)L_1S_1T_1;n\Lambda,bN_2[f_2](\lambda_2\mu_2)L_2S_2T_2\{\mathfrak{L}\}\rangle$$
$$\times\,|A-bN_1[f_1](\lambda_1\mu_1)L_1S_1T_1;\,n\Lambda,bN_2[f_2](\lambda_2\mu_2)L_2S_2T_2\{\mathfrak{L}\}:LST\rangle, \tag{15}$$

contains only the sp states of the $A-b$ nucleons $|A-bN_1[f_1](\lambda_1\mu_1)L_1S_1T_1\rangle$. Taking account of the fact that the states of the nucleon groupings $A-b$ and "b" enter in the development (15) symmetrically, we conclude that (15) contains also only the sp states of the cluster "b". It is on this account that we are especially interested in the sp states. We note that in view of (6) and (12) the construction of the sp states is given unambiguously by the quantum numbers L,S,T; $[f]$ is the complete scheme of Young; $(\lambda\mu)$ is the symbol for the SU(3) symmetry which at the same time defines the Young scheme for p nucleons $[f_p]=[f_{p_1},f_{p_2},f_{p_3}]$, $(\lambda=f_{p_1}-f_{p_2},\ \mu=f_{p_2}-f_{p_3},\ f_{p_1}+f_{p_2}+f_{p_3}=A-\nu)$.

The sp states with different values of these quantum numbers are evidently orthogonal to one another. The number of the sp states is either equal to the total number of the shell-model states of the configuration $s^{\nu}p^{A-\nu}$ or less than this number. This occurs when some of the states (6) are entirely spurious. We do not pay special attention to them, for the calculation by formula (10) will automatically yield for them $\Omega_A=0$. Finally, let the phases of all sp states be chosen so that $\Omega_A\geq 0$.

IV. FRACTIONAL PARENTAGE COEFFICIENTS FOR TRANSITION TO HOLE STATES OF FINAL NUCLEUS

We now extend formula (4) to the case when the final nucleus $A-b$ is formed in the sp state with $N_1\geq N_1^{\min}$. To do so, consider the subsidiary integral

$$I\equiv\langle\Psi_{000}(\vec{R}_A),AN^{\min}[f](\lambda\mu)LST\,|\,\Psi_{000}(\vec{R}_{A-b}),A-bN_1[f_1](\lambda_1\mu_1)L_1S_1T_1;\Psi_{n\Lambda}(\vec{R}_b),bN_2[f_2](\lambda_2\mu_2)L_2S_2T_2\{\mathfrak{L}\}:LST\rangle. \tag{16}$$

If we take R_A and the Jacobi coordinates for the nucleus A as the variables of integration, we have[2,11]

$$I=(-1)^n\left(\frac{A}{A-b}\right)^{-n/2}\langle AN^{\min}[f](\lambda\mu)LST\,|A-bN_1[f_1](\lambda_1\mu_1)L_1S_1T_1;\,n\Lambda,bN_2[f_2](\lambda_2\mu_2)L_2S_2T_2\{\mathfrak{L}\}\rangle. \tag{16'}$$

On the other hand, if we substitute Eq. (3) for the left-hand side of the integral I and substitute the expansion (12) for the nucleus $A-b$ in the right-hand side, then, integrating over the single-particle variables $\vec{r}_1,\vec{r}_2,\ldots,\vec{r}_A$, we find that

$$I=\Omega_{A-b}\sum_{(\lambda_2'\mu_2')}\langle s^4p^{A-4}[f](\lambda\mu)LST\,|s^{\nu_1}p^{N_1}[f_1](\lambda_1\mu_1)L_1S_1T_1;s^{\nu_2}p^{N_2+n}[f_2](\lambda_2'\mu_2')\mathfrak{L}S_2T_2\rangle$$
$$\times\langle s^{\nu_2}p^{N_2+n}[f_2](\lambda_2'\mu_2')\mathfrak{L}S_2T_2\,|n\Lambda,bN_2[f_2](\lambda_2\mu_2)L_2S_2T_2\rangle, \tag{17}$$

where $\nu_1+N_1=A-b$, $\nu_2+N_2+n=b$. The right-hand side of (17) contains a two-shell FPC of the type $\langle s^4p^{A-4}\,|s^{\nu_1}p^{N_1};s^{\nu_2}p^{N_2+n}\rangle$, and CC for the configuration is $s^{\nu_2}p^{N_2+n}$. The second and remaining terms in the development (12) do not contribute to the integral I, because the configuration in the left-hand side of the integral does not contain nucleons in the $2s$-$2d$ and higher shells. Comparing (16') and (17), we obtain the formula relating FPC in TISM to the usual FPC:

