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Momentum distribution in the nucleus. II†

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We calculate the single particle momentum distribution $n(q)$ for a one-dimensional model with δ forces. There is a domain of q for which $n(q)$ has an exponential falloff, but after allowance is made for the nonsaturation in the model, that domain does not grow significantly with particle number. The relation of this result to large momentum scattering from the nucleus and to the Hartree approximation is briefly discussed.

[NUCLEAR STRUCTURE Single particle momentum distribution in a one-dimensional model and its large momentum behavior.]

I. INTRODUCTION

The momentum distribution in the nucleus $n(q)$, gives the probability density for finding a nucleon in the nucleus of momentum q with respect to the center of mass. Although $n(q)$ is important in a wide variety of medium energy and high energy physics situations, very little is known theoretically or empirically about the form or the constraints on $n(q)$ for large q. We have recently shown[1] the important role $n(q)$ plays in the remarkable high momentum transfer scattering experiments of Frankel *et al.*,[2] where $n(q)$ seems to have an exponential falloff in q. We have also shown[3,4] that for local two-body potentials v, $n(q)$ goes, for very large q, like $q^{-4}\tilde{V}^2(q)$ where $\tilde{V}$ is the Fourier transform of v and would be expected to have a power law falloff in q. An important question, therefore, is whether the exponential experimental falloff and the theoretical power law can be shown to be compatible. In the related problem of the form factor $F(q)$ we showed[3,4] that for an A-particle system the asymptotic dependence of $F(q)$ must be $[\tilde{V}(q)q^{-2}]^{A-1}$, but that for this to be the dominant form it is q/A that must be large compared to typical momenta of the system. For only q large, the form factor does fall exponentially. Is there a similar intermediate domain of q for $n(q)$? It is very difficult to answer this question in general. If there is, it must arise, as does the intermediate domain in the form factor case, from features of the many-body wave function. We must, therefore, examine this question in the context of an interacting many-body system. For example, it is easy to see both formally and intuitively that the limiting form of $n(q)$ $[q^{-4}\tilde{V}^2(q)]$ comes from pair correlations. This arises because for sufficiently large momentum the easiest way for the nucleus to have a particle of momentum $\vec{q}$ with respect to the center of mass is to have another of momentum $-\vec{q}$. But for smaller momenta might it not be easier for one nucleon to acquire a momentum $\vec{q}$ via n coherent collisions with n nucleons in which it gives each only $-\vec{q}/n$, so long as q/A is small? This could be more likely since there are so many more ways for this to occur.

To investigate $n(q)$ in this regime we need a model rich enough to permit such complex correlations, yet simple enough to permit analysis. Such a model is N bosons moving in one dimension and interacting via δ function forces.[5] The many-body bound state of this model is known and the purpose of this paper is to present a calculation of its momentum distribution. The model has the many-body correlations that we seek built into it and it obeys the asymptotic form $n(q)\sim q^{-4}[\tilde{V}^2(q)]$ but for δ functions $\tilde{V}(q)=1$. The specialization to one dimension loses logarithmic corrections that are probably not essential to understanding the large q behavior of $n(q)$, but the restriction to bosons certainly affects the asymptotic behavior since the Pauli principle will cause the true $n(q)$ to have higher momentum components than the Bose system.

It is in the one-dimensional δ function system that we showed that the form factor has an intermediate regime of exponential falloff.[3,4] This model has also been investigated extensively by Colagero and Degasperis[6] in order to elucidate the validity of the Hartree method. In the Hartree approximation they find an exponentially falling form factor that agrees with our "intermediate regime" form factor. They also find, in the Hartree approximation, an exponentially falling $n(q)$. Since the Hartree approximation is a large A approximation, we are encouraged to believe that the exact $n(q)$, at least for large A systems, may also have an important regime of exponential falloff. We therefore undertake an exact calculation of $n(q)$ for the one-dimensional system with δ function interactions.

