PHYSICAL REVIEW C VOLUME 15, NUMBER 1 JANUARY 1977

K-matrix formalisms in multiparticle scattering*

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(Received 23 August 1976)

The Kouri-Levin K-matrix formalism for multiparticle scattering is examined in the particular case of three-particle scattering. It is shown that in order for this method to generate transition amplitudes satisfying unitarity, much more stringent constraints must be imposed upon the K operators in addition to the zero discontinuity condition. The imposition of these constraints seriously limits the usefulness of this formalism as a unitarization technique; this conclusion holds for multiparticle scattering as well. The relationship of the Kouri-Levin method to other realizations of the K-matrix idea is studied. In the three-particle case it is found that the Kouri-Levin method is not preferable on grounds of simplicity to any of the extant techniques even ignoring the constraints required in addition to the zero discontinuity condition.

[NUCLEAR REACTIONS Channel coupling array theory, K operators, unitary approximations.]

I. INTRODUCTION

Approximation methods for multiparticle scattering processes abound although most of these have been formulated and justified mainly upon intuitive grounds. The availability in recent years of several sets of well-defined integral equations for nonrelativistic N-particle scattering[1] presents the opportunity of developing approximation procedures in more systematic ways than had been possible previously.

Among this new generation of relatively well-founded techniques is the K-matrix formalism developed by Kouri and Levin (KL) in connection with their interesting and unified method for generating connected-kernel multiparticle scattering integral equations.[2] The essential features of the KL approach are independent of particle number. Therefore, the three-particle realization, which was recently developed in detail by Kouri, Levin, and Sandhas (KLS),[3] serves as an instructive example of the suitability of this K-matrix approach to arbitrary scattering processes.

The work of Refs. 2 and 3 reveals that the use of the channel coupling array trick[4] for formulating scattering integral equations leads to a K-matrix formalism which is quite distinct in several respects from those based upon a Faddeev-type approach.[5-9] The primary difference, however, is that the latter are completely well-defined unitarization techniques while the former, as originally proposed, is not.

It is stated in Refs. 2 and 3 that the introduction of K operators with zero discontinuities across the unitary cut(s) will, by means of an appropriate set of Heitler (damping) equations, generate a set of scattering amplitudes satisfying the correct multiparticle discontinuity relations and, presumably, thereby multiparticle unitarity. We find that without the imposition of further and nontrivial constraints upon these K operators both of these statements are false. The reason for this in the three-particle case is the fact that above the breakup threshold those scattering amplitudes which correspond to processes with three-free particles in the initial and/or final states do not have direct identifications with the T operators generated by the KL Heitler equations. Similar difficulties evidently realize in the general multiparticle case as well.

Apart from the preceding, we also establish that the apparent simplicity[3] of the KL K-matrix approach relative to other methods[6-9] is illusory. Moreover, the imposition of the required constraints upon the K operator will, in general, increase the complexity even further.

In Sec. II we develop the coupled channel array formalism in a somewhat different manner than in Refs. 2 and 3. This is done to facilitate the detailed discussion of the KL unitary program in Sec. III. Finally, in Sec. IV we compare the diverse K-matrix approaches which have been proposed for multiparticle scattering.

II. COUPLED CHANNEL ARRAY EQUATIONS

Of all the various formulations of three-particle scattering the one proposed by Alt, Grassberger, and Sandhas (AGS)[10] possesses the most symmetrical structure. This attribute is singularly useful for discussions involving unitarity and to this end it is convenient to rederive the work of KL[1,2,11]

in AGS form.

The AGS scattering operators $U_{\beta\alpha}(z)$ are defined in terms of the full Green function,

$$G(z) = (z - H)^{-1},$$

and the channel Green functions,

$$G_\alpha(z) = (z - H_\alpha)^{-1},$$

by

$$G(z) = G_\beta(z)\,\delta_{\beta\alpha} + G_\beta(z)\,U_{\beta\alpha}(z)\,G_\alpha(z) \tag{2.1}$$

for $\alpha = 0, 1, 2,$ and 3, where the channel index notation is standard[6,8,10,12] and z is a complex parameter. The complete Hamiltonian H can be decomposed in terms of the channel Hamiltonians H_α and the residual interactions V^α, such that

$$H = H_\alpha + V^\alpha$$

which holds for all α. For the sake of simplicity we confine ourselves only to pairwise interactions V_α with $V_0 = 0$. Then

$$H_\alpha = H_0 + V_\alpha,$$

where H_0 is the kinetic energy operator,

$$V^\alpha = \sum_\gamma \bar{\delta}_{\alpha\gamma} V_\gamma$$

and $\bar{\delta}_{\alpha\beta} = 1 - \delta_{\alpha\beta}$.

