PHYSICAL REVIEW C VOLUME 15, NUMBER 2 FEBRUARY 1977

Analysis of four-body final states: Nonrelativistic*

Sadhan K. Adhikari
Departmento de Fisica, Universidade Federal de Pernambuco, Recife, Pe. Brazil

R. D. Amado
Departmento de Fisica, Universidade Federal de Pernambuco, Recife, Pe. Brazil
and Department of Physics, University of Pennsylvania, Philadelphia, Pennsylvania 19174†
(Received 21 September 1976)

The constraints of unitarity and analyticity on four-body final state amplitudes are studied. It is shown that unitarity alone forces the amplitudes to be coherent and have rapid (singular) behavior. The implementation of unitarity yields a set of linear integral equations for the four-body amplitudes that are the minimal set consistent with quantum mechanics, but are also equivalent to the full dynamical equation with separable interactions.

[NUCLEAR REACTIONS Four-body final state interaction theory; development and implementation of unitarity and analyticity constraints.]

I. INTRODUCTION

In three recent papers (referred to as A1, A2, and A3) one of us (with Aaron)[1-3] showed that the elementary constraints of quantum mechanics—unitarity and analyticity—when applied to three-body final states in the quasiparticle picture, force singularity structure and interdependence of amplitudes usually taken as independent and constant in phenomenological analysis. It was further shown that implementation of these constraints along with some simple ideas about total energy analyticity leads to a set of integral equations for the three-body amplitudes that are essentially identical to the usual separable potential equations.

In this paper we turn to a corresponding analysis of the four-body problem. We formulate the four-body final state amplitudes in the language of the sequential decay or quasiparticle picture and apply unitarity to them, focusing particularly on two-body unitarity. We find that unitarity alone forces the amplitudes to vary over the phase space, be singular on its edge, and be interrelated. The unitarity relation itself can be used to determine the numerical importance of these effects in any problem. If they are important, they must be implemented by considering analyticity as well. Since we take only two-body unitarity contributions into account, the full implementation of unitarity is ambiguous. We choose among these ambiguities in such a way as to preserve the total energy analyticity of the amplitude as well. In this way we are led to an integral equation for the four-body amplitudes. We show that that equation is in fact a full dynamical scheme.[4] Hence, as in the three-body case, the implementation of the minimal constraints of quantum mechanics leads all the way back to a full dynamical equation. Unfortunately, in the four-body case that equation is in two variables and is difficult to solve. It is important, therefore, to use the unitarity relations to test the numerical importance of the subenergy variation in four-body applications.

From a practical point of view this work permits both a study of when unitarity effects are important in a particular case and also points the way to incorporating these effects—although the way is difficult. In the three-body case we find situations where the unitarity corrections are crucial,[5] where they are significant,[6] and where they are negligible.[7] No doubt a similar spectrum of four-body examples exists and this paper is a first step toward providing a framework to examine them. As in the three-body problem, it is also probable that systematic examination of four-body examples will lead to useful approximation techniques.

In Sec. II we review our conventions for unitarity and in Sec. III we derive the unitarity constraints for the four-body amplitudes. In Sec. IV we discuss their implementation and their relation to dynamics, stressing the importance of the "arbitrary" choices that are made there. In Sec. V we summarize our results and discuss possible applications.

II. UNITARITY

Before applying unitarity to the four-body system, we review some of our conventions and definitions. (For a more complete review, see Ref. 1.) We define the S matrix and T matrix by

$$S = 1 - 2\pi i\,\delta(E)T\,, \tag{1}$$

so that

$$\mathrm{Im}\langle\alpha|T|\beta\rangle = -\pi\sum_{\gamma}\langle\alpha|T|\gamma\rangle\langle\gamma|T|\beta\rangle^{*}\delta(E_\alpha - E_\gamma) \tag{2}$$

as long as $E_\alpha = E_\beta$. For the two-body t matrix we write (neglecting spin, isospin, etc.)