In Sec. II we review briefly the one-dimensional model and calculate its $n(q)$ first for four particles in order to establish our Feynman graph methods, and then for N particles. Some technical material related to these calculations is relegated to the Appendices, but by and large the entire section is technical and the reader interested only in results may wish to skim it to the result Eqs. (12) and (14). In Sec. III we analyze the result, primarily numerically. In Sec. IV we give our conclusions. In summary these are that $n(q)$ has a limited region of exponential falloff that grows only slowly with particle number. The use of a phenomenological momentum distribution with an exponential tail to interpret the inclusive scattering results of Frankel *et al.*,[2] while not strongly supported by the one-dimensional model, cannot be ruled out since more realistic models may well have a far more extended exponential region.

II. MOMENTUM DISTRIBUTION

In this section we review briefly the major features of the one-dimensional model and the graphical methods for calculating the momentum distribution; we apply these methods to the four-body case as an example, and then to the general N-body case. A number of technical aspects are relegated to the Appendices.

A. Model

For N particle moving in one dimension and interacting with δ forces, the Hamiltonian is ($\hbar = 2m = 1$)

$$H=-\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2}-g\sum_{i<j=1}^{N}\delta(x_i-x_j),\quad g>0\ . \qquad (1)$$

The center of mass solution for the bound state is[5]

$$\psi(x_1,x_2,\ldots,x_N)=M\exp\Big(-\tfrac{1}{4}g\sum_{i<j=1}^{N}|x_i-x_j|\Big), \qquad (2)$$

where M is the normalization. The binding energy corresponding to this state is

$$E_N=-\tfrac{1}{48}g^2N(N^2-1)\ . \qquad (3)$$

The single particle momentum distribution $n(q)$ is the probability of finding a particle of momentum q in that state. In terms of the wave function (2) it can be written

$$n(q)=\int e^{iqx'}\psi^*(x_1+\tfrac{1}{2}x',x_2,\ldots,x_N)\psi(x_1-\tfrac{1}{2}x',x_2,\ldots,x_N)\delta\Big(\frac{1}{N}\sum_{i=1}^{N}x_i\Big)dx'\prod_{i=1}^{N}dx_i\ . \qquad (4)$$

The wave function (2) is a product of exponential factors of the form $e^{-\xi|x|}$. These have a particularly simple Fourier transform

$$e^{-\xi|x|}=2\xi\int\frac{dk}{2\pi}\frac{e^{ikx}}{k^2+\xi^2}\ . \qquad (5)$$

Thus each factor $\exp(-\xi|x|)$ corresponds to a propagator of "mass" $i\xi$. (This is no surprise since Green's functions are intimately related to δ functions as sources.) Introducing (2) in (4) using the wave function forms (5), we obtain[4]

$$\begin{aligned}n(q)={}&M^2\gamma^{(N-1)(3N-4)}2^{2(N-1)(3N-5)}\\&\times\int\frac{dk}{2\pi}\prod_{\substack{i>j\\j=2}}^{N}\Big(\frac{dk_{ij}}{2\pi}\frac{1}{k_{ij}^2+4\gamma^2}\Big)\prod_{i=2}^{N}\Big(\frac{dk_{1i}}{2\pi}\frac{dk'_{1i}}{2\pi}\frac{1}{(k_{1i}^2+\gamma^2)(k'^2_{1i}+\gamma^2)}\Big)\\&\times2\pi\delta\Big(2q+\sum_{i=2}^{N}(k'_{1i}-k_{1i})\Big)2\pi\delta\Big(\frac{k}{N}+\sum_{i=2}^{N}(k'_{1i}+k_{1i})\Big)\\&\times\prod_{l=2}^{N}\delta\Big(\frac{k}{N}-k_{1l}-k'_{1l}+\sum_{i=l+1}^{N}k_{li}-\sum_{j=2}^{l-1}k_{jl}\Big),\end{aligned} \qquad (6)$$

where under the integrals we have put $\frac{1}{4}g=\gamma$ to simplify the propagators. From now on we will call γ or 2γ a mass (dropping the factor i). Equation (6) can be interpreted as an integral corresponding to a Feynman graph with $N-1$ vertices $2,3,\ldots,N$, and two vertices 1 and 1′. Each of the pair of vertices $2,3,\ldots,N$ is connected by a propagator of mass 2γ while from 1 (or 1′) to each of $2,3,\ldots,N$ there is a propagator of mass γ. Momentum q enters at 1 and leaves at 1′. The δ functions ensure momentum conservation at each vertex while the "extra" k integral removes the over-all momentum conserving δ function one normally associates with a Feynman graph. After