Equations (2.1) for all α, β imply that the $U_{\beta\alpha}(z)$ satisfy the AGS integral equations:

$$U_{\beta\alpha}(z) = \bar{\delta}_{\beta\alpha} G_0(z)^{-1} + \sum_\gamma \bar{\delta}_{\beta\gamma} V_\gamma G_\gamma(z)\,U_{\gamma\alpha}(z) \tag{2.2a}$$

$$= \bar{\delta}_{\beta\alpha} G_0(z)^{-1} + \sum_\gamma U_{\beta\gamma}(z)\,G_\gamma(z)\,V_\gamma \bar{\delta}_{\gamma\alpha}. \tag{2.2b}$$

It is clear that the zero-indexed operators[13] $U_{0\alpha}$ and $U_{\beta 0}$, $\alpha, \beta \neq 0$, can be regarded either as dependent quantities derived from the $U_{\beta\alpha}$ with $\alpha, \beta \neq 0$ or as autonomous objects defined by integral equations. Both options are useful. Note, however, that U_{00} is defined entirely in terms of the other U's. The following identities are useful in this connection:

$$U_{0\alpha} = (\bar{\delta}_{0\alpha} - \bar{\delta}_{\lambda\alpha})\,G_0^{-1} + f_\lambda^L U_{\lambda\alpha}, \tag{2.3a}$$

$$U_{\beta 0} = (\bar{\delta}_{\beta 0} - \bar{\delta}_{\beta\lambda})\,G_0^{-1} + U_{\beta\lambda} f_\lambda^R \tag{2.3b}$$

which hold for all α, β, and λ, where

$$f_\lambda^L \equiv (1 + V_\lambda G_\lambda),$$

$$f_\lambda^R \equiv (1 + G_\lambda V_\lambda).$$

In particular, we note that

$$U_{00} = f_\lambda^L U_{\lambda 0} - \bar{\delta}_{\lambda 0} G_0^{-1} \tag{2.4a}$$

$$= U_{0\lambda} f_\lambda^R - \bar{\delta}_{\lambda 0} G_0^{-1}. \tag{2.4b}$$

The G_0^{-1} terms in Eqs. (2.3) and (2.4) will make no contributions to the relevant on shell scattering amplitudes.

The limits of any operator, $\mathcal{O}(z)$, which is a function of z, on either side of the real z axis are denoted by

$$\mathcal{O}(\pm) = [\mathcal{O}(z)]_{z = E \pm i0},$$

where E is a real energy; the discontinuity across this axis is, therefore,

$$\Delta\mathcal{O} \equiv \mathcal{O}(+) - \mathcal{O}(-).$$

Since $G(z^*) = G(z)^\dagger$, we see from (2.1) that

$$U_{\beta\alpha}(z)^\dagger = U_{\alpha\beta}(z^*),$$

which means that the U's satisfy the Hermitian analyticity conditions

$$U_{\beta\alpha}(-) = U_{\alpha\beta}(+)^\dagger. \tag{2.5}$$

The on shell[14] discontinuity relations for the U's follow immediately from Eqs. (2.2):

$$\Delta U_{\beta\alpha} = -2i\pi \sum_{\gamma=0}^{3} U_{\beta\gamma}(\pm)\,D_\gamma\,U_{\gamma\alpha}(\mp), \tag{2.6}$$

where

$$D_\alpha \equiv \sum_{E'_\alpha, \eta_\alpha} |\,\phi_\alpha(\eta_\alpha, E'_\alpha)\rangle\,\delta(E - E'_\alpha)\,\langle\phi_\alpha(\eta_\alpha, E'_\alpha)\,|,$$

the channel states $|\,\phi_\alpha(\eta_\alpha, E_\alpha)\rangle$ are eigenstates of H_α and η_α refers to any other labels needed to specify the asymptotic configuration including the enumeration of the various bound states of the relevant pair of particles. Equations (2.5) in conjunction with Eqs. (2.6) imply the constraints imposed by unitarity on the scattering amplitudes. The fact that the discontinuity relations (2.6) alone do not suffice to ensure unitarity is highly pertinent to our subsequent discussion.

Next, let us introduce the coupled channel array[2-4,11] of real numbers $W_{\lambda\gamma}$ which satisfy

$$\sum_\gamma W_{\lambda\gamma} = 1.$$

Two distinct types of W arrays have been employed in the three-particle case.[2,3,11] In the first instance λ and γ are allowed to range over only 1 to 3 while in the second case this restriction is not imposed. Until indicated explicitly we do not distinguish between these two cases.