$$\langle\vec{p}_1,\vec{p}_2|T|\vec{p}_1{}',\vec{p}_2{}'\rangle = (2\pi)^3\delta(\vec{p}_1+\vec{p}_2-\vec{p}_1{}'-\vec{p}_2{}')\langle\vec{q}_{12}|\tau|\vec{q}_{12}{}'\rangle\,, \tag{3}$$

where $(m_1+m_2)q_{12} = m_2 p_1 - m_1 p_2$. Decomposing in partial waves gives

$$\langle\vec{q}|\tau|\vec{q}\,'\rangle = \sum_{l,m} Y_{lm}(\hat{q})\tau_l(q^2/2\mu)Y^*_{lm}(\hat{q}')\,, \tag{4}$$

where $\hat{q}$ is a unit vector and $\mu(m_1+m_2) = m_1 m_2$. Unitarity (2) gives

$$\mathrm{Im}\tau_l(E) = \frac{-\mu}{8\pi^2}q\,|\tau_l(E)|^2\,. \tag{5}$$

If we write (suppressing l) $\tau = N/D$, Eq. (5) becomes

$$\mathrm{Im}D(E) = \frac{\mu q}{8\pi^2}N(E)\,. \tag{6}$$

III. FOUR-BODY UNITARITY

To derive the constraints on a four-particle amplitude required by two-body subenergy unitarity, consider an amplitude $T_{2,4}$ for going from a stable state of two particles to a four-body state. Assuming only two-, three-, and four-body intermediate states are energetically allowed, unitarity for $T_{2,4}$ can be written

$$\begin{aligned}\mathrm{Im}T_{2,4} = &-\pi\sum_{2'} T_{2,2'}\delta(E-E_{2'})T^*_{2',4}\\ &-\pi\sum_{3'} T_{2,3'}\delta(E-E_{3'})T^*_{3',4}\\ &-\pi\sum_{4'} T_{2,4'}\delta(E-E_{4'})T^*_{4',4}\,.\end{aligned} \tag{7}$$

Contributions to the pair subenergy discontinuity of $T_{2,4}$ will come from the disconnected parts of unitarity and hence we decompose the amplitudes in Eq. (7) into disconnected and totally connected parts. Equation (7) and this decomposition are shown schematically in Fig. 1. The last term in Fig. 1 clearly gives the discontinuity across the pair subenergy branch cut since it has the appropriate threshold.[8] The next to last term in the figure also appears to give a pair discontinuity, but it is easy to see that in this term a given pair threshold depends on the energy of the other pair and hence this term does not contribute to the pair subenergy cut. We shall see later how, by implementing the unitarity constraint with analyticity, we arrive at an amplitude that satisfies all of

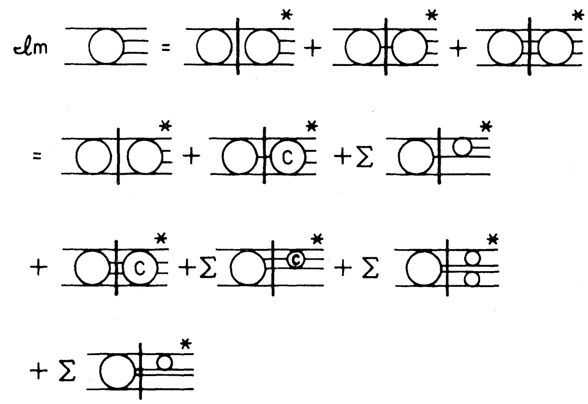

FIG. 1. Schematic representation of the unitarity relation of Eq. (2). The second line shows the amplitudes decomposed into fully connected (represented by a C) and disconnected parts.

unitarity and in particular the constraint implied by this next to last term. It is clear from a consideration of thresholds that no other term in Fig. 1 contributes to the pair discontinuity.