$$\langle AN^{\min}[f](\lambda\mu)LST\,|\,A-bN_1[f_1](\lambda_1\mu_1)L_1S_1T_1;nA,bN_2[f_2](\lambda_2\mu_2)L_2S_2T_2\{\mathfrak{L}\}\rangle$$
$$=(-1)^n\left(\frac{A}{A-b}\right)^{n/2}\Omega_{A-b}\sum_{(\lambda_2'\mu_2')}\langle s^4p^{A-4}[f](\lambda\mu)LST\,|\,s^{\nu_1}p^{N_1}[f_1](\lambda_1\mu_1)L_1S_1T_1;s^{\nu_2}p^{N_2+n}$$
$$\times[f_2](\lambda_2'\mu_2')\mathfrak{L}S_2T_2\rangle\langle s^{\nu_2}p^{N_2+n}[f_2](\lambda_2\mu_2)\mathfrak{L}S_2T_2\,|\,nA,bN_2[f_2](\lambda_2\mu_2)L_2S_2T_2\rangle. \quad (18)$$

The expression (4) is a particular case of this relationship. The two-shell FPC in (18) can be factorized into the orbital and spin-isospin parts:

$$\langle s^4p^{A-4}[f](\lambda\mu)LST\,|\,s^{\nu_1}p^{N_1}[f_1]L_1S_1T_1;s^{\nu_2}p^{N_2+n}[f_2](\lambda_2'\mu_2')L_2S_2T_2\rangle$$
$$=\left(\frac{n_f}{n_{f_1}n_{f_2}}\right)^{1/2}\sum_{\epsilon}\langle s^4p^{A-4}[f](\lambda\mu)L\,|\,s^{\nu_1}p^{N_1}[f_1](\lambda_1\mu_1)L_1;s^{\nu_2}p^{N_2+n}[f](\lambda_2'\mu_2')\mathfrak{L}\rangle_\epsilon$$
$$\times\langle(st)^A[\tilde{f}]ST\,|\,(st)^{\nu_1+N_1}[\tilde{f}_1]S_1T_1;(st)^{\nu_2+N_2+n}[\tilde{f}_2]S_2T_2\rangle_\epsilon. \quad (19)$$

Here n_f is the dimension of the irreducible representation $[f]=[f_1,f_2,\ldots,f_m]$ for a permutation group of A particles

$$n_f=\frac{A!\mathfrak{D}(f_1f_2\cdots f_m)}{h_1!h_2!\cdots h_m!},\quad h_i=f_i+m-i,\quad \mathfrak{D}(f_1f_2\cdots f_m)=\prod_{i<j}^{m}(f_i-f_j+j-i) \quad (20)$$

(see, for example, Ref. 16). The sum over ϵ occurs when $[f]$ is contained more than once in the outer product of Young's schemes $[f_1]\times[f_2]$. In the case of the p-shell such combinations of Young schemes are, for example, $[f_1]=[f_2]=[21]$, $[f]=[321]$. The Young scheme $[\tilde{f}]$ in the spin-isospin FPC in (19) is derived from $[f]$ by substituting columns for rows. The orbital two-shell FPC in (19) may be expressed through the orbital for the p shell if we use the formula proved in Ref. 2:

$$\langle s^4p^{A-4}[f](\lambda\mu)L\,|\,s^{\nu_1}p^{N_1}[f_1](\lambda_1\mu_1)L_1,s^{\nu_2}p^{N_2+n}[f_2](\lambda_2'\mu_2')\mathfrak{L}\rangle_\epsilon$$
$$=\binom{4}{\nu_1}^{1/2}\binom{A-4}{N_1}^{1/2}\binom{A}{b}^{-1/2}(-1)^{\nu_2N_1}\left(\frac{n_{f_{p_1}}n_{f_{p2}}n_f}{n_{f_1}n_{f_2}n_{f_p}}\right)^{1/2}$$
$$\times\langle p^{A-4}(\lambda\mu)L\,|\,p^{N_1}(\lambda_1\mu_1)L_1,\,p^{N_2+n}(\lambda_2'\mu_2')\mathfrak{L}\rangle_\epsilon\begin{pmatrix}[\tilde{\nu}_1] & [\tilde{\nu}_2] & [\tilde{4}]\\ [\tilde{f}_{p_1}] & [\tilde{f}_{p_2}] & [\tilde{f}_p]\\ [\tilde{f}_1] & [\tilde{f}_2] & [\tilde{f}]\end{pmatrix}_\epsilon. \quad (21)$$