The α, β degeneracy in definition (2.1) can be exploited to derive integral equations for the U's distinct from (2.2). We find, depending upon the

use of the index λ on $W_{\lambda\gamma}$,[3,11] that

$$U_{\beta\alpha} = \bar{\delta}_{\beta\alpha} G_\alpha^{-1} + \sum_\gamma \bar{\delta}_{\beta\gamma} V_\gamma W_{\gamma\alpha} + \sum_{\gamma,\sigma} \bar{\delta}_{\beta\gamma} V_\gamma W_{\gamma\sigma} G_\sigma U_{\sigma\alpha} \tag{2.7a}$$

or

$$U_{\beta\alpha} = \bar{\delta}_{\beta\alpha} G_\alpha^{-1} + V^\beta W_{\lambda\alpha} + V^\beta \sum_\gamma W_{\lambda\gamma} G_\gamma U_{\gamma\alpha} . \tag{2.7b}$$

We assume, henceforth, that the arrays are such that Eqs. (2.7) are well-defined integral equations. The properties of such arrays have been studied in detail by Kouri, Levin, and Tobocman.[2,11]

In the case of (2.7a) the only acceptable choice for $W_{\lambda\gamma}$ is found to be[3,11]

$$W_{\lambda\gamma} = \delta_{\lambda\gamma} . \tag{2.8}$$

Using (2.8) Eqs. (2.7a) reduce to the AGS equations (2.2). For (2.7b) several choices of W are possible.[3,11] However, whatever the choice of the W array, provided Eqs. (2.7) are well-defined integral equations, the U's determined by the latter are precisely the same operators appearing in Eqs. (2.1) and (2.2). This is in contrast to the operators $T_2^{\beta\alpha}$ introduced by KLS[3] and KL[11] and defined below. The latter operators are different for each choice of a W array.

Let T_1 be the solution of

$$T_1 = \bar{\delta} V + \bar{\delta} V \hat{G} T_1 \tag{2.9a}$$

$$= \bar{\delta} V + T_1 \hat{G} \bar{\delta} V , \tag{2.9b}$$

where we have introduced a matrix notation with respect to the channel indices:

$$(T_1)_{\beta\alpha} = T_1^{\beta\alpha}, \quad (V)_{\beta\alpha} = V_\beta \delta_{\beta\alpha} ,$$

$$(\hat{G})_{\beta\alpha} = G_\beta \delta_{\beta\alpha}, \quad (\bar{\delta})_{\beta\alpha} = 1 - \delta_{\beta\alpha} .$$

In this notation the AGS equations become, e.g.,

$$U = \bar{\delta} G_0^{-1} + \bar{\delta} V \hat{G} U ,$$

so we see immediately that

$$T_1 = U G_0 V . \tag{2.10}$$

A somewhat more useful expression for U in terms of T_1 is

$$U = T_1 \hat{G} \bar{\delta} G_0^{-1} + \bar{\delta} G_0^{-1} . \tag{2.11}$$

From either (2.10) or (2.11) it is clear that U and T_1 are half-off-shell equivalent for those reactions initiating from $\alpha \neq 0$ channels. We note that all of the $T_1^{\beta\alpha}$ for $\alpha \neq 0$ correspond to scattering amplitudes and that $T_1^{\beta 0}$ vanishes for all β.

If we let

$$(\hat{V})_{\beta\alpha} = V^\beta \delta_{\beta\alpha} ,$$

$$(W)_{\beta\alpha} = W_{\lambda\alpha} ,$$

then Eqs. (2.7b) can be rewritten in the compact form

$$U = \bar{\delta} \hat{G}^{-1} + \hat{V} W + \hat{V} W \hat{G} U . \tag{2.7b'}$$

Corresponding to (2.7b') we define T_2 as the solution of

$$T_2 = \hat{V} W + \hat{V} W \hat{G} T_2 \tag{2.12a}$$

$$= \hat{V} W + T_2 \hat{G} \hat{V} W . \tag{2.12b}$$

Comparing Eqs. (2.7b') and (2.12) we see that

$$U = T_2 \hat{G} (1 + \bar{\delta}) \hat{G}^{-1} + \bar{\delta} \hat{G}^{-1} \tag{2.13}$$

or, in component form,

$$U_{\beta\alpha} = \sum_\gamma T_2^{\beta\gamma} G_\gamma G_\alpha^{-1} + \bar{\delta}_{\beta\alpha} G_\alpha^{-1} . \tag{2.13'}$$

It is obvious from Eqs. (2.13) that for all reactions initiating from a channel $\alpha \neq 0$, $U_{\beta\alpha}$ and $T_2^{\beta\alpha}$ are half-off-shell equivalent.