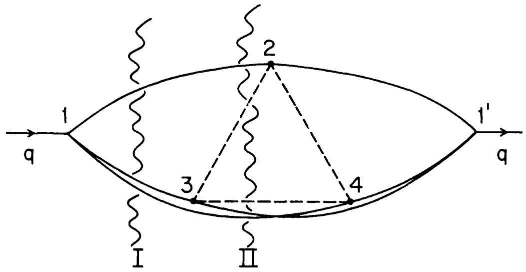

FIG. 1. Graphical representation of the four-body momentum distribution. The lines I and II indicate two inequivalent ways of cutting the graph.

eliminating the δ functions, the remaining integrals can be done by contour integration since all k integrals run from $-\infty$ to $+\infty$. (This is the great advantage of the one-dimensional case. In three dimensions the presence of angular integrals and/or a lower limit on the k integrals complicates the analysis and introduces logarithmic factors.) Since the integrand has only simple poles, the result of the successive k integrations leads to a meromorphic function of q (that is, a ratio of polynomials). We have shown that $n(q) \sim q^{-4}$ for large q. Such a meromorphic function can be written in a Laurent expansion as a sum of nth order poles with residues. Thus the problem of doing the integral (4) is equivalent to finding all the poles of (6) and the coefficient of each. Locating all the poles of a Feynman graph is a standard problem using the methods of Landau.[7] The coefficient of each singularity is also easily found by the methods of Landau and Cutkosky for most cases, but there is a special technical problem that $n(q)$ possesses. In general $n(q)$ has both first and second order poles at the same location. The coefficient of the first order pole "under" the second order pole is difficult to obtain directly. We solve this problem by introducing different masses into (6); this makes all poles first order and it is straightforward to calculate the residues. The masses are then put equal (by a carefully limiting procedure that introduces second order poles). We now proceed to illustrate this in the four-body and then N-body cases.

B. Four-body case as as example

For four particles the momentum distribution is represented by the graph shown in Fig. 1. The dotted lines represent propagators of mass 2γ and the full lines of mass γ. As discussed above, all singularities of $n_4(q)$ are poles. The Landau equations[7] state that a Feynman graph has a singularity when each line of the graph is either contracted out of the graph (this is equivalent to putting its Feynman parameter equal to zero) or is on mass shell. The position of the singularity is determined by the "invariant mass" of the lines that are on mass shell. The nature of the singularity can be determined by calculating the reduced graph consisting of those lines which on shell produce the singularity. The wavy lines in Fig. 1 correspond to the two inequivalent ways of "cutting" the graph (the cut lines go on mass shell at the singular point). Cut I cuts three lines and gives a singularity at $q^2 = -(3\gamma)^2$, while II cuts three γ lines and two 2γ lines to give a singularity at $q^2 = -(3\gamma + 4\gamma)^2$. These singularities are both single and double poles. For example, for the singularity at $q^2 = -9\gamma^2$, a contraction of the central (dotted) triangle gives a pure double pole, while a contraction of *all* lines coming from the right (or left) vertex gives a single pole. These two reduced graphs are shown in Fig. 2. There is also a single and double pole at $q^2 = -49\gamma^2$. The major technical problem is to get the correct residue at the single pole, which is "under" the double pole. To do this we break the symmetry of the graph by making the masses different. This makes all poles single poles. We calculate the position and residue of all the poles and then reintroduce the symmetry. All masses coming from the left vertex we call λ, all those coming from the right vertex we call ρ, and all those associated with the central dotted lines we call μ. After calculating the graph we let $\lambda = \rho = \gamma$, $\mu = 2\gamma$.