If we keep only the last term on the right in Fig. 1 we no longer have the imaginary part of $T_{2,4}$, but only its discontinuity across the pair subenergy cut. We call this Disc$T_{2,4}$, and write schematically

$$\mathrm{Disc}T_{2,4} = -\pi\sum T_{2,4'}\delta(E-E')T^*_{2',2}\delta\delta\,, \tag{8}$$

where $T_{2,2}$ is the two-body amplitude and the δ's represent the two "fly-by" particles. To proceed further we postulate a simple form for $T_{2,4}$ based on a similar form used in the three-body case. This form is suggested by the sequential decay or quasiparticle models of nuclear physics and the isobar model of particle physics. We assume the four-body final state is dominated by pair interactions and that these are in turn each dominated by a particular important partial wave or quasiparticle state. The four-body state is then due to a sum of terms in which first a pair quasiparticle is produced (along with two other particles) and it subsequently propagates and decays. This propagation is given by the appropriate pair D function while the decay is proportional to the pair vertex function, the square of which is the two-body N function. All this is as in the three-body case. We then have for a reaction of two particles of relative momentum $\vec{k}$ going (in the center of mass) to four particles of momentum $\vec{p}_i$ at total energy E,

$$\langle\vec{k}|T_{2,4}(E)|\vec{p}_1,\vec{p}_2,\vec{p}_3,\vec{p}_4\rangle = \tfrac{1}{4}(2\pi)^3\delta(\vec{p}_1+\vec{p}_2+\vec{p}_3+\vec{p}_4)\sum_{ijkl}\frac{\langle\vec{k}|F_{kl}(E)|\vec{p}_i,\vec{p}_j\rangle v_{kl}(q_{kl}{}^2)}{D_{kl}(q_{kl}{}^2)(4\pi)^{1/2}}\,, \tag{9}$$

where we have assumed that all pair interactions are in s waves and that the particles do not carry spin, isospin, etc., in order to reduce the complexity of the equation. Following the form for the three-body case given in A1, it is easy to remove those restrictions if need be. v_{kl} is the vertex or penetrability factor for the kl pair related to N of Eq. (6) by $N=v^2$. D is the D function or propagator of the pair defined in Eq. (6). F is defined by Eq. (9) but is clearly the quasi-three-body amplitude for going from the initial state to a state of the correlated pair kl and particles i and j with momentum $\vec{p}_i$ and $\vec{p}_j$. The factor of $\frac{1}{4}$ in front accounts for the fact that $D_{kl}=D_{lk}$, $v_{kl}=v_{lk}$, and $\langle\vec{k}|F_{kl}|\vec{p}_i,\vec{p}_j\rangle=\langle\vec{k}|F_{kl}|\vec{p}_j,\vec{p}_i\rangle$; the $(4\pi)^{-1/2}$ is simply Y_{00} and is there to agree with the conventions of A1. Other forms for the quasiparticle amplitude involving the full two-body t matrix or the N function rather than v/D as used here have been considered in the three-body case.[2] The various problems they cause, primarily with total energy analyticity when implementing unitarity, lead us to consider only this form here. We now substitute Eq. (9) into Eq. (8) to obtain

$$\frac{(2\pi)^3\delta(\sum\vec{p}_i)}{4(4\pi)^{1/2}}\sum_{ijkl}\Bigg[\operatorname{Disc}\frac{\langle\vec{k}|F_{kl}(E)|\vec{p}_i,\vec{p}_j\rangle v_{kl}(q_{kl})}{D_{kl}(q_{kl}^2)}$$
$$=-\frac{\pi}{4}\int\sum_{abcd}\frac{\langle\vec{k}|F_{cd}(E)|\vec{p}_a',\vec{p}_b'\rangle}{D_{cd}(q_{cd}'^2)}v_{cd}(q_{cd}')\,\delta(\vec{p}_i-\vec{p}_i')\,\delta(\vec{p}_j-\vec{p}_j')\,\delta(\vec{p}_k+\vec{p}_l-\vec{p}_k'-\vec{p}_l')\,\delta\Big(E-\sum\frac{p_i'^2}{2m_i}\Big)$$
$$\times\frac{1}{(2\pi)^3}\Big(\prod_{i=1}^{4}d^3p_i'\Big)\frac{v_{kl}^2(q_{kl})}{4\pi D_{kl}^*(q_{kl}^2)}\Bigg]\,, \tag{10}$$