The last factor is the $9j$ symbol for the SU(4) group.[17] The relation between $(\lambda\mu),(\lambda_1\mu_1),(\lambda_2'\mu_2')$ and $[f_p],[f_{p_1}],[f_{p_2}]$ has been described above. The index ϵ labels the degenerated Young's schemes $[f]$ ($[f_p]$) in the outer product $[f_1]\times[f_2]$ ($[f_{p_1}]\times[f_{p_2}]$). We note that CC is independent of the quantum numbers S_2, and T_2, so they may be omitted from the notation of this coefficient. Also, let us isolate the SU(3) scalar part of CC:

$$\langle s^{\nu_2}p^{N_2+n}[f_2](\lambda_2'\mu_2')\mathfrak{L}\,|\,nA(\vec{R}_b),bN_2[f_2](\lambda_2\mu_2)L_2\rangle$$
$$=\langle(n0)A\times(\lambda_2\mu_2)L_2\,|\,(\lambda_2'\mu_2')\mathfrak{L}\rangle\langle s^{\nu_2}p^{N_2+n}(\lambda_2'\mu_2')[f_2]\,|\,(n0),bN_2[f_2](\lambda_2\mu_2)\rangle. \quad (22)$$

The first factor of this expression is the Clebsch-Gordan coefficient for the SU(3) group, while the second factor is the scalar part of CC. Now it remains only to consider the computation of the coefficients Ω and CC, and the problem of finding the complete set of FPC in the TISM for the low-lying states of the p-shell nuclei will be solved.

V. CLUSTER COEFFICIENTS AND COEFFICIENTS Ω

In the expression (22) for CC we represent the wave function for the center of masses of a cluster in the form[18]:

$$\Psi_{nAM}(\vec{R}_b)=Q_{nA}(\underline{A}^\dagger\underline{A}^\dagger)^{(n-A/2)}\mathcal{Y}_{AM}(\underline{A}^\dagger)\Psi_{000}(\vec{R}_b), \quad (23)$$

$$Q_{n\Lambda} = \sqrt{4\pi}(-1)^{(n-\Lambda)/2}[(n-\Lambda)!!\,(n+\Lambda+1)!!\,]^{-1/2},$$

$$\underline{A}^{\dagger} = \frac{1}{\sqrt{b}} \sum_{i=1}^{b} \underline{a}_i^{\dagger}. \tag{24}$$

Here $\mathcal{Y}_{\Lambda M}(\underline{A}^{\dagger})$ is constructed from the operators $A_x^{\dagger}$, $A_y^{\dagger}$, and $A_z^{\dagger}$ just as the function $r^{\Lambda}Y_{\Lambda M}(\theta,\varphi)$ is constructed from x, y, and z. Making use of the development (12) for the cluster "b", we write the CC of Eq. (22) as

$$\Omega_b \langle s^{\nu_2} p^{N_2+n}[f_2](\lambda_2'\mu_2')\mathcal{L}\,|\,Q_{n\Lambda}(\underline{A}^{\dagger}\underline{A}^{\dagger})^{(n-\Lambda)/2}\mathcal{Y}_{\Lambda M}(\underline{A}^{\dagger}), s^{\nu_2+n}p^{N_2}[f_2](\lambda_2\mu_2)L_2\rangle. \tag{25}$$

Again, the second and remaining terms of the development (12) do not contribute to CC, since they generate configurations that are not contained in the left-hand side of the matrix element [Eqs. (22), and (25)]. In Eq. (25) it is implied that the angular momenta $\vec{\Lambda}$ and $\vec{L}_2$ are coupled to angular momentum $\vec{L}$. The operators $\underline{a}_i^{\dagger}$ transfer p nucleons into the $2s$-$2d$ shell, but there are no such nucleons in the left-hand side of (25). Consequently, only those terms will be nonvanishing which correspond to the transition of nucleons from the s shell to the p shell. The final expression for the SU(3) scalar part of CC has the form

$$\langle s^{\nu_2}p^{N_2+n}(\lambda_2'\mu_2')[f_2]\,|\,(n0), bN_2[f_2](\lambda_2\mu_2)\rangle$$

$$= \Omega_b \binom{\nu_2+n}{n}^{1/2} \binom{N_2+n}{n}^{1/2} \left(\frac{n}{b}\right)^{n/2} \left(\frac{n_{f_{p_2}}}{n_{f'_{p_2}}}\right)^{1/2} U([\tilde{\nu}_2][\tilde{n}][\tilde{f}_2][\tilde{f}_{p_2}]; [\nu_2'\tilde{+}n][\tilde{f}'_{p_2}])K_n, \tag{26}$$

where

$$K_n = \langle p^n[n]\Lambda\,|\,n\Lambda, A=n0[n](00)\rangle = (n!/n^n)^{1/2}, \tag{27}$$

$[f_{p_2}]$ and $[f'_{p_2}]$ are the Young orbital schemes for the configurations p^{N_2} and p^{N_2+n}, respectively, whose form is given unambiguously by the values of $(\lambda_2\mu_2)$ and $(\lambda_2'\mu_2')$. The derivation of Eq. (26) is given in Appendix. It will be noted that at $n=0$ CC of Eq. (26) is equal to Ω_b. In the general case, the coefficients Ω may be calculated in the following manner.

