Off-energy-shell generalization of the results in potential scattering*

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Within the framework of nonrelativistic potential scattering theory, an off-energy-shell generalization of the various solutions and the potential matrix elements is carried out. Some relations between these solutions and between the matrix elements are derived. The off-energy-shell generalization of the Jost-Pais theorem is proved for a general nonlocal interaction. A constraint on the t matrix elements of a local potential is obtained. For given t matrix elements, the value of a certain expression gives a measure of the nonlocality of the underlying interaction.

[NUCLEAR REACTIONS Off-energy-shell potential scattering theory, relations between the solutions and between the potential matrix elements. Off-energy-shell Jost-Pais theorem. Constraint on t matrix.]

I. INTRODUCTION

An understanding of off-energy-shell[1] potential scattering is quite basic to the study of any system involving more than two interacting particles. But, to our knowledge, no systematic attempt has been made to obtain formal relations similar to those existing in the on-shell case. In nuclear physics, for example, the emphasis has been on exploring the arbitrariness in the off-shell t matrix starting from given on-shell data, and on the development of numerical techniques to handle various types of potentials.[2] Recently Fuda[3] has developed an off-shell generalization of the Jost function and expressed off-shell t matrix elements in terms of them. He also related the off-shell outgoing scattering solution to the off-shell irregular solutions. Our aim in this paper is to obtain (i) an off-shell version of the various solutions and potential matrix elements and their mutual relationships, and (ii) the off-shell generalization of Jost-Pais theorem.[4] A constraint on off-shell t matrix elements corresponding to a local potential is also obtained.

We shall consider in Sec. II an inhomogeneous form of the Schrödinger equation. This equation contains two momenta, k and q, where k is an on-shell momentum related to the energy, and q is an off-shell momentum. When $q=k$ the equation reduces to the usual Schrödinger equation. We shall obtain regular, irregular, outgoing, and standing wave scattering solutions of the inhomogeneous equation corresponding to the various possible choices of the boundary conditions. A comparison of these solutions yields relations between potential matrix elements involving these solutions. All the matrix elements are expressible in terms of the functions $Y_l(p,q;k^2)$ introduced by Fuda.[3]

Section III is devoted to generalizing the well known result[5-8] expressing the Jost function as a ratio of the Fredholm determinants of the integral equations corresponding to outgoing wave scattering solution and regular solution respectively. We prove that the off-shell Jost function is the ratio of a generalized Fredholm determinant of the integral equation for the off-shell outgoing scattering solution to that of the on-shell regular solution. For a local potential, the determinant in the denominator is unity. As an illustration the results of this section are applied to the case of a one-term separable potential.

All the formal results of Secs. II and III go over into corresponding results of the usual (on-shell) potential scattering theory.

In Sec. IV we address a different aspect of off-shell scattering. It is known from inverse scattering analysis that, for any given set of on-shell t matrix elements, the extrapolation to off-shell matrix elements is rather arbitrary and depends on infinitely many types of possible nonlocalities of the underlying interaction. In case the potential is local, this extrapolation is unique. We shall obtain a constraint on off-shell t matrix elements in such a situation.

Throughout this paper we work in units in which $\hbar^2/2m$ is unity.

II. OFF-ENERGY-SHELL SOLUTIONS, MATRIX ELEMENTS, AND THEIR RELATIONS

The outgoing scattering wave function for the on-shell two-particle scattering in the center of mass system is given by the solution of the Lippmann-Schwinger equation

$$|\Psi^{(+)}(\vec{k},k^2)\rangle = |\vec{k}\rangle + G^{(+)}(k^2)V\,|\Psi^{(+)}(\vec{k},k^2)\rangle\,, \quad (1)$$

where $|\vec{k}\rangle$ represents a plane wave and satisfies the free particle equation

$$(k^2 - H_0)\,|\vec{k}\rangle = 0\,, \tag{2}$$

and

$$G^{(+)}(k^2) = (k^2 - H_0 + i\epsilon)^{-1} \tag{3}$$

is the appropriate free particle Green's function. Other solutions (regular, irregular, and standing wave) correspond to different choices for the Green's function and the free solution, depending on the imposed boundary conditions. In order to discuss its off-shell continuation, we consider the integral equation

$$|\,\zeta(k^2,\vec{q})\rangle = |\,\xi(\vec{q})\rangle + G(k^2)V\,|\,\zeta(k^2,\vec{q})\rangle\,, \tag{4}$$