The operators $T_{1,2}$ satisfy the discontinuity relations

$$\Delta T_{1,2} = -2i\pi T_{1,2}(\pm) \hat{D} T_{1,2}(\mp) , \tag{2.14}$$

where

$$\hat{D}_\alpha = D_\alpha \bar{\delta}_{\alpha 0} + f_\alpha^R(\pm) D_0 f_\alpha^L(\mp) .$$

Unlike the discontinuity relations (2.6) for the U's, above the threshold for breakup (2.14) always involves off-shell matrix elements of the T's. Also, except on shell, the $T_{1,2}(\pm)$ do not satisfy a Hermitian analyticity relation. These two circumstances are distinctive features of the KL[2] unitary program.

III. KOURI-LEVIN UNITARY PROGRAM

Kouri and Levin[2] have proposed a unitary program for multiparticle scattering based upon the introduction of a K-matrix formalism; the detailed realization of this program in the case of three particles is considered in Ref. 3. However, the essential aspects of any unitary program which is based upon a set of generally approximate integral equations for the KL T operators consist of the following:

(1) There must exist definitive connections between the T's and the various scattering operators, i.e., those operators which on shell are to be identified with the appropriate scattering amplitudes;

(2) if (1) is assumed, then the off-shell discontinuity relations (2.14) must imply the satisfaction of the on-shell equations (2.6) for the scattering operators;

(3) the definitions of the scattering operators in terms of the T's must be such that the constraints

(2.5) are satisfied on shell.

Given the connections (2.11) and (2.13) between the U's and the T's and the stipulation that these operators satisfy their relevant (exact) integral equations, it is straightforward to show that conditions (1)–(3) are satisfied.[3] However, this is not the situation of interest relative to a unitary program. Rather, one is concerned with the circumstance where the T's are, in general, approximate. In this case the arguments of KLS[3] are inapplicable and the entire unitarity question must be reconsidered. It is convenient to do this separately for the two cases of the three- and four-coupled-channel arrays.

Case (i): Three channels

In this case the indices on $T^{\beta\alpha}$ and $W_{\lambda\gamma}$ run only from 1 to 3. Also it is unnecessary for us to differentiate between T_1 and T_2.

We identify

$$\langle\phi_\beta|\, U_{\beta\alpha}(\pm)\,|\,\phi_\alpha\rangle = \langle\phi_\beta|\, T^{\beta\alpha}(\pm)\,|\,\phi_\alpha\rangle \tag{3.1}$$

on shell for $\alpha,\beta\neq 0$. We also assume that the $T(\pm)$'s satisfy the discontinuity relations (2.14). There remains the question of defining the various zero-index scattering operators in terms of the T's.

When the U's and the T's are the exact solutions of their defining integral equations Eqs. (2.2) and Eqs. (2.9) and (2.12), respectively, it follows that

$$U_{0\alpha} = \hat{U}_{0\alpha}(\lambda) + f^L_\lambda\left[\bar{\delta}_{\lambda\alpha} + \sum_{\gamma=1}^{3} T^{\lambda\gamma} G_\gamma \bar{\delta}_{\gamma\alpha}\right] G_\alpha{}^{-1}, \qquad \alpha,\lambda\neq 0, \tag{3.2a}$$

$$U_{\beta 0} = G_0{}^{-1} + \sum_{\gamma=1}^{3} T^{\beta\gamma} f^R_\gamma, \quad \beta\neq 0, \tag{3.2b}$$

$$U_{00} = \sum_{\gamma=1}^{3} \hat{U}_{0\gamma}(\lambda) f^R_\gamma - G_0{}^{-1}, \tag{3.2c}$$

where

$$\hat{U}_{0\alpha}(\lambda) \equiv \delta_{\lambda\alpha} G_0{}^{-1} + f^L_\lambda T^{\lambda\alpha}, \quad \lambda,\alpha\neq 0. \tag{3.3}$$

Equations (3.2) and (3.3) imply that

$$f^L_\beta U_{\beta 0} = U_{00} + \bar{\delta}_{\beta 0} G_0{}^{-1} \tag{3.4}$$

which holds for all β; the validity of (3.4) is independent of whether or not the U's satisfy the AGS equations.