Let us expand the exponent (11) in a power series, representing each term of the expansion as

$$:(\underline{A}^{\dagger}\underline{A})^r: = 4\pi(-1)^r \sum_l \frac{1}{(2l+1)}^{1/2} G(r,l)(\underline{A}^{\dagger}\underline{A}^{\dagger})^{(r-l)/2} \times [\mathcal{Y}_l(\underline{A}^{\dagger})\mathcal{Y}_l(\underline{A})]_{00}(\underline{A}\,\underline{A})^{(r-l)/2}, \tag{28}$$

where

$$G(r,l) = (2l+1)r!\,Q_{rl}^{\,2}/4\pi. \tag{29}$$

A similar transformation for the usual scalar product $(\vec{r}_1\cdot\vec{r}_2)^r$ is performed in Ref. 19. However, it is valid for the normalized product of the operators $(\underline{A}^{\dagger}\underline{A})^r$, too. Taking into account the relation (28) and using the closure properties of shell-model wave functions we may write Eq. (10) as follows:

$$\Omega_A^{\,2} = \sum_{r,l,L',(\lambda'\mu')} (-1)^l Q_{rl}^{\,2} \langle s^{\nu}p^{A-\nu}[f](\lambda\mu)L\,|\,(\underline{A}^{\dagger}\underline{A}^{\dagger})^{(r-l)/2}\mathcal{Y}_l(\underline{A}^{\dagger})\,|\,s^{\nu+r}p^{A-\nu-r}[f](\lambda'\mu')L'\rangle^2$$

$$= \sum_{r,[f_p]} (-1)^r \binom{\nu+r}{r}\binom{A-\nu}{r}\left(\frac{r}{A}\right)^r K_r^{\,2} \frac{n_{f'p}}{n_{fp}} U^2([\tilde{\nu}][\tilde{r}][\tilde{f}][\tilde{f}'_p]; [\nu\tilde{+}r][\tilde{f}_p]). \tag{30}$$

Here the antisymmetric states $|s^{\nu+r}p^{A-\nu-r}[f](\lambda'\mu')L'\rangle$ have been used as the intermediate states. Therefore, the sum over r is finite: $\nu+r\leq 4$. In the process, we have come to the matrix elements of type (25), for which formula (26) has been used. As before, $[f_p] = [f_{p_1}, f_{p_2}, f_{p_3}]$ is the Young orbital scheme for the configuration $p^{A-\nu}$, which is defined by the symbol $(\lambda\mu)$: $\lambda = f_{p_1} - f_{p_2}$, $\mu = f_{p_2} - f_{p_3}$, $A-\nu = f_{p_1} + f_{p_2} + f_{p_3}$. Similarly, from the symbol $(\lambda'\mu')$ we determine the Young scheme $[f'_p]$ for the configuration $p^{A-\nu-r}$.

Formulas (18)–(21), (26), (27), and (30) settle the question of calculating the complete set of FPC in the TISM for nuclei with $A \leq 16$ and $N = N^{\min}$, and as a result the corresponding CSF can be found. As an illustration of these formulas, we give an explicit expression for the FPC for the separation of one particle in the state of zero oscillations relative to the remaining $A-1$ nucleons:

$$\langle AN^{\min}[f](\lambda\mu)L\,|\,A-1N^{\min}[f_1](\lambda\mu)L;00\rangle$$
$$=\frac{2}{\sqrt{A}}\left(\frac{n_f}{n_{f_1}}\right)^{1/2}U([\tilde{f}_p][\bar{3}][\tilde{f}]|[1];[\tilde{f}_1][\bar{4}])\left[1-\frac{4(A-4)}{A-1}\frac{n_{f_{p_1}}}{n_{f_p}}U^2([\bar{3}][1][\tilde{f}_1][\tilde{f}_{p_1}];[\bar{4}][\tilde{f}_p])\right]^{1/2}. \quad (31)$$