or its equivalent

$$(k^2 - H_0 - V)\,\langle\vec{r}\,|\,\zeta(k^2,\vec{q})\rangle = (k^2 - q^2)\,\langle\vec{r}\,|\,\xi(\vec{q})\rangle \tag{5}$$

with appropriate boundary conditions. $|\xi\rangle$ is a solution of the free equation. Assuming that the potential is spherically symmetric and the particles have no intrinsic angular momentum, the partial wave decomposition can be easily carried through to give

$$\mathcal{L}_l(k)\zeta_l(k^2,q,r) = (k^2 - q^2)\xi_l(q,r)\,, \tag{6}$$

where

$$\mathcal{L}_l(k) \equiv k^2 + \frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} - V(r)\,, \tag{7}$$

$$\int Y^*_{lm}(\hat{r})\langle\vec{r}\,|\,\xi(\vec{q})\rangle\,d\Omega = (2/\pi)^{1/2}(qr)^{-1}\xi_l(q,r)\,, \tag{8}$$

$$\int Y^*_{lm}(\hat{r})\langle\vec{r}\,|\,\zeta(k^2,\vec{q})\rangle\,d\Omega = (2/\pi)^{1/2}(qr)^{-1}\zeta_l(k^2,q,r)\,. \tag{9}$$

Note that if $k=q$, Eq. (6) reduces to the usual Schrödinger equation. It is well known[9] that the free Green's functions appropriate to the various types of solution are the following:

$$G_k^{(\pm)}(r,r') = -\frac{e^{\mp i\pi l}}{k}u_l(kr_<)w_l^{(\pm)}(kr_>)\,, \tag{10}$$

for the outgoing/incoming wave scattering solution;

$$G_k^P(r,r') = \frac{1}{k}u_l(kr_<)v_l(kr_>)\,, \tag{11}$$

for the standing wave scattering solution;

$$G_k^R(r,r') = G_k^{(+)}(r,r') + \frac{e^{-il\pi}}{k}u_l(kr)w_l^{(+)}(kr')\,, \tag{12}$$

for the regular solution;

$$\begin{aligned} G_k^I(r,r') &= G_k^{(+)}(r,r') + \frac{e^{-il\pi}}{k}u_l(kr')w_l^{(+)}(kr)\,,\\ &= \tilde{G}_k^R(r,r')\,, \end{aligned} \tag{13}$$

for Jost solutions, where $r_<$ indicates the smaller of r and r' and $r_>$ the larger. The partial wave index l on G has been suppressed and[10]

$$u_l(x) = xj_l(x) \to \begin{cases} x^{l+1}/(2l+1)!!\,, & x\sim 0\,;\\ \sin(x-\tfrac{1}{2}l\pi)\,, & x\sim\infty\,;\end{cases} \tag{14}$$

$$v_l(x) = xn_l(x) \to \begin{cases} -x^{-l}(2l-1)!!\,, & x\sim 0\,;\\ -\cos(x-\tfrac{1}{2}l\pi)\,, & x\sim\infty\,;\end{cases} \tag{15}$$

$$\left.\begin{aligned} w_l^{(+)}(x) &= ie^{il\pi}xh^{(1)}(x)\\ w_l^{(-)}(x) &= -ixh_l^{(2)}(x)\end{aligned}\right] \to e^{\pm ix + il\pi/2}. \tag{16}$$

The corresponding natural choices for the free wave functions are

$$\xi_l(q,r) = \begin{cases} u_l(qr)\,, & (17)\\ u_l(qr)\,, & (18)\\ u_l(qr)(2l+1)!!\,q^{-l-1}\,, & (19)\\ e^{-il\pi/2}w_l^{(\pm)}(qr)\,. & (20)\end{cases}$$

We are now in a position to consider the various off-shell solutions.