where the extra factor of $\frac{1}{4}$ on the right comes from the symmetry under interchange of the pair ij and kl as well as ab and cd. Using $\operatorname{Disc}(F/D)=F\operatorname{Disc}(1/D)+(1/D^*)\operatorname{Disc}F$ as in the three-body case,[1] noting that $\operatorname{Disc}(1/D)=\operatorname{Im}(1/D)$, and using two-body unitarity for $\operatorname{Im}(1/D)$, Eq. (6), we find the $\operatorname{Im}(1/D)$ terms on the left just cancel the terms with $(ab)=(ij)$ and $(cd)=(kl)$ on the right. Equating the appropriate coefficients, taking account of symmetries, and canceling common factors, then gives

$$\operatorname{Disc}\langle\vec{k}|F_{kl}(E)|\vec{p}_i,\vec{p}_j\rangle=\frac{1}{4}\int\frac{d^3p_k'\,d^3p_l'}{(2\pi)^3}\,\delta(\vec{p}_k+\vec{p}_l-\vec{p}_k'-\vec{p}_l')\,\delta\Big(\frac{p_k^2-p_k'^2}{2m_k}+\frac{p_l^2-p_l'^2}{2m_l}\Big)v_{kl}(q_{kl})$$
$$\times\Bigg[\frac{\langle\vec{k}|F_{ij}(E)|\vec{p}_k',\vec{p}_l'\rangle v_{ij}(q_{ij})}{D_{ij}(q_{ij})}+\frac{\langle\vec{k}|F_{jl}(E)|\vec{p}_i,\vec{p}_k'\rangle v_{jl}(q_{jl}')}{D_{jl}(q_{jl})}+\frac{\langle\vec{k}|F_{jk}(E)|\vec{p}_i,\vec{p}_l'\rangle v_{jk}(q_{jk}')}{D_{jk}(q_{jk}')}$$
$$+\frac{\langle\vec{k}|F_{il}(E)|\vec{p}_j,\vec{p}_k'\rangle v_{il}(q_{il}')}{D_{il}(q_{il}')}+\frac{\langle\vec{k}|F_{ik}(E)|\vec{p}_j,\vec{p}_l'\rangle v_{ik}(q_{ik}')}{D_{ik}(q_{ik}')}\Bigg]\,. \tag{11}$$

This equation is represented graphically in Fig. 2. For the special case of identical particles with $2m=1$, this simplifies to

$$\operatorname{Disc}\langle\vec{k}|F(E)|\vec{p},\vec{p}'\rangle=-\frac{1}{4}\int\frac{d^3p''}{(2\pi)^3}\,\delta(E-p^2-p'^2-p''^2-(\vec{p}+\vec{p}'+\vec{p}'')^2)$$
$$\times\Bigg[2\Bigg(\frac{\langle\vec{k}|F(E)|\vec{p},\vec{p}''\rangle v(\vec{p}'+\frac{1}{2}(\vec{p}+\vec{p}''))v(\vec{p}''+\frac{1}{2}(\vec{p}+\vec{p}'))}{D(E-p^2-p''^2-\frac{1}{2}(\vec{p}+\vec{p}'')^2)}$$
$$+\frac{\langle\vec{k}|F(E)|\vec{p}',\vec{p}''\rangle v(\vec{p}+\frac{1}{2}(\vec{p}'+\vec{p}''))v(\vec{p}''+\frac{1}{2}(\vec{p}+\vec{p}'))}{D(E-p'^2-p''^2-\frac{1}{2}(\vec{p}'+\vec{p}'')^2)}\Bigg)$$
$$+\frac{\langle\vec{k}|F(E)|\mathrm{p}'',-(\vec{p}+\vec{p}'+\vec{p}'')\rangle v(\vec{p}''+\frac{1}{2}(\vec{p}+\vec{p}'))v(\frac{1}{2}(\vec{p}-\vec{p}'))}{D(E-\frac{3}{2}p''^2-\frac{3}{2}(\vec{p}+\vec{p}'+\vec{p}'')^2+\vec{p}''\cdot(\vec{p}''+\vec{p}'+\vec{p}))}\Bigg]\,. \tag{12}$$

As in the three-body case, these unitarity relations show that the constraints of quantum mechanics introduce interrelations and variation on amplitudes usually taken as independent and constant in phenomenological analysis. In particular these amplitudes have a branch point at the edge of phase space. It is easy to see that, just as we expect from a two-body cut, and just as occurs in the three-body case,[3] this is a square root branch point. The position of the branch point makes it