The graph of Fig. 1 now has four simple poles. These come at $q^2 = -(3\lambda)^2$ corresponding to the I cut of Fig. 1, $q^2 = -(2\mu + 2\lambda + \rho)^2$ corresponding to the II cut of Fig. 1 and at $q^2 = -(3\rho)^2$ and $q^2 = -(2\mu + 2\rho + \lambda)^2$, corresponding to the interchange of λ with ρ. The graph can be evaluated (in terms of a Laurent expansion) by calculating the residue at each of these poles. To calculate the residue we reduce the graph by making contractions until only the singular part is left. The residue is given in terms of the weight of the reduced graph (the number of ways the contractions can be made), the value of the contracted pieces evaluated at the momentum corresponding to the singularity, and the reduced graph itself. For the graph of Fig. 1 there is only one way to make the I contraction. The reduced graph is shown in Fig. 3(a); it can be evaluated by the formulas developed in Appendix A. The part contracted out to obtain the reduced graph is shown in Fig. 3(b), where the momenta appropriate to the singularity in question are shown flowing in and out the graph. The general

FIG. 2. Reduced graphs corresponding to double and single poles at $q^2 = -9\gamma^2$.

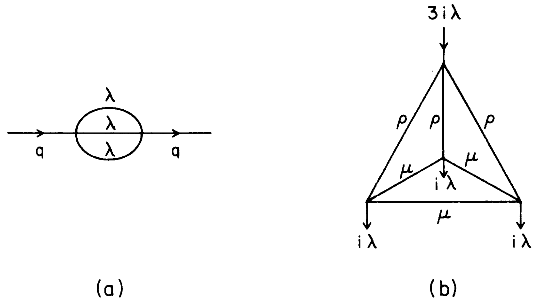

FIG. 3. (a) The reduced graph for the singularity at $q^2=-9\gamma^2$. (b) The factor graph contracted out of Fig. 1 to produce the reduced graph (a).

expression for such a factor graph is obtained in Appendix B. The total contribution of this contraction is then

$$\frac{3\pi^2}{\lambda^2}\frac{1}{q^2+9\lambda^2}\left(\frac{\pi}{\mu}\right)^3 \times \frac{(2\rho+\mu)^2(\rho+\mu)}{2\rho^2(\rho^2-\lambda^2)[(\rho+\mu)^2-\lambda^2][(\rho+2\mu)^2-\lambda^2]}\,. \tag{7}$$

There are three ways to make the II contraction. One such way is shown in Fig. 4(a). The lines that go on shell at the singular point are indicated with an x. One line coming from the left is contracted and it is shown carrying the appropriate momentum. The loop that must be contracted out from the right is shown in Fig. 4(b) and a reduced graph is shown in Fig. 4(c). All parts can now be evaluated by using Appendix A or Appendix B and we get for the contribution of this contraction

$$\frac{3\pi^5}{2\lambda^2\rho^2\mu^4}\frac{(2\lambda+2\mu+\rho)}{q^2+(2\lambda+2\mu+\rho)^2} \times\left(\frac{1}{\rho^2+(\mu+\lambda)^2}-\frac{1}{(\mu+\rho)^2-(\mu+\lambda)^2}\right) \times\left(\frac{1}{\lambda^2-(\rho+2\mu)^2}\right). \tag{8}$$

There are two more contributions from interchanging λ and ρ. In each of (7) and (8) there is a denominator that vanishes when $\lambda=\rho$. But when the contribution from $\lambda \leftrightarrow \rho$ is added in, the numerator will also be seen to vanish. Hence the $\lambda=\rho$ limit must be taken by using l'Hospital's rule. That will introduce second order poles. The final result we obtain for the four-body momentum distribution is

$$n_4(q)=\frac{\pi^5}{\gamma^{11}2^8}\left(\frac{49\gamma^2}{3(q^2+49\gamma^2)^2}+\frac{27\gamma^2}{(q^2+9\gamma^2)^2} +\frac{1}{q^2+49\gamma^2}-\frac{1}{q^2+9\gamma^2}\right) \tag{9}$$