A. Jost solutions

The off-shell Jost solutions satisfy the differential equation

$$\mathcal{L}_l(k)f_l^{(\pm)}(k,q,r) = (k^2 - q^2)e^{-il\pi/2}w_l^{(\pm)}(qr) \tag{21}$$

with the asymptotic boundary conditions

$$f_l^{(\pm)}(k,q,r) \underset{r\to\infty}{\sim} e^{\pm iqr}. \tag{22}$$

The integral equation incorporating this boundary condition is

$$f_l^{(\pm)}(k,q,r) = e^{-il\pi/2}w_l^{(\pm)}(qr) - \int_r^\infty dr'\,G_k^I(r,r')V(r')f_l^{(\pm)}(k,q,r')\,. \tag{23}$$

Since

$$w_l^{(+)}(-x) = e^{il\pi}w_l^{(+)*}(x) = w_l^{(-)}(x)\,, \tag{24}$$

it immediately follows from Eq. (21) that

$$f_l^{(+)}(k,-q,r) = f_l^{(+)*}(k,q,r) = f_l^{(-)}(k,q,r) \tag{25}$$

for real k, q, and r.

In general, for small r, the centrifugal term in Eq. (21) dominates over the potential; the Jost solutions $f_l^{(\pm)}(k,q,r)$ are expected to behave as r^{-l} as $r\to 0$ [see Eq. (23) also]. Let us define, following Fuda,[3] an off-shell Jost function, analogous to the on-shell case, by

$$f_l^{(\pm)}(k,q) = \frac{e^{-il\pi/2}q^l}{(2l-1)!!}\lim_{r\to 0} r^l f_l^{(\pm)}(k,q,r)\,. \tag{26}$$

From Eq. (25) it follows that

$$f_l^{(+)}(k,-q)=f_l^{(+)*}(k,q)=f_l^{(-)}(k,q) \tag{27}$$

for real k and q. Using the definition (26) in Eq. (23), one obtains an integral representation for the off-shell Jost function[11]

$$f_l(k,q)=1+\frac{1}{k}(q/k)^l e^{-il\pi/2}\langle u_l(k)\,|\,V\,|f_l(k,q)\rangle\,. \tag{28}$$

The Jost function $f_l(k,q)$ can also be expressed in terms of the on-shell Jost functions by taking the limit (26) in the solution

$$f_l(k,q,r)=f_l(q,r)+(k^2-q^2)\int_r^\infty dr'\mathcal{G}_k^I(r,r')[e^{-il\pi/2}w_l^{(+)}(qr')-f_l(qr')]\,, \tag{29}$$

where the interacting Green's function $\mathcal{G}^I$ is given by[9]

$$\mathcal{G}_k^I(r,r')=\begin{cases}\dfrac{k^l e^{il\pi/2}}{f_l^{(+)}(k)(2l+1)!!}[f_l(k,r)\phi_l(k,r')-f_l(k,r')\phi_l(k,r)]\,, & r<r'\,;\\ 0\,, & ,r>r'\,;\end{cases} \tag{30}$$

and the on-shell regular function ϕ_l by

$$\phi_l(k,r)=\frac{k^{-l}(2l+1)!!}{2ik}[e^{-il\pi/2}f_l(-k)f_l(k,r)-e^{il\pi/2}f_l(k)f_l(-k,r)]\,. \tag{31}$$

We get

$$f_l(k,q)=f_l(q)+\frac{(k^2-q^2)q^l}{(2l+1)!!}\int_0^\infty dr'\phi_l(k,r')[w_l^{(+)}(qr')-e^{il\pi/2}f_l(q,r')]\,, \tag{32}$$

which is similar to the relation[12,13]

$$t_l(p,k;k^2)=\left(\frac{p}{k}\right)^l t_l(k,k;k^2)+\frac{(k^2-p^2)}{pk}\int_0^\infty dr'u_l(pr')[\psi_l^{(+)}(k,r')-\psi_l^{(+)}{}_{\text{asym}}(k,r')] \tag{33}$$

expressing the half-shell K matrix element in terms of the on-shell quantities.

B. Outgoing wave scattering solution $\psi_l^{(+)}(k,q,r)$

It satisfies the integral equation

$$\psi_l^{(+)}(k,q,r)=u_l(qr)+\int_0^\infty dr'G_k^{(+)}(r,r')V(r')\psi_l^{(+)}(k,q,r')\,, \tag{34}$$

or equivalently the differential equation

$$\mathcal{L}_l(k)\psi_l^{(+)}(k,q,r)=(k^2-q^2)u_l(qr)=\frac{(k^2-q^2)}{2i}[e^{-il\pi}w_l^{(+)}(qr)-w_l^{(-)}(qr)] \tag{35}$$

with the asymptotic boundary condition

$$\psi_l^{(+)}(k,q,r)\underset{r\to\infty}{\sim}\sin(qr-\tfrac{1}{2}l\pi)-qt_l(k,q;k^2)e^{i(kr-l\pi/2)}\,. \tag{36}$$