The Racah coefficients for the unitary groups are calculated by the standard methods.[20] It is helpful to note that the first Racah coefficient in this formula is equal to $(N_{\tilde{f}_1}/4N_{\tilde{f}_p})^{1/2}$ and the second one is equal to $(N_{\tilde{f}_p}/4N_{\tilde{f}_{p_1}})^{1/2}$, except for the phase factor ±1. Here N_ν is the dimension of the irreducible representation $[\nu]=[\nu_1\nu_2\nu_3\nu_4]$ for the group SU(4) (Ref. 16):

$$N_\nu=\frac{\mathfrak{D}(\nu_1\nu_2\nu_3\nu_4)}{\mathfrak{D}(0000)}. \quad (32)$$

The Young schemes $[f_p]$ and $[f_{p_1}]$ in (31) are obtained from $[f]$ and $[f_1]$ by eliminating the upper row consisting of four squares. In these calculations it is important to bear in mind the following properties of the nj symbols for the unitary groups: (1) The value of any nj symbol for the group U(m) is independent of the group rank m, but depends only on the form of the Young schemes it contains. (2) Any nj symbol for the group U(m) coincides with the element of Kaplan's transformation matrix for the permutation group including the same Young schemes,[17] while the squared elements of these matrices remain unchanged when all the Young schemes $[f]$ are replaced by the conjugate schemes $[\tilde{f}]$, where columns have been substituted for rows. (As regards the choice of phase factors, see the work by Kramer.[21])

VI. EXAMPLES

a. Spectroscopic factors for the excited α clusters. In analyzing the cluster knock-out reactions it is important to know the CSF for both the ground state and excited clusters. Table I shows CC of α particles in all allowed sp states and the other quantities required for calculating CSF, as well as numerical values of the latter for the O^{16} nucleus. From the table one can see that, if we take into account the excited cluster states, the number of virtual clusters taking part in the process of knock-out increases from $N_{\text{eff}}=13.7$ to $N_{\text{eff}}=35.0$. (N_{eff} is equal to the sum of CSF for all allowed states of the cluster and final nucleus.) The TISM FPC defining the structure of the α-particle sp states are given in Table II. The fractional parentage development of these states has the form

$$|A=4N_2[4](\lambda\mu)\rangle=\sum_{N_1'N_2'}\langle 4N_2[4](\lambda\mu)\|2N_1';(n0),2N_2':(\lambda'\mu')\rangle\,|2N_2'(\vec{r}_{12});(n0)(\vec{R}_{12}-\vec{R}_{34}),2N_2'(\vec{r}_{34})(\lambda'\mu'):(\lambda\mu)\rangle. \quad (33)$$

Let us illustrate the technique of calculating these FPC by taking the p^4 configuration as an example. The sp state of an α cluster with four quanta of the internal excitation has, according to (9), the form

$$\Psi_{000}(\vec{R}_\alpha)\,|A=4N=4[4](\lambda\mu)=(40)L\rangle=\frac{1}{\Omega_\alpha}P_{000}\,|p^4[4]L\rangle. \quad (34)$$

Here only the orbital parts of the corresponding wave functions are indicated. Let us perform a scalar multiplication on both sides of this equality by the function

$$\Psi_{000}(\vec{R}_\alpha)\,|N_1L_1(\vec{r}_{12});n\Lambda(\vec{R}_{12}-\vec{R}_{34}),N_2L_2(\vec{r}_{34})\{\mathfrak{L}\}:L\rangle, \quad (35)$$

and let us apply the projection operator P_{000} of (34) to the function $\Psi_{000}(\vec{R}_\alpha)$ in (35). Then we have

$$\langle A=4N=4[4](\lambda\mu)L\,|\,2N_1L_1;n\Lambda,2N_2L_2\{\mathfrak{L}\}\rangle$$
$$=\frac{1}{\Omega_\alpha}\sum_{\substack{L_1'L_2'N'L'\\N''L''L_3}}\langle p^4[4]L\,|\,p^2[2]L_1',p^2[2]L_2'\rangle\langle 1\,1,1\,1:L_2'\,|\,N''L'',N_2L_2:L_2'\rangle\langle 1\,1,1\,1:L_1'\,|\,N'L',N_1L_1:L_1'\rangle(-1)^{L_1'-L_1-L'}$$
$$\times\langle N'L',N''L'':\Lambda\,|\,00,n\Lambda:\Lambda\rangle U(L_1'L''LL_2;L_3L_2')U(L_1L'L_3L'';L_1'\Lambda)U(L_1\Lambda LL_2;L_3\mathfrak{L}). \quad (36)$$