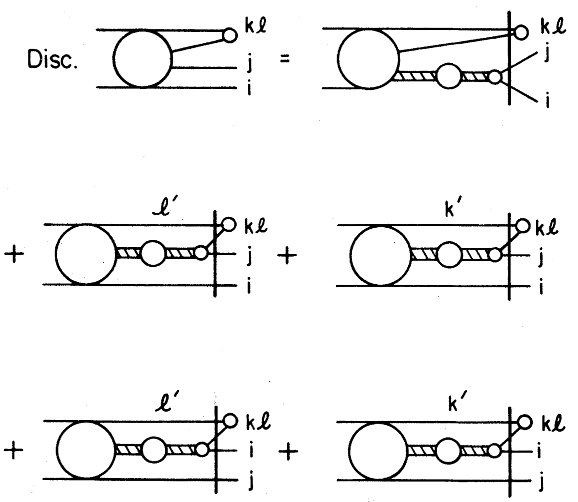

FIG. 2. Schematic representation of Eq. (11), the vertical lines represent propagators replaced by δ functions.

particularly important for threshold enhancement. A test of the importance of the unitarity constraint in a particular problem is easily made using Eq. (11). If the F's are assumed constant, the right-hand side of Eq. (11) or Eq. (12) is calculated. If it generates a small discontinuity for F (measured against the assumed scale of F itself) the assumption of constant F is a good approximation. If Disc$F$ is large by the scale of F and by the degree of accuracy required for the phenomenology, the constraints of unitarity cannot be ignored. We turn now to their implementation.

IV. IMPLEMENTATION

In the last section we saw that unitarity alone forces the quasi-three-body amplitude F to have important (singular) dependence in two-particle pair subenergy. This is not surprising if we look back into the corresponding analysis of three-body final states. The next question is then how to implement unitarity in any four-particle final state phenomenology.

In order to be able to implement the conditions of unitarity let us examine the analytic structure of $\langle \vec{k} | F(E) | \vec{p}, \vec{q} \rangle$ for the identical particle case. It has a simple square root branch cut in two-particle pair subenergy $\sigma = E - \frac{3}{2}p^2 - \frac{3}{2}q^2 - \vec{p}\cdot\vec{q}$ in the interval $0 < \sigma < \infty$. In general, it also has left-hand cuts corresponding to the vertex functions. It also has more complicated branch cuts in the three-particle subenergies (these three branch cuts are in three different variables), and of course it has total energy singularities corresponding to the four-body thresholds, and other lower thresholds. The best way to implement Eq. (12) in order that it gives us information about the discontinuity across the $\sigma = E - \frac{3}{2}p^2 - \frac{3}{2}q^2 - \vec{p}\cdot\vec{q}$ cut for fixed p^2, q^2, and $\vec{p}\cdot\vec{q}$ is to disperse in E as in A2 and A3. Because the essential features of the discontinuity of Eq. (12) are two simple δ functions in E, it is trivial to disperse in E.

We write the dispersion relation for $F(E)$ in schematic partial wave form. If we assume $F(E)$ goes to zero sufficiently rapidly as $E \to \infty$ we can write a dispersion relation for partial wave $F(E)$ as

$$F(E) = R(E) + \frac{1}{\pi}\int \frac{dE'}{E' - E}\,\mathrm{Disc}F(E')\,, \tag{13}$$

where $R(E)$ is a term that does not have the discontinuity. Schematically, let us assume that

$$\mathrm{Disc}F(E') = \pi f(E')\,\delta(E' - E_0)\,. \tag{14}$$

Hence we have from Eq. (13) that

$$\begin{aligned} F(E) &= R(E) + \int \frac{dE'}{E' - E} f(E')\,\delta(E' - E_0) \\ &= R(E) + \frac{f(E_0) + \mathfrak{F}(E, E_0)}{E_0 - E}\,, \end{aligned} \tag{15}$$

where $\mathfrak{F}(E_0, E_0) = 0$, and hence does not contribute to the discontinuity of $F(E)$. $\mathfrak{F}$ is arbitrary except for this condition, and could be included in the definition of R, but it is more convenient (as was discussed in Refs. 2 and 3) to keep it explicitly. The dispersion integral essentially puts the argument of the δ function in the denominator and the numerator becomes as shown in Eq. (15).