The half-shell t matrix element $t_l(k,q;k^2)$ derived from Eq. (34) is given by

$$t_l(k,q;k^2)=\frac{1}{kq}\langle u_l(k)\,|\,V\,|\,\psi_l^{(+)}(k,q)\rangle\,. \tag{37}$$

From Eqs. (35)–(37) it follows that

$$\psi_l^{(+)}(k,q,r)=\frac{1}{2i}[e^{-il\pi/2}f_l(k,q,r)-e^{il\pi/2}f_l(k,-q,r)]-qt_l(k,q;k^2)e^{-il\pi/2}f_l(k,r)\,. \tag{38}$$

From expression (38) follows a relation between the fully off-shell t matrix elements and the matrix elements involving $f_l(k,\pm q,r)$:

$$t_l(p,q;k^2)=\frac{1}{pq}\langle u_l(p)\,|\,V\,|\,\psi_l^{(+)}(k,q)\rangle$$

$$=t_l(k,q;k^2)[1-Y_l(p,k;k^2)]+\frac{1}{2iq}\left(\frac{k}{q}\right)^l[Y_l(p,q;k^2)-Y_l(p,-q;k^2)]\,, \tag{39}$$

where

$$Y_l(p,q;k^2)=1+\frac{1}{p}\left(\frac{q}{k}\right)^l e^{-il\pi/2}\langle u_l(p)\,|\,V\,|f_l(k,q)\rangle\,. \tag{40}$$

Equation (40) is really a generalization of Eq. (28) for $f_l(k,q)$ and can also be written as[3]

$$Y_l(p,q;k^2)=f_l(k,q)+\frac{(k^2-p^2)}{p}\left(\frac{q}{k}\right)^l e^{-il\pi/2}\int_0^\infty dr\,u_l(pr)[f_l(k,q,r)-e^{-il\pi/2}w_l^{(+)}(qr)]\,. \tag{41}$$

For $p=k$, the relation (39) reduces to a relation expressing half-shell t-matrix elements in terms of $f_l(k,\pm q)$,

$$t_l(k,q;\,k^2)=\left(\frac{k}{q}\right)^l\;\frac{f_l(k,q)-f_l(k,-q)}{2iqf_l(k)}\,. \tag{42}$$

The relation (42) can also be obtained by using the boundary conditions $\psi_l^{(+)}\to 0$ as $r\to 0$, and Eq. (26).

C. Standing wave solution $\psi_l^P(k,q,r)$

It satisfies the integral equation

$$\psi_l^P(k,q,r)=u_l(qr)+\int_0^\infty dr' G_k^P(r,r')V(r')\psi_l^P(k,q,r')\,, \tag{43}$$

and the same differential equation as satisfied by $\psi_l^{(+)}(k,q,r)$ but with the boundary condition

$$\psi_l^P(k,q,r)\underset{r\to\infty}{\sim}\sin(qr-\tfrac{1}{2}l\pi)-qK_l(k,q;k^2)\cos(kr-\tfrac{1}{2}l\pi)\,. \tag{44}$$

From Eqs. (43) and (44), half-shell K-matrix element $K_l(k,q;k^2)$ is given by

$$K_l(k,q;k^2)=\frac{1}{kq}\,\langle u_l(k)\,|\,V\,|\,\psi_l^P(k,q)\rangle\,. \tag{45}$$

It immediately follows that

$$\psi_l^P(k,q,r)=\frac{1}{2i}\,[e^{-il\pi/2}f_l(k,q,r)-e^{il\pi/2}f_l(k,-q,r)]-\frac{q}{2}\,K_l(k,q;k^2)[e^{-il\pi/2}f_l(k,r)+e^{il\pi/2}f_l(-k,r)] \tag{46}$$

and

$$K_l(p,q;k^2)=\frac{1}{pq}\,\langle u_l(p)\,|\,V\,|\,\psi_l^{(P)}(k,q)\rangle$$

$$=K_l(k,q;k^2)[2-Y_l(p,k;k^2)-Y_l(p,-k;k^2)]+\frac{1}{2iq}\left(\frac{k}{q}\right)^l[Y_l(p,q;k^2)-Y_l(p,-q;k^2)]\,. \tag{47}$$