On shell we have

$$\langle\phi_0|\, U_{0\alpha}|\,\phi_\alpha\rangle = \langle\phi_0|\, \hat{U}_{0\alpha}(\lambda)\,|\,\phi_\alpha\rangle, \quad \alpha,\lambda\neq 0, \tag{3.5a}$$

$$\langle\phi_0|\, U_{0\alpha} f^R_\alpha|\,\phi_0\rangle = \langle\phi_0|\, \hat{U}_{0\alpha}(\lambda) f^R_\alpha|\,\phi_0\rangle, \quad \alpha,\lambda\neq 0, \tag{3.5b}$$

$$\langle\phi_\beta|\, U_{\beta 0}|\,\phi_0\rangle = \sum_{\gamma=1}^{3}\langle\phi_\beta|\, T^{\beta\gamma} f^R_\gamma|\,\phi_0\rangle, \quad \beta\neq 0, \tag{3.5c}$$

$$\langle\phi_0|\, U_{00}|\,\phi_0\rangle = \sum_{\gamma=1}^{3}\langle\phi_0|\, \hat{U}_{0\gamma}(\lambda) f^R_\gamma|\,\phi_0\rangle. \tag{3.5d}$$

It is evident that $\hat{U}_{0\alpha}(\lambda)$ can be regarded as independent of λ when it appears in on shell matrix elements of the form (3.5a) and (3.5b) and therefore in (3.5d).

Our subsequent work is simplified by the introduction of the operators:

$$T^{LR}(\pm) \equiv DT(\pm)D, \tag{3.6a}$$

$$T^{L}(\pm) \equiv f^L(\pm)\,T(\pm)D, \tag{3.6b}$$

$$T^{R}(\pm) \equiv DT(\pm)f^R(\pm), \tag{3.6c}$$

$$\tau(\pm) \equiv f^L(\pm)\,T(\pm)f^R(\pm). \tag{3.6d}$$

Then in accord with KLS[3] and Eqs. (3.1) and (3.5) we *identify* for the purposes of the unitary program:

$$U_{\beta\alpha}(\pm) \equiv T^{LR}_{\beta\alpha}(\pm), \quad \beta,\alpha\neq 0, \tag{3.7a}$$

$$U_{0\alpha}(\pm) \equiv T^{L}_{\alpha\alpha}(\pm), \quad \alpha\neq 0, \tag{3.7b}$$

$$U_{\beta 0}(\pm) \equiv \sum_{\gamma=1}^{3} T^{R}_{\beta\gamma}(\pm), \quad \beta\neq 0, \tag{3.7c}$$

$$U_{00}(\pm) \equiv \sum_{\gamma=1}^{3}[\tau_{\gamma\gamma}(\pm) + t_\gamma(\pm)], \tag{3.7d}$$

where t_γ is the two-particle transition operator

$$t_\gamma = V_\gamma + V_\gamma G_\gamma V_\gamma.$$

We see that the disconnected parts of $U_{00}(\pm)$ are correctly reproduced. A crucial part of the preceding identification prescription is the supposition that, as it appears in Eqs. (3.5), $\hat{U}_{0\alpha}(\lambda)$ is independent of λ. Thus, in writing Eqs. (3.7) we have simply set $\lambda=\alpha$ in which case

$$\hat{U}_{0\alpha} = f^L_\alpha T_{\alpha\alpha} + G_0{}^{-1}.$$

The presumed λ independence of $\hat{U}_{0\alpha}(\lambda)$ implies, however, the following constraints upon the T's:

$$T^L_{\lambda\alpha}(\pm) = \bar{\delta}_{\lambda\alpha} G_0(\pm)^{-1} D_\alpha + T^L_{\alpha\alpha}(\pm), \tag{3.8a}$$

$$\tau_{\lambda\alpha}(\pm) = \bar{\delta}_{\lambda\alpha}[G_0(\pm)^{-1} + t_\alpha(\pm)] + \tau_{\alpha\alpha}(\pm). \tag{3.8b}$$

Then it is easily deduced using Eqs. (3.7) and (3.8) that if Eqs. (2.14) are valid then so are Eqs. (2.6) for the on-shell matrix elements of the scattering operators defined by Eqs. (3.7). We remark that the $G_0{}^{-1}$ terms in (3.8b) and (3.8c) are not essential to this proof since these terms vanish on shell.