The right-hand side of this equation contains the generalized Talmi coefficients.[19] Table II shows the SU(3) scalar parts of FPC related to the FPC in (36) as follows

$$\langle A=4N[4](\lambda\mu)L\,|\,2N_1L_1;n\Lambda,2N_2L_2\{\mathfrak{L}\}\rangle$$
$$=\sum_{(\lambda'\mu')}\langle A=4N[4](\lambda\mu)\|2N_1;(n0),2N_2:(\lambda'\mu')\rangle\langle(n0)\Lambda\times(N_20)L_2\,|\,(\lambda'\mu')\mathfrak{L}\rangle\langle(N_10)L_1\times(\lambda'\mu')\mathfrak{L}\,|\,(\lambda\mu)L\rangle. \quad (37)$$

TABLE I. Cluster spectroscopic factors for transitions $^{16}O \rightarrow {}^{12}C + \alpha^*$.

N_1	N_2	$\Omega_{^{12}C}$	CC	S_α^2
8	0	1	$\frac{1}{4}\sqrt{\frac{3}{2}}$	$\frac{40}{9}$
	2		$\frac{1}{4}\sqrt{3}$	5
	3		$\frac{1}{2}$	5
	4		$\frac{1}{4}\sqrt{\frac{15}{2}}$	$\frac{225}{32}$
9	0	$1/\sqrt{2}$	$\frac{1}{4}\sqrt{6}$	$\frac{40}{9}$
	2		$\frac{1}{4}\sqrt{6}$	$\frac{5}{2}$
	3		$\frac{1}{2}$	$\frac{5}{4}$
10	0	$\frac{1}{2}\sqrt{\frac{3}{2}}$	$\frac{1}{2}\sqrt{3}$	3
	2		$\frac{1}{2}$	$\frac{9}{16}$
11	0	$\frac{5}{12}\sqrt{2}$	1	$\frac{25}{18}$
12	0	$\frac{1}{12}\sqrt{\frac{109}{2}}$	1	$\frac{109}{288}$

b. Cluster coefficients for the unexcited complex clusters ^{6}Li, ^{7}Li, ^{8}Be in the p^b configuration. Consider CC in (4) for the ground states of clusters, belonging to the p shell such as the nuclei ^{6}Li, ^{7}Li, and ^{8}Be, which are of interest for the theory of transfer reactions induced by heavy ions. Measurements of CSF for the ejection of clusters ^{5}Li and ^{6}Li from the nuclei ^{11}B, ^{12}C, ^{16}O, and ^{19}F in the reactions $(p, {}^6\text{Li}), (p, {}^7\text{Be})$ have recently been made by Kato *et al.*[22].

In analogy with Eq. (25) we may write the following expression of CC for the shell-model wave function $\Psi^{\text{shell}}_{bNLST}$

$$\langle \Psi^{\text{shell}}_{bNLST} | Q_{n\Lambda}(\underline{A^+}\underline{A^+})^{(n-\Lambda)/2}\, y_\Lambda(\underline{A^+}), s^4p^{b-4}(\lambda\mu)L_1ST\rangle .$$

Here N is the total number of quanta corresponding to $\Psi^{\text{shell}}_{bNLST}$ and L, S, and T are the orbital angular momentum, spin, and isospin for this state, $n = N - b - 4$, $|s^4p^{b-4}(\lambda\mu)L_1ST\rangle$—the shell-model function for the ground state of the cluster *"b"*. Making use of the standard shell-model formalism, we can readily calculate the matrix elements (25) from combinations of the operator A^+ for the wave function Ψ^{shell}. Especially simple is the case when the cluster is ejected from the p shell, i.e., $\Psi^{\text{shell}} = |p^b[f]LST\rangle$. Then we may resort to formula (26). The results of relevant calculations are presented in Table III.

TABLE II. SU(3) invariant part of FPC in TISM for α cluster $\langle 4N_2[4](\lambda\mu)|2N_1'; (n0)2N_2':(\lambda'\mu')\rangle$.

N_1	n	N_2	$N=0$	$N=2$	$N=3$	$N=4$
0	0	0	1			
2	0	0		$1/\sqrt{3}$		
0	2	0		$1/\sqrt{3}$		
0	0	2		$1/\sqrt{3}$		
2	1	0			$1/\sqrt{2}$	
0	1	2			$1/\sqrt{2}$	
2	0	2				$\sqrt{\frac{8}{15}}$
2	2	0				$-\sqrt{\frac{2}{15}}$
0	2	2				$-\sqrt{\frac{2}{15}}$
0	4	0				$1/\sqrt{5}$

The above examples show that the formalism developed in the present work is an effective tool for calculating the structural factors for various nuclear reactions in which nucleon clusters take part. Although most of the specific formulas given above refer to the p-shell nuclei, the approach of this work may be extended to the heavier nuclei as well.