For $p=k$, Eq. (47) reduces to

$$K_l(k,q;k^2)=\frac{1}{iq}\left(\frac{k}{q}\right)^l\;\frac{f_l(k,q)-f_l(k,-q)}{f_l(k)+f_l(-k)}\quad, \tag{48}$$

which leads to the well-known result[9]

$$K_l(k,q;k^2)=\frac{2f_l(k)}{f_l(k)+f_l(-k)}\;t_l(k,q;k^2)$$

$$=\frac{t_l(k,q;k^2)}{1-ikt_l(k,k;k^2)}\,. \tag{49}$$

D. Regular solution $\phi_l(k,q,r)$

It satisfies the integral equation

$$\phi_l(k,q,r)=\frac{(2l+1)!!}{q^{l+1}}\,u_l(qr)+\int_0^r dr'\,G_k^R(r,r')V(r')\phi_l(k,q,r')\,, \tag{50}$$

which corresponds to the differential equation

$$\mathcal{L}_l(k)\phi_l(k,q,r)=(k^2-q^2)\,\frac{(2l+1)!!}{q^{l+1}}\,u_l(qr) \tag{51}$$

with the boundary condtion

$$\lim_{r\to 0} r^{-l-1}\phi_l(k,q,r)=1\,. \tag{52}$$

From Eq. (50), the asymptotic form of ϕ_l is given by

$$\phi_l(k,q,r)\underset{r\to\infty}{\sim}\frac{(2l+1)!!}{q^{l+1}}\sin(qr-\tfrac{1}{2}l\pi)-\frac{1}{k}\left[\langle u_l(k)\,|\,V\,|\,\phi_l(k,q)\rangle\cos(kr-\tfrac{1}{2}l\pi)+\langle v_l(k)\,|\,V\,|\,\phi_l(k,q)\rangle\sin(kr-\tfrac{1}{2}l\pi2)\right]. \tag{53}$$

It follows from Eqs. (51) and (53) that

$$\phi_l(k,q,r)=\frac{(2l+1)!!}{2iq^{l+1}}\left[e^{-il\pi/2}f_l(k,q,r)-e^{il\pi/2}f_l(k,-q,r)\right]$$
$$+\frac{1}{2ik}\left[e^{il\pi/2}\langle w_l^{(+)}(k)\,|\,V\,|\,\phi_l(k,q)\rangle f_l(k,r)-e^{-il\pi/2}\langle w_l^{(+)*}(k)\,|\,V\,|\,\phi_l(k,q)\rangle f_l(-k,r)\right]. \tag{54}$$

The matrix element $\langle w_l^{(+)*}(k)\,|\,V\,|\,\phi_l(k,q)\rangle$ can be easily related to the Jost functions by condition $\phi_l(k,q,r=0)=0$ and comparison to the on-shell solution (31). We get

$$1+\frac{q^l}{(2l+1)!!}\left(\frac{q}{k}\right)^{l+1}e^{-il\pi}\langle w_l^{(+)*}(k)\,|\,V\,|\,\phi_l(k,q)\rangle$$
$$=f_l(k)+[f_l(k,q)-f_l(k)]/f_l(-k)\,. \tag{55}$$

The relation (55) can also be used to express t and K matrix elements in terms of $\langle w_l^{(+)}(k)\,|\,V\,|\,\phi_l(k,q)\rangle$.

In general, we have

$$\langle\phi_l(k,q)\,|\,V\,|w_l^{(+)}(p)\rangle$$
$$=\frac{(2l+1)!!}{q^{l+1}}\,\langle(1-G_k^RV)^{-1}u_l(q)\,|\,V\,|w_l^{(+)}(p)\rangle$$
$$=\frac{(2l+1)!!}{q^{l+1}}\,\langle u_l(q)\,|\,V(1-G_k^IV)^{-1}\,|w_l^{(+)}(p)\rangle$$
$$=\frac{(2l+1)!!}{q^{l+1}}\,e^{il\pi/2}\langle u_l(q)\,|\,V\,|f_l(k,p)\rangle\,. \tag{56}$$

Using Eq. (56) for $p=k$ in Eq. (55) gives a simple relation between $Y_l(q,k;k^2)$ and $Y_l(k,q;k^2)$:

$$\left(\frac{q}{k}\right)^{l+1}[Y_l(q,k;k^2)-1]-[Y_l(k,k;k^2)-1]$$
$$=[Y_l(k,q;k^2)-Y_l(k,k;k^2)]/Y_l(k,-k;k^2)\,. \tag{57}$$