From this it is clear that if we disperse the discontinuity in Eq. (12) in E, we get

$$\begin{aligned} \langle \vec{k} | F(E) | \vec{p}, \vec{p}' \rangle = {} & \langle \vec{k} | R(E) | \vec{p}, \vec{p}' \rangle \\ & + \int \frac{d^3p''}{(2\pi)^4} \frac{v(2(\vec{p}'' + \frac{1}{2}(\vec{p} + \vec{p}')))}{E - p^2 - p'^2 - p''^2 - (\vec{p} + \vec{p}' + \vec{p}'')^2} \left[\frac{\langle \vec{k} | F(E) | \vec{p}, \vec{p}'' \rangle v(\vec{p} + \frac{1}{2}(\vec{p}'' + \vec{p}))}{D(E - \frac{3}{2}p^2 - \frac{3}{2}p''^2 - \vec{p}\cdot\vec{p}'')} \right. \\ & + \frac{\langle \vec{k} | F(E) | \vec{p}', \vec{p}'' \rangle v(\vec{p} + \frac{1}{2}(\vec{p}' + \vec{p}''))}{D(E - \frac{3}{2}p''^2 - \frac{3}{2}p'^2 - \vec{p}'\cdot\vec{p}'')} \\ & \left. + \frac{\frac{1}{2}\langle \vec{k} | F(E) | \vec{p}'', -\vec{p} + \vec{p}' + \vec{p}'' \rangle v(\frac{1}{2}(\vec{p} - \vec{p}'))}{D(E - \frac{3}{2}p''^2 - \frac{3}{2}(\vec{p} + \vec{p}' + \vec{p}'')^2 + \vec{p}''\cdot(\vec{p}'' + \vec{p}' + \vec{p}))} \right]. \end{aligned} \tag{16}$$

In the first term in Eq. (16) we chose

$$\mathfrak{F}(E,E_0)=\left[\frac{\langle \vec{k}|F(E)|\vec{p},\vec{p}''\rangle}{D(E-\frac{3}{2}p^2-\frac{3}{2}p''^2-\vec{p}\cdot\vec{p}'')}-\frac{\langle \vec{k}|F(E_0)|\vec{p},\vec{p}''\rangle}{D(E_0-\frac{3}{2}p^2-\frac{3}{2}p''^2-\vec{p}\cdot\vec{p}'')}\right]v(\vec{p}''+\tfrac{1}{2}(\vec{p}+\vec{p}'))\,v(\vec{p}'+\tfrac{1}{2}(\vec{p}''+\vec{p}))\,, \quad (17)$$

apart from factors of π and the phase space integral. Here

$$E_0=p^2+p'^2+p''^2+(\vec{p}+\vec{p}'+\vec{p}'')^2\,. \quad (18)$$

This choice is motivated by the fact that in the final integral equation we wish to get D and F as functions of E, while the v's should not depend on E. We write v's as functions of momentum and not as functions of E because then the left-hand cuts corresponding to v do not get involved in the p' integration of Eq. (16) when E is kept fixed. These choices are of course all equivalent at the pole so that the discontinuity is independent of this choice. This ambiguity and our form for resolving it is discussed in much greater detail in the three-body case (A2 and A3). $\mathfrak{F}$ is chosen similarly in the other three terms. This is straightforward and we do not give expressions for $\mathfrak{F}$; we stress that any choice of $\mathfrak{F}$ subject to the condition $\mathfrak{F}(E_0,E_0)=0$ will lead to an F that satisfies two-body subenergy unitarity [Eq. (8)] and analyticity [Eq. (13)], but our particular choice also satisfies total energy analyticity. Another reason for this particular choice is that we then get a set of dynamical equations for F and not just an integral representation for F.