The Hermitian analyticity requirements (2.5) impose the additional constraints on the various matrix elements of the T's:

$$T^{LR}_{\beta\alpha}(\pm) = T^{LR}_{\alpha\beta}(\mp)^{\dagger}, \tag{3.9a}$$

$$T^{L}_{\alpha\alpha}(\pm) = \sum_{\gamma=1}^{3}[T^{R}_{\alpha\gamma}(\mp)]^{\dagger}, \tag{3.9b}$$

$$\sum_{\gamma=1}^{3}\tau_{\gamma\gamma}(\pm) = \sum_{\gamma=1}^{3}\tau_{\gamma\gamma}(\mp)^{\dagger}. \tag{3.9c}$$

The final step in the KL unitary program consists in the introduction of a K-matrix formalism for the operators defined by Eqs. (3.6). One of the appealing features of the KL formalism is the relative ease by which this can be done for the operators $T_{1,2}$. The latter satisfy Eqs. (2.9) and (2.12) both of which can be written in the generic forms

$$T = \mathcal{V} + \mathcal{V}\hat{G}T \tag{3.10a}$$

$$= \mathcal{V} + T\hat{G}\mathcal{V}. \tag{3.10b}$$

The Green function $\hat{G}(\pm)$ can be decomposed into its principal value, $\hat{G}^{P}$, and δ function, $\hat{D}$, parts:

$$\hat{G}(\pm) = \hat{G}^{P} \mp i\pi\hat{D}.$$

If we define K as the solution of Eqs. (3.10) with $\hat{G}$ replaced by $\hat{G}^{P}$ we see that $\Delta K = 0$ and in addition

$$T(\pm) = K \mp i\pi K\hat{D}T(\pm) \tag{3.11a}$$

$$= K \mp i\pi T(\pm)\hat{D}K. \tag{3.11b}$$

Obviously for any operators $K_{\beta\alpha}$ such that $\Delta K_{\beta\alpha} = 0$, Eqs. (3.11) generate T operators satisfying Eqs. (2.14). Unfortunately, that is not quite good enough since Eqs. (2.14) do not, by themselves, ensure unitarity.

The completion of the KL unitarity program consists in the stipulation that the $K_{\beta\alpha}$ be such that in addition to $\Delta K_{\beta\alpha} = 0$, the solutions of Eqs. (3.11) satisfy the constraints embodied in Eqs. (3.8) and (3.9). It is not at all obvious, however, what constraints, in turn, must be imposed upon the input $K_{\beta\alpha}$ in order to guarantee this. At this point, however, it is clear that the KL unitarity program is far less simple than it initially appears to be.

Case (ii): Four channels

We now let the indices on the T's and W's run from 0 to 3. There are a few minor differences in detail but the major conclusions concerning the unitary program are the same as in three-channel case.

The case of T_1 is entirely trivial since

$$T_1^{\beta 0} = 0$$

and

$$T_1^{0\alpha} = V_\alpha + \sum_{\gamma=1}^{3} V_\gamma G_\gamma T_1^{\gamma\alpha}$$

$$= V_\alpha + \sum_{\gamma=1}^{3} T_1^{0\gamma} G_\gamma \bar{\delta}_{\gamma\alpha} V_\alpha$$

so that the $T_1^{\beta\alpha}$ for $\alpha,\beta \neq 0$ are uncoupled from the zero index T_1 operators and as a consequence the former are precisely the same as in the three-channel case. Thus, the extension to four channels in this instance is trivial. We remark, however, that

$$T_1^{0\alpha} = \delta_{\lambda\alpha} V_\alpha + f^{L}_{\gamma} T_1^{\lambda\alpha}$$

which for all purposes of the unitary program is identical to $\hat{U}_{0\alpha}(\lambda)$.

The extension to four channels does lead to something new in the case of T_2. Here all the components $T_2^{\beta\alpha}$ are coupled and for $\alpha, \beta \neq 0$ these operators are distinct from the corresponding quantities in the three-channel case. Nonetheless, Eq. (2.13′) assures us that the identifications (3.1) remain valid.

The counterpart of (3.2a) is, for all α and λ,

$$U_{0\alpha} = \hat{U}_{0\alpha}(\lambda) + f^{L}_{\lambda}\left[\bar{\delta}_{\lambda\alpha} + \sum_{\gamma=0}^{3} T_2^{\lambda\gamma} G_\gamma \bar{\delta}_{\gamma\alpha}\right] G_\alpha^{-1}, \tag{3.2a′}$$

where

$$\hat{U}_{0\alpha}(\lambda) = (\delta_{\lambda\alpha} - \delta_{0\alpha}) G_0^{-1} + f^{L}_{\lambda} T_2^{\lambda\alpha}. \tag{3.3′}$$

In place of (3.2b) we have, for all β,

$$U_{\beta 0} = \bar{\delta}_{\beta 0} G_0^{-1} + \sum_{\gamma=0}^{3} T_2^{\beta\gamma} f^{R}_{\gamma}. \tag{3.2b′}$$

For U_{00} we can use (3.2b′) for $\beta = 0$ or, alternatively, we find that

$$U_{00} = \sum_{\gamma=0}^{3} \hat{U}_{0\gamma}(\lambda) f^{R}_{\gamma}. \tag{3.2c′}$$

From (3.2b′), (3.2c′), and (3.3′) one again obtains Eq. (3.4).