The case $q=k$ in Eq. (56) leads to an alternative integral representation for $f_l(k,q)$:

$$f_l(k,q)=1+\frac{q^l}{(2l+1)!!}\,e^{-il\pi}\langle\phi_l(k)\,|\,V\,|w_l^{(+)}(q)\rangle\,. \tag{58}$$

Recently we noticed a derivation of the off-shell Jost function expression Eq. (58) by Fuda.[14] He arrived at the result using a rather different method. After inserting a complete set of functions $u_l(pr)$ between V and $w^{(+)}$ in Eq. (58) and using the definition of t matrix elements, one can easily obtain his second result, the momentum representation of $f_l(k,q)$.

III. OFF-ENERGY-SHELL GENERALIZATION OF JOST-PAIS THEOREM

In this section we attempt to generalize the on-shell result[6,7] expressing the Jost function as the ratio of the Fredholm determinants of the Lippmann-Schwinger equations corresponding to the outgoing scattering solution and the regular solution, respectively.

Consider the propagator

$$(1-G_k^RV-F_{k,q}V)^{-1}\,,$$

where

$$F_{k,q}(r,r')=C(k,q)u_l(kr)w_l^{(+)}(qr')\,. \tag{59}$$

The normalization constant C will be chosen later. This propagator can be expanded as

$$(1-GV-FV)^{-1}=(1-GV)^{-1}+C(1-GV)^{-1}\,|u\rangle\langle w^{(+)*}|\,V(1-GV)^{-1}+C^2(1-GV)^{-1}\,|u\rangle\,\langle w^{(+)*}\,|\,V(1-GV)^{-1}\,|u\rangle$$
$$\times\langle w^{(+)*}\,|\,V(1-GV)^{-1}+\cdots=(1-GV)^{-1}+C\,\frac{(1-GV)^{-1}\,|u\rangle\langle w^{(+)*}\,|\,V(1-GV)^{-1}}{1-C\langle w^{(+)*}\,|\,V(1-GV)^{-1}|u\rangle}\,. \qquad (60)$$

The indices R, q, k, and l have been dropped for clarity. If C is taken as

$$C(k,q)=-\frac{1}{k}\left(\frac{q}{k}\right)^l e^{-il\pi}\,, \qquad (61)$$

the denominator in the second term of Eq. (60) reduces to

$$1+\frac{1}{k}\left(\frac{q}{k}\right)^l e^{-il\pi}\langle w^{(+)*}\,|\,V(1-GV)^{-1}|\,u\rangle=1+\frac{q^l e^{-il\pi}}{(2l+1)!!}\,\langle w^{(+)*}\,|\,V\,|\,\phi\rangle$$
$$=1+\frac{q^l e^{-il\pi}}{(2l+1)!!}\,\langle\phi(k)\,|\,V\,|w^{(+)}(q)\rangle=f_l(k,q)\,. \qquad (62)$$

The last step in the above equation follows from Eq. (58). Let us consider now

$$\mathrm{Tr}(GV+FV)\,(1-GV-FV)^{-1}-\mathrm{Tr}\,GV(1-GV)^{-1}$$
$$=\mathrm{Tr}\,FV(1-GV)^{-1}+\mathrm{Tr}(GV+FV)\,(1-GV)^{-1}FV(1-GV)^{-1}/[1-C\langle w^{(+)*}\,|\,V(1-GV)^{-1}|u\rangle]$$
$$=\mathrm{Tr}(1-GV)^{-1}FV(1-GV)^{-1}/f_l(k,q)$$
$$=-\frac{1}{k}\left(\frac{q}{k}\right)^l e^{-il\pi}\langle w_l^{(+)*}(q)\,|\,V(1-GV)^{-2}\,|u_l(k)\rangle/f_l(k,q)\,. \qquad (63)$$

Replacing $V\to\gamma V$ where γ is the strength of the interaction, Eq. (63) yields

$$\mathrm{Tr}(G_k^R V+F_{k,q}V)(1-\gamma G_k^R V-\gamma F_{k,q}V)^{-1}-\mathrm{Tr}\,G_k^R V(1-\gamma G_k^R V^{-1}$$
$$=-\frac{1}{k}\left(\frac{q}{k}\right)^l e^{-il\pi}\langle w_l^{(+)*}(q)\,|\,V(1-\gamma G_k^R V)^{-2}\,|u_l(k)\rangle/f_l(k,q)\,. \qquad (64)$$