The authors are thankful to Professor V. G. Neudatchin for stimulating discussions.

APPENDIX

The derivation of formula (26). We calculate the matrix element (25) using the complete antisymmetric wave function. Let us represent the antisymmetric two-shell wave function in the right-hand side of the matrix element (25) as

TABLE III. Cluster coefficients for a nonexcited complex cluster for the p^b configuration.

Nucleus	$[f_2]$	CC
^{6}Li	[42]	$1/\sqrt{10}$
^{7}Li	[43]	$2\sqrt{15}/49$
^{8}Be	[44]	$\sqrt{15}/16\sqrt{2}$

$$|s^{\nu_2+n}p^{N_2}(\lambda_2\mu_2)L_2:[f_2]L_2ST\rangle$$
$$= \sum_{S_1T_1S_2T_2} \langle [\nu_2\tilde{+}n]S_1T_1, [\tilde{f}_{p_2}]S_2T_2 | [\tilde{f}_2]ST\rangle \binom{b}{N_2}^{-1/2} \hat{A} |s^{\nu_2+n}[\nu_2+n]0S_1T_1; p^{N_2}[f_{p_2}](\lambda_2\mu_2)L_2S_2T_2:LST\rangle , \quad \text{(A1)}$$

where $\hat{A}$ is the antisymmetrization operator, the functions on the right of it being no longer antisymmetrized; s^{ν_2+n} depends on the arguments $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_{\nu_2+n}$, whereas p^{N_2} depends on the variables of the remaining particles. The first factor in the right-hand side of (A1) represents the Clebsch-Gordan coefficient for the group SU(4), which coincides with the spin-isospin fractional parentage coefficient.[17] Let us apply the operator $\hat{A}$ to the function standing on the left of it, taking into account that the operator in the matrix element is symmetric over all particles. We begin the calculation of Eq. (25) by integrating over the variables $\vec{r}_{\nu_2+n+1} \ldots \vec{r}_{\nu_2+N_2+n}$, bearing in mind that those terms in the operator $\hat{\hat{R}} = (\underline{A}^+\underline{A}^+)^{(n-\Lambda)/2}\mathcal{Y}_\Lambda(\underline{A}^+)Q_{n\Lambda}$ which change the state of p nucleons do not contribute to Eq. (25). Therefore, the state of the grouping of nucleons p^{N_2} in (A1) is not changed by the operator and it is easy to integrate over their variables. As a result Eq. (25) takes the form

$$\Omega_b \binom{N_2+n}{n}^{1/2} \sum_{S_1T_1S_2T_2} \langle [\nu_2\tilde{+}n]S_1T_1, [\tilde{f}_{p_2}]S_2T_2 | [\tilde{f}_2]ST\rangle$$
$$\times \langle s^{\nu_2}p^{N_2+n}(\lambda_2'\mu_2'):[f_2]\mathfrak{L}ST | s^{\nu_2}p^n[n]:[\nu_2+n]\Lambda S_1'T_1'; p^{N_2}(\lambda_2\mu_2)L_2S_2T_2\rangle$$
$$\times \langle s^{\nu_2}p^n[n]:[\nu_2+n]\Lambda S_1T_1 | \hat{R} | s^{\nu_2+n}[\nu_2+n]0S_1T_1\rangle . \quad \text{(A2)}$$

In the left-hand side of the last matrix element we again resort to the representation in terms of the nonantisymmetrized functions of type (A1) and we take into account that the operators $a_i^\dagger$ acting on the left thus annihilate the states of s nucleons to zero.

Then we have

$$\langle s^{\nu_2}p^n[n]:[\nu_2+n]\Lambda S_1T_1 | \hat{\hat{R}} | s^{\nu_2+n}[\nu_2+n]0S_1T_1\rangle$$
$$= \binom{\nu_2+n}{n}^{1/2} \sum_{S_1'T_1'S_2'T_2'} \langle [\tilde{\nu}_2]S_1'T_1', [\tilde{n}]S_2'T_2' | [\nu_2\tilde{+}n]S_1T_1\rangle \langle p^n[n]\Lambda S_2'T_2' | \hat{\hat{R}} | s^n[n]0S_2'T_2'\rangle . \quad \text{(A3)}$$

Now the operator $\hat{\hat{R}}$ actually contains not $\underline{A}^\dagger = (1/\sqrt{b})\sum_{i=1}^b a_i^\dagger$ but $\underline{A}^\dagger = (1/\sqrt{b})\sum_{i=1}^n a_i^\dagger$, therefore, the last matrix element in (A3) coincides with the CC K_n of Eq. (27) except for the factor $(n/b)^{n/2}$. If we now substitute (A3) in (A2) and separate the FPC into the orbital and spin-isospin parts, the latter may be transformed into the Racah coefficients for the group SU(4) indicated in Eq. (26). Finally, we separate out the scalar part of the CC, thus obtaining the general formula (26).