The on-shell equations (3.5) remain valid with minor modifications. Equations (3.5a) and (3.5b) now hold for any λ, the sums in Eqs. (3.5c) and (3.5d) now run from 0 to 3, and (3.5c) is true for all β. We infer that

$$\hat{U}_{0\alpha}(\lambda) = \hat{U}_{0\alpha}(0) = T_2^{0\alpha}, \quad \alpha \neq 0, \tag{3.12}$$

when $\hat{U}_{0\alpha}(\lambda)$ appears in matrix elements of the type (3.5a) and (3.5b). However, the generalization of (3.5d) implies that (3.12) holds for all α, so

$$f^{L}_{\alpha} T_2^{\lambda\alpha} = \bar{\delta}_{\lambda\alpha} G_0^{-1} + f^{L}_{\alpha} T_2^{\alpha\alpha}$$

for all λ, α.

We retain the form of the identifications (3.6) in the four-channel case. However, in view of the fact that $f_0^L = f_0^R = 1$, in order to avoid a redundancy in the zero-index case we take the matrix D to be

$$(D)_{\beta\alpha} = \delta_{\beta\alpha}\bar{\delta}_{0\alpha}D_\alpha. \tag{3.13}$$

With this understood the identifications (3.7) hold precisely as they stand except for the summations in (3.7c) and (3.7d) which now run from 0 to 3.

The constraints (3.8) take the same form as before. Then the proof of Eqs. (2.6) for the operators defined by (3.7) assuming the discontinuity relations (2.14) follows just as before. The Hermitian analyticity constraints (3.9) and the K-matrix equations (3.11) retain the same forms. All comments concerning the KL unitary program made in connection with case (i) apply to the four-channel case as well.

IV. K-MATRIX EQUATIONS

We now elaborate upon the K-matrix formalism of Refs. 2 and 3 and compare it with other techniques.[5-9] We define, analogously to Eqs. (3.6), the operators

$$K^{LR} \equiv DKD, \tag{4.1a}$$

$$K^L(\pm) \equiv f^L(\pm)KD, \tag{4.1b}$$

$$K^R(\pm) \equiv DKf^R(\pm), \tag{4.1c}$$

$$\kappa(\pm) \equiv f^L(\pm)Kf^R(\pm), \tag{4.1d}$$

and, in addition,

$$\bar{\kappa}(\pm) \equiv f^L(\pm)Kf^R(\mp). \tag{4.1e}$$

D is defined by Eq. (3.13) and therefore (3.6) and (4.1) apply for both three and four channels. In either case, the Heitler equations (3.11) become in terms of the operators (3.6):

$$T^{LR}(\pm) = K^{LR} \mp i\pi K^{LR}T^{LR}(\pm) \mp i\pi K^R(\mp)D_0T^L(\pm) \tag{4.2a}$$

$$= K^{LR} \mp i\pi T^{LR}(\pm)K^{LR} \mp i\pi T^R(\pm)D_0K^L(\mp), \tag{4.2b}$$

$$T^L(\pm) = K^L(\pm) \mp i\pi K^L(\pm)T^{LR}(\pm) \mp i\pi\bar{\kappa}(\pm)D_0T^L(\pm) \tag{4.3a}$$

$$= K^L(\pm) \mp i\pi T^L(\pm)K^{LR} \mp i\pi\tau(\pm)D_0K^L(\mp), \tag{4.3b}$$

$$T^R(\pm) = K^R(\pm) \mp i\pi K^{LR}T^R(\pm) \mp i\pi K^R(\mp)D_0\tau(\pm) \tag{4.4a}$$

$$= K^R(\pm) \mp i\pi T^{LR}(\pm)K^R(\pm) \mp i\pi T^R(\pm)D_0\bar{\kappa}(\mp), \tag{4.4b}$$

$$\tau(\pm) = \kappa(\pm) \mp i\pi K^L(\pm)T^R(\pm) \mp i\pi\bar{\kappa}(\pm)D_0\tau(\pm) \tag{4.5a}$$

$$= \kappa(\pm) \mp i\pi T^L(\pm)K^R(\pm) \mp i\pi\tau(\pm)D_0\bar{\kappa}(\mp). \tag{4.5b}$$

Equations (4.2)–(4.5) and the identifications (3.7) constitute the K-matrix formalism of KL and KLS for a given input defined by Eqs. (4.1). Often one only needs to consider a subset of Eqs. (4.2)–(4.5). For example, the nucleon-deuteron scattering problem entails the consideration of only the coupled integral equations (4.2a) and (4.3a) for the operators T^{LR} and T^L which correspond to the elastic rearrangement and breakup amplitudes, respectively.