Consider Eq. (58) for the Jost function $f_l(k,q)$. For the interaction γV it can be written as

$$f_l(k,q,\gamma)=1+\gamma\,\frac{q^l e^{-il\pi}}{(2l+1)!!}\,\langle\phi_l(k,\gamma)\,|\,V\,|w_l^{(+)}(q)\rangle$$
$$=1+\frac{\gamma}{k}\left(\frac{q}{k}\right)^l e^{-il\pi}\langle w_l^{(+)*}(q)\,|\,V(1-\gamma G_k^R V)^{-1}|u_l(k)\rangle\,. \qquad (65)$$

Differentiating Eq. (65) with respect to γ and comparing the result with Eq. (63), one arrives at the following relation:

$$\frac{1}{f_l(k,q,\gamma)}\,\frac{d}{d\gamma}\,f_l(k,q,\gamma)=\mathrm{Tr}\,G_k^R V(1-\gamma G_k^R V)^{-1}-\mathrm{Tr}(G_k^R V+F_{k,q}V)\,(1-\gamma G_k^R V-\gamma F_{k,q}V)^{-1}\,. \qquad (66)$$

Integrating this, one obtains on putting $\gamma=1$

$$\ln f_l(k,q)=\mathrm{Tr}\ln(1-G_k^R V-F_{k,q}V)-\mathrm{Tr}\ln(1-G_k^R V)\,, \qquad (67)$$

leading to the following determinantal equation for the Jost function:

$$f_l(k,q)=\frac{\det(1-G_k^R V-F_{k,q}V)}{\det(1-G_k^R V)}\,. \qquad (68)$$

If $q=k$ (on-shell case),

$$G_k^R(r,r')+F_{k,k}(r,r')=G_k^R(r,r')-\frac{e^{-il\pi}}{k}\,u_l(kr)w_l^{(+)}(kr')$$
$$=G_k^{(+)}(r,r')\,, \qquad (69)$$

and Eq. (68) reduces to

$$f_l^{(+)}(k)=\frac{\det(1-G_k^{(+)}V)}{\det(1-G_k^R V)}\,, \qquad (70)$$

the expression derived by Warke and Bhaduri.[6]

Example: We illustrate the results of this section by calculating $f(k,q)$ in the case of a one-term separable potential with Yamaguchi form factor in the $l=0$ state.[15] In configuration space, this potential is given by

$$V(r,r')=\lambda g(r)g(r')\,, \qquad (71)$$

where λ is a constant and

$$g(r)=e^{-\alpha r}\,. \qquad (72)$$

For a one-term separable potential

$$\det(1-G_k^R V)=\exp[\operatorname{Tr}\ln(1-G_k^R V)]$$
$$=1-\operatorname{Tr}G_k^R V. \qquad (73)$$

Therefore, from Eq. (68),

$$f(k,q)=(1-\operatorname{Tr}G_k^R V-\operatorname{Tr}F_{k,q}V)/(1-\operatorname{Tr}G_k^R V). \qquad (74)$$

Now

$$1-\operatorname{Tr}G_k^R V=1-\lambda\int_0^\infty dr\int_0^r dr'\,[u_0(kr')v_0(kr)-u_0(kr)v_0(kr')]e^{-\alpha r}e^{-\alpha r'}$$
$$=1-\lambda\int_0^\infty dr\int_0^r dr'\sin k(r-r')e^{-\alpha(r+r')}=1-\frac{\lambda}{2\alpha(\alpha^2+k^2)}\equiv D(k). \qquad (75)$$

Similarly,

$$\operatorname{Tr}F_{k,q}V=-\frac{\lambda}{k}\int_0^\infty dr\int_0^r dr'\sin kr e^{iqr'}e^{-\alpha r}e^{-\alpha r'}$$
$$=-\frac{\lambda(\alpha+iq)}{(\alpha^2+k^2)(\alpha^2+q^2)}. \qquad (76)$$