[1]V. V. Balashov, V. G. Neudatchin, Yu. F. Smirnov, and N. P. Yuding, Zh. Eksp. Teor. Fiz. 37, 1385 (1959) [JETP 37, 983 (1960)]; Yu. F. Smirnov and D. Chlebowska, Nucl. Phys. 26, 306 (1961).

[2]V. G. Neudatchin and Yu. F. Smirnov, *Nucleon clusters in the light nuclei* (Nauka, Moscow, 1969), (in Russian).

[3]I. Rotter, Fortschr. Phys. 16, 195 (1968); Nucl. Phys. A122, 567 (1968); A135, 378 (1969); R. Bock and H. Yoshida, Nucl. Phys. A189, 177 (1972).

[4]I. Rotter and M. A. Jusupov, Ann. Phys. (Leipzig) 17, 57 (1966); A. N. Boyarkina, *Structure of 1p-shell nuclei* (Moscow State U., 1973) (in Russian); D. Kurath, Nucl. Phys. A222, 1 (1974), A238, 269 (1975).

[5]M. Ichimura, A. Arima, E. C. Halbert, and T. Terasawa, Nucl. Phys. A204, 225 (1973).

[6]Yu. A. Kudeyarov, I. V. Kurdyumov, V. G. Neudatchin, and Yu. F. Smirnov, Nucl. Phys. A163, 316 (1971).

[7]N. F. Golovanova, I. M. Il'in, V. G. Neudatchin, Yu. F. Smirnov, and Yu. M. Tchuvil'sky, Zh. Eksp. Teor. Fiz. Pis'ma Red. 20, 674 (1974); 22, 112 (1975) [JETP Lett. 20, 310 (1974); 22, 50 (1975)]; Yad. Fiz. 23, 63 (1976); V. V. Balashov, in Proceedings of the Second International Conference on Clustering Phenomena in Nuclei, University of Maryland, 1975 (unpublished).

[8]N. Anyas-Weiss *et al.*, Phys. Rep. 12C, 201 (1974).

[9]N. F. Golovanova, N. S. Zelenskaya, and N. El-Nagar, Nucl. Phys. A111, 1 (1968).

[10]P. Beregi, N. S. Zelenskaya, V. G. Neudatchin, and Yu. F. Smirnov, Nucl. Phys. 66, 513 (1965).

[11]I. V. Kurdyumov, Yu. F. Smirnov, K. V. Shitikova, and S. H. El-Samarae, Nucl. Phys. A145, 593 (1970).

[12]H. A. Jahn and H. van Wieringen, Proc. R. Soc. London A209, 502 (1951); J. P. Elliott, J. Hope, and H. A. Jahn, Phil. Trans. Roy. Soc. London A246, 241 (1953); D. Chlebowska, Acta Phys. Pol. 25, 313 (1964); I. Rotter, Ann. Phys. (Leipzig) 16, 242 (1965).

[13]K. T. Hecht and D. Branschweig, Nucl. Phys. A244, 365 (1975); M. N. Aljadir, Nucl. Phys. A251, 156 (1975).

[14]J. P. Eliott and T. H. R. Skyrme, Proc. Roy. Soc.

London A232, 561 (1955).
[15]P. Federman, B. Girand, and D. Zaikin, Nucl. Phys. A102, 81 (1967).
[16]H. Weyl, *The Classical Groups* (Princeton U. P. 1946); I. G. Kaplan, *Symmetry of many-electron systems* (Nauka, Moscow, 1969) (in Russian); H. Horie, J. Phys. Soc. Jpn. 19, 1783 (1964).
[17]V. I. Kukulin, L. Majling, and Yu. F. Smirnov, Nucl. Phys. A103, 681 (1969).
[18]M. Moshinsky, *Harmonic Oscillator in Modern Physics* (Gordon and Breach, New York, 1970).
[19]Yu. F. Smirnov, Nucl. Phys. 27, 177 (1959).
[20]R. Reznikoff, J. Math. Phys. 8, 63, 79 (1967).
[21]P. Kramer, Z. Phys. 205, 181 (1967); 216, 68 (1968).
[22]S. Kato *et al.*, in Proceedings of the INS-IPCR, Symposium on Cluster Structure of Nuclei and Transfer Reactions Induced by Heavy Ions, Tokyo, March 17–22, 1975 (unpublished), p. 604.