Equation (16) is the "minimal" implementation of subenergy unitarity and analyticity, provides a useful phenomenology, and does not contain any spurious singularities in E. The equation is arbitrary to the extent of choice of $\mathfrak{F}$. But the simple choice of $\mathfrak{F}$ gives a set of equations which is the full dynamical set for the four-particle system.[4] It has been shown in the three-particle case that by a prescription similar to this we get the full dynamical equation for the three-particle system. We have obtained this set of equations with two-body subenergy unitarity and analyticity and the assumption of separable interaction in disguise. The assumption of separable interaction comes through the introduction of v's. This assumption gives a set of equations in F free from spurious E singularities.

In order to understand the content of Eq. (16) we make some assumption about $\langle \vec{k}|R(E)|\vec{p},\vec{p}'\rangle$ and use the iteration scheme. We take the case of a decay to four particles with a simple Born term corresponding to R where one particle decays to two free particles and a quasiparticle state of two particles. We show the diagrammatic representation of Eq. (16) in Fig. 3(a). Here the crossed double line represents the quasiparticle states of two particles and the circle the propagator (D) of that line. Figure 3(b) shows the terms we get after the first iteration of Eq. (16). It is clear that if we make more iterations, we expect three types of terms. First we have a totally connected four-body amplitude. Secondly there will be terms where we have a four-body amplitude times a totally connected three-body amplitude, and finally we have terms where we have a four-body amplitude times the amplitude for two independent pairs interacting. Typical diagrammatic representations of these three terms are shown in Figs. 4(a)–4(c). It is easy to see that all the terms we get out of iterations of the equation in Fig. 3 will fall into one of these classes. It is also clear that this set is in fact a full dynamical scheme corresponding to starting with separable potentials in the Schrödinger equation.[4] We note in particular that the sum of terms of the kind shown in Fig. 4(c) is just what is needed to satisfy the term in unitarity represented by the next to last term in Fig. 1. It is important to stress that starting from two-body subenergy discontinuity and analyticity we derive this set of dynamical equations for four particles.

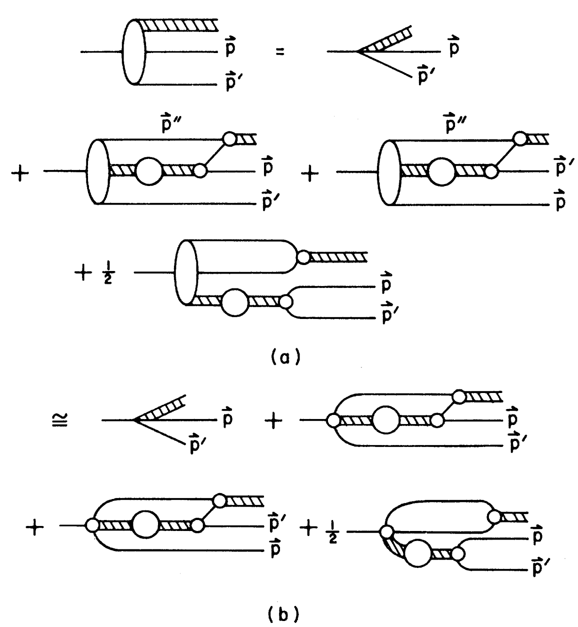

FIG. 3. (a), Schematic representation of Eq. (16). (b) Representation of the first iteration of Eq. (16).

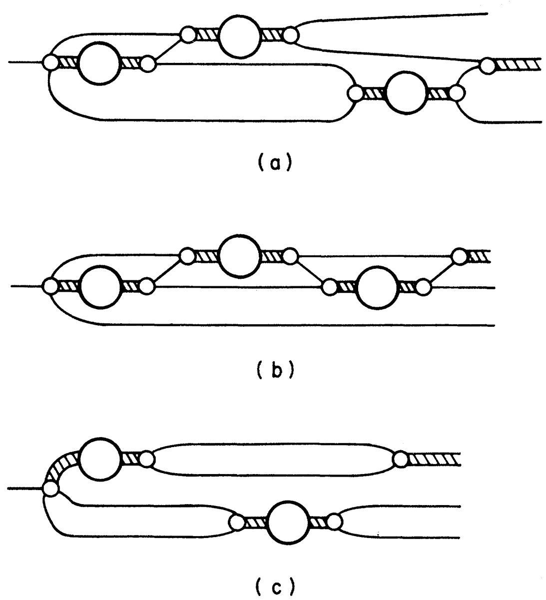

FIG. 4. Schematic representation of each of the types of terms generated by iterations of Eq. (16): (a) connected four-particle terms; (b) connected three-particle terms; (c) independent pair interactions.