Using Eqs. (75) and (76) in Eq. (74), we get the final result

$$f(k,q)=\left[D(k)+\frac{\lambda(\alpha+iq)}{(\alpha^2+k^2)(\alpha^2+q^2)}\right]\Big/D(k)$$
$$=1+\frac{2\lambda\alpha}{(\alpha-iq)[2\alpha(\alpha^2+k^2)-\lambda]}. \qquad (77)$$

IV. CONSTRAINT ON THE t MATRIX ELEMENTS OF A LOCAL POTENTIAL

This section is devoted to finding some constraint on t matrix elements in the case of a local potential. By a local potential we mean locality in the usual sense. For example, we admit spin-orbit or l^2-dependent interactions which are local in any partial wave. Angular momentum independence (same interaction in all partial waves) is not enforced as it is uninteresting and leads to rather too stringent restrictions. It is well known that for a local potential, the Wronskian of $f_l(\pm k,r)$ and $\phi_l(k,r)$ is independent of r and the Fredholm determinant $D(k)\equiv\det(1-G_k^R V)$ of the regular solution is unity. These conditions are not very useful as they involve quantities which are not physically observable. A relation involving t matrices is expected to be more useful as they are more closely related to observed physical processes. Procedures based on the inverse scattering problem to calculate off-shell t matrix elements directly from elastic scattering data (scattering phase shifts) in the case of a local potential have been given earlier,[9,16-18] but these are rather too complicated to be used as a constraint on the t matrix. For a separable interaction, t matrix elements are known to have a simple structure.

Consider $\operatorname{Tr}G_k^R V$. It is really the asymptotic value[19] of $1-D(k)$. One can easily see that it vanishes for a local potential.[6] In momentum space it leads to

$$\operatorname{Tr}G_k^R V=\frac{\pi}{2}\int_0^\infty\frac{V(k,q)-V(q,q)}{q^2-k^2}\,q^2dq, \qquad (78)$$

where the potential matrix element

$$V(p,q)=\frac{1}{pq}\int_0^\infty dr\int_0^\infty dr'\,u_l(pr)V(r,r')u_l(qr') \qquad (79)$$

can be expressed in terms of t matrix elements[2,20,21]

$$V(p,q)=t_l(p,q;q^2)+\frac{2}{\pi}\int_0^\infty dk'\,\frac{t_l(p,k';k'^2)t_l^*(q,k';k'^2)}{k'^2-q^2-i\epsilon}\,k'^2$$
$$=t_l(p,q;k^2)+\frac{2}{\pi}\int_0^\infty dk'\,\frac{t_l(p,k';k'^2)t_l^*(q,k';k'^2)}{k'^2-k^2-i\epsilon}\,k'^2. \qquad (80)$$

It is assumed, for simplicity, that the potential does not support a bound state. Using Eq. (80), the optical theorem, and the principal value integrations, Eq. (78) yields

$$I(k)\equiv\operatorname{Tr}G_k^R V$$
$$=\operatorname{Im}\int dk'\int dq\,k'q^2\,\frac{t_l(q,q;k'^2)-t_l(k,q;k'^2)}{(q^2-k^2)(k'^2-k^2+i\epsilon)}. \qquad (81)$$

Expression (81) is the desired result. If the interaction is local, the t matrix elements are constrained to make the right-hand side of Eq. (81) vanish identically. The value of I can be taken as a measure of nonlocality of the underlying interaction.

Equation (81) can be simplified by considering a particular value of k, say $k=0$.

V. CONCLUSION

An off-shell generalization of the various solutions in on-shell potential scattering is carried out. In analogy with the on-shell matrix elements, off-shell potential matrix elements with respect to these solutions are introduced. Some relations between these matrix elements and between the solutions are derived. All the matrix elements are expressed in terms of the generalized off-shell Jost function $Y_l(p,q;k^2)$ introduced by Fuda.[3] When the off-shell and on-shell momenta are equal, all these solutions, matrix elements, and their relations go over to the well-known results of scattering theory. The off-shell generalization of the Jost-Pais theorem is proved. It states that the off-shell Jost function is the ratio of a generalized Fredholm determinant of the integral equation for the off-shell outgoing wave scattering solution to that of the on-shell regular solution. It may be a useful result to study its analytic properties. An expression involving the t matrix elements is obtained. It can be used, in principle, as a measure of nonlocality of the underlying interaction for given t matrix elements.

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[1]For brevity the off-energy-shell, half-off-energy-shell, and on-energy-shell matrix elements shall be called off-shell, half-shell, and on-shell, respectively.

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