(with arbitrary normalization). It should be noted that although these are single as well as double poles, the $1/q^2$ terms cancel for large q so that $n_4(q)\sim 1/q^4$ for large q—as we have established in general. It is a general property of $n(q)$ that the coefficients of the single pole terms sum to zero so as to assure the correct large q behavior for $n(q)$.

C. N-Body case

We now calculate the momentum distribution for the N-body case. The method is the same as in the four-body case. We take the masses to be different and put them equal at the end. The general graph has $N-1$ central points connected in pairs by propagators mass μ. Each of these $N-1$ points is connected to point 1 (or 1′) by a propagator of mass λ (or ρ). Momentum q flows in at 1 and out at 1′. In the graph we calculate the normalization is set equal to one and after all momentum conserving δ functions have been eliminated, each independent momenta integral occurs without a factor of 2π. Successive contractions of the graph are made by taking $0, 1, 2, \ldots, \frac{1}{2}(N-2)$ (we take N to be even) central points to the left and contracting the remainder to the right. If $s-1$ points are taken to be left, the pole is at $q^2=-[(s-1)\rho+(N-s)\lambda+(N-s)(s-1)\mu]^2$ and there are $(N-1)![(s-1)!(N-s)!]^{-1}$ ways to make such a contraction. We can calculate the singular part of the graph (with $s-1$ ρ lines, $N-s$ ρ lines, and $(N-s)(s-1)$ μ lines) by the methods of Appendix

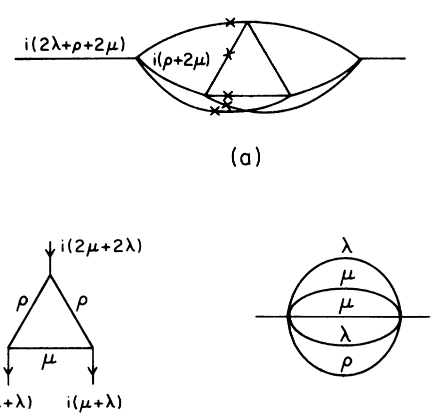

FIG. 4. (a) The four-body graph showing a particular way of making the II contraction (see text) giving a singularity at $q^2=-(2\lambda+\rho+2\mu)^2$. The lines marked × are on shell at the singular point. The line carrying momentum $i(\rho+2\mu)$ is contracted from the left. (b) The loop that must be contracted out of the four-body graph from the right. (c) The resulting reduced graph.

A. There will be two contracted factor graphs of the kind discussed in Appendix B. The one on the left will have $s-1$ λ legs and carry a momentum $Q^2=-(s-1)^2[\rho+(n-s)\mu]^2$, while the one on the right will have $n-s$ ρ legs and carry $Q^2=-(N-s)^2[\lambda+(s-1)\mu]^2$. There is, of course, another completely equivalent contraction obtained by interchanging λ and ρ, and these two are to be added. The general term, C_s, with $s-1$ lines contracted is given by

$$C_s=\frac{(N-1)!}{(s-1)!(N-s)!}\,\frac{\pi^{\frac{1}{2}N(N-1)-1}}{\rho^{N-3}\lambda^{N-3}\mu^{1/2(N-2)(N-1)}}\,\frac{1}{2^{N-3}}\,\frac{(s-1)\rho+(N-s)\lambda+(N-s)(s-1)\mu}{q^2+[(s-1)\rho+(N-s)\lambda+(N-s)(s-1)\mu]^2}$$
$$\times\prod_{i=1}^{s-2}(i\mu+2\lambda)\prod_{i=1}^{N-s-1}(i\mu+2\rho)\prod_{i=0}^{s-2}\left(\{(\lambda+i\mu)^2-[\rho+(N-s)\mu]^2\}\{(\rho+i\mu)^2-[\lambda+(s-1)\mu]^2\}\right)^{-1}$$
$$\times\prod_{i=s}^{N-s-1}\{(\rho+i\mu)^2-[\lambda+(s-1)\mu]^2\}^{-1}\{[\rho+(s-1)\mu]^2-[\lambda+(s-1)\mu]^2\}^{-1}+(\lambda\leftrightarrow\rho)\,. \tag{10}$$