V. SUMMARY AND APPLICATION

We have considered the effects of unitarity, particularly two-body unitarity, on four-body final states. We have seen that in the framework of a sequential decay, quasiparticle, or isobar model, where the four-body state is assumed to be formed by decay of two-body correlated pairs or resonances, the amplitudes for forming these pairs though usually taken as constant (up to kinematical factors) in phenomenology in fact have square root branch points, and the discontinuity associated with the branch cut for a given pair is linearly related to the amplitudes for producing other pairs. This is a manifestation for four particles of the "coherence" of the full final state amplitude. The unitarity relation permits a quantitative measure of the importance of the singular part of the amplitude and hence permits one to determine in a particular case if neglect of the unitarity effect is justified. If it is not, unitarity can be implemented by writing a dispersion relation for the amplitude in terms of its discontinuity. This gives a set of coupled linear integral equations for the pair amplitudes. These equations form the minimal set required to satisfy unitarity and analyticity. They are also, as are the corresponding three-body minimal set, equivalent to the full dynamical equations with separable interactions. Since they are equations in two vector variables, even after partial wave decomposition, they are very complicated to solve. (The variable complication cannot be simplified by solving first the three-body parts unless the three-body solutions are themselves separable.[9]) Hence the equations are hard to solve. It is certainly interesting that the full dynamical scheme emerges from the limited consideration of only unitarity and analyticity, but it is also sobering that the only way to implement these constraints fully is to solve the very complicated dynamical equation that results. It may be that as particular situations are considered and as simple cases are attacked, useful approximation schemes will emerge as they have to some extent in the three-body case, but for the present the full dynamics is all we have if the unitarity constraints are important.

We expect these effects to be important for two classes of strongly overlapping final state interactions, threshold enhancements as one encounters in the nucleon-nucleon system, particularly for $S=0$ pairs, and resonance interactions. Thus these effects could be important in reactions such as $d+d\rightarrow 2n+2p$, and other similar reactions leading to four nucleons in the final state or in reactions leading to a final state such as $2n+p+\alpha$ where the nucleon-α resonances play a role. In nuclear physics one can construct hundreds of examples, but it remains to test them in unitarity through Eq. (11) to see if the full machinery of unitarity and analyticity is important. In medium energy physics, and in particle physics, with pions, the number of possibilities is even more interesting. We plan to present results for the relativistic problem separately.[10]

One of us (RDA) is grateful to the Physics Department of the Federal University of Pernambuco and particularly to Professor H. Coelho for the hospitality extended to him during this work.

---

*Supported in part by the National Research Council (Brazil) and the National Science Foundation (United States).
†Permanent address.
[1]R. D. Amado, Phys. Rev. C 11, 719 (1975), referred to as A1.
[2]R. D. Amado, Phys. Rev. C 12, 1354 (1975), referred to as A2.
[3]R. Aaron and R. D. Amado, Phys. Rev. D 13, 2581 (1976), referred to as A3.
[4]C.f. V. V. Komarov and A. M. Popova, Nucl. Phys. 69, 253 (1965); A90, 635 (1967); and P. Grassberger and W. Sandhas, Nucl. Phys. B2, 181 (1967).

[5]C.f. S. K. Adhikari and R. D. Amado, Phys. Rev. D 9, 1467 (1974).
[6]C.f. R. Aaron, R. H. Thompson, R. D. Amado, R. A. Arndt, D. C. Teplitz, and V. L. Teplitz, Phys. Rev. D 12, 1984 (1975).
[7]T. Takahashi (unpublished).
[8]See A1; A3; and also R. Eden, P. V. Landshoff, D. Olive, and J. C. Polkinghorne, *An Analytic S Matrix* (Cambridge U.P., Cambridge, 1966).
[9]C.f. E. O. Alt, P. Grassberger, and W. Sandhas, Phys. Rev. C 1, 85 (1970).
[10]S. K. Adhikari (unpublished).