The actual complexity of Eqs. (4.2)–(4.5) is about the same as one obtains in other unitary formalisms.[5-9] The latter are, however, superficially more elaborate; for example, it is shown in Ref. 6 that the formalisms of Refs. 7 and 8 involve a hierachy of two sets of equations such as (4.3) for a given input. The reason for this is elementary and if anything this doubling of equations simplifies the solution of the problem. References 7 and 8 both make use of the so-called reduced K-matrix technique which has the effect of decoupling, say, Eqs. (4.2a) and (4.3a).[15] The latter, after a partial wave decomposition, reduce to a set of one-dimensional integral equations; the decoupling halves the number of these and the other equations become algebraic equations.

We remark also that the Heitler integral equations of Refs. 7–9 can be easily expressed in connected-kernel forms. This is not the case for Eqs. (4.5) although it is not evident whether connectedness is a necessary or attractive attribute of the kernels of Heitler or, in general, phase-space types of integral equations.

With regard to the preceding points there is not much reason to prefer one of the various unitary formalisms over another. The primary difference lies in the rather simple constraints (Hermiticity) which need to be imposed upon the input operators in the formalisms reviewed in Ref. 6 in order to guarantee unitary scattering amplitudes compared with the unknown and certainly complicated constraints on K which are required to guarantee Eqs. (3.8) and (3.9). This is a major drawback of the KL method and one which evidently obtains independently of the number of particles involved.

Our final remarks concern the rather ambiguous nomenclature attendant to the different usages of K matrices and operators. We are only concerned with the differences which appear in the definition of these objects for multiparticle scattering beyond the free three-particle thresholds; most extant formalisms coincide in the elementary case of only two-particle channels.

The only definition of a K operator which seems

to be unequivocal and of universal applicability is that which relates K to the scattering operator S in the Heisenberg picture,

$$S = 1 - 2\pi i T , \tag{4.6}$$

as the solution of the integral equations

$$K = T + i\pi KT \tag{4.7a}$$

$$= T + i\pi TK . \tag{4.7b}$$

The necessary and sufficient condition for S to be unitary is that K be Hermitian. The definition of K encompassed in Eqs. (4.6) and (4.7) holds for arbitrary systems, relativistic or not, involving arbitrary numbers of particles.

A detailed realization of the preceding definition in the case of the nonrelativistic scattering of three particles was given in Ref. 8. The transformation of Eqs. (4.6) and (4.7) to the interaction picture was considered there as well. In that case the Heitler equations (4.7) involve only the operators D_α, $\alpha = 0-3$, in the kernels in contrast to $\hat{D}_\alpha$ which appears in the Heitler-like equations (3.11) in the KL formalism. The latter type of Heitler equation and corresponding K operator have wide usage in nuclear scattering theory.[16] The relationship of this sort of K-matrix theory, which is always described by the replacement of all Green's functions in the various scattering equations by their principal values and relating the solutions of the two types of equations, to the one derived from the general definition (4.7) is, for multiparticle scattering, somewhat remote.

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*This work was supported, in part, by the National Science Foundation.

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[12]The channels are designated by the possible asymptotic configurations of the three-particle system. That is, either one has a state consisting of a particle $\alpha(=1, 2, 3)$ moving freely and where the other two particles are in one of their possible bound states or, in the case where $\alpha = 0$, a state of three free particles. In the absence of bound states between any or all of the particle pairs α, for $\alpha \neq 0$, serves only as a pairing index.

[13]We suppress the parametric dependence upon z when no ambiguity is likely to arise.

[14]This means that Eqs. (2.6) are to be evaluated between states $\langle\varphi_\beta|$ and $|\varphi_\alpha\rangle$ of equal energy E, See Refs. 3, 6–8, and 10 for further details.

[15]The general reduced K-matrix formalism is discussed in Ref. 8. The specific choice of the projection operator P of Ref. 8 which is made there for the three-particle problem involves the sum over all asymptotic (in or out) two-particle states while the choice made in Ref. 7 corresponds to the sum over the asymptotic (in or out) three-particle states.

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