The complete graph is given by

$$C_N=\sum_{s=1}^{N/2}C_s\,. \tag{11}$$

In (10) we have explicitly shown, as the last factor, the denominator that vanishes as $\rho=\lambda=\gamma$. It is the only term that vanishes. It is clear that when we add in the $\lambda\leftrightarrow\rho$ term, it will have the same factor, but of opposite sign. Thus the $\lambda=\rho=\gamma$, $\mu=2\gamma$ limit must be taken carefully, by using l'Hospital's rule. Doing this, we get for the momentum distribution, with our normalization (and even N)

$$n_N(q)=\frac{\pi^{\frac{1}{2}N(N-1)-1}}{\gamma^{\frac{1}{2}N(N+3)-3}}\,\frac{1}{2^{\frac{1}{2}N(N+1)-3}(N-1)!}$$
$$\times\sum_{s=1}^{N/2}\left[\frac{1}{q^2+\gamma^2[(N-1)+2(N-s)(s-1)]^2}\right.$$
$$\left.\times\left(\frac{(N-2s+1)^2[N-1+2(N-s)(s-1)]^2\gamma^2}{q^2+\gamma^2[N-1+2(N-s)(s-1)]^2}+\tfrac{1}{2}[2(N-s)(s-1)+N-1-(N-2s+1)^2]\right)\right] \tag{12}$$

As expected, this is a sum of single and double poles. Since $n(q)\sim q^{-4}$ for large q, the sum of the coefficients of the single pole terms must be zero. It is straightforward to verify that in fact

$$\sum_{s=1}^{N/2}[2(N-s)(s-1)+N-1-(N-2s+1)^2]=0\,. \tag{13}$$

If N is odd, the summation in (12) would extend from $s=1$ to $\frac{1}{2}(N-1)$ and an additional single pole contribution

$$\tfrac{1}{8}(N^2-1)\{q^2+\tfrac{1}{4}[\gamma^2(N^2-1)^2]\}^{-1} \tag{14}$$

would be added.

III. NUMERICAL RESULTS

The final expression (12) is far too complicated for direct analysis. It is clear that it falls like q^{-4} for large q but is there an intermediate domain of far faster decrease? This could be most easily studied by writing $n(q)$ as a ratio of polynomials

$$n(q)=N(q)/D(q)\,. \tag{15}$$

Because of the asymptotic form for n, $D(q)$ must have two more powers of q^2 than $N(q)$. Furthermore, we know from our Feynman graph analysis that $D(q)$ has single and double poles for negative q^2. The number and location of these poles moves out in $|q^2|$ for increasing numbers of particles. The polynomial $N(q)$ must have zeros. It is easy to see that these must come for q^2 negative or complex. If its zeros interlace the poles of D, the asymptotic form of n will set in for small q^2, but if, for example, all the zeros of $N(q)$ were to occur beyond the last pole of D, $N(q)$ would effectively be a constant for a large part of the domain of positive q^2 and since D has more and more poles with increasing particle number, n would appear to fall exponentially until q got large enough for the zeros of $N(q)$ to become significant. For example, for the three-body case we have from (12) and (14) or direct integration

$$n_3(q)=\frac{q^2+52\gamma^2}{(q^2+4\gamma^2)^2(q^2+16\gamma^2)}\,. \tag{16}$$

(Note—this result was misprinted in Ref. 4) and we see that the zero of $N(q)$ does indeed come beyond the poles of $D(q)$.

However, even for the four-body case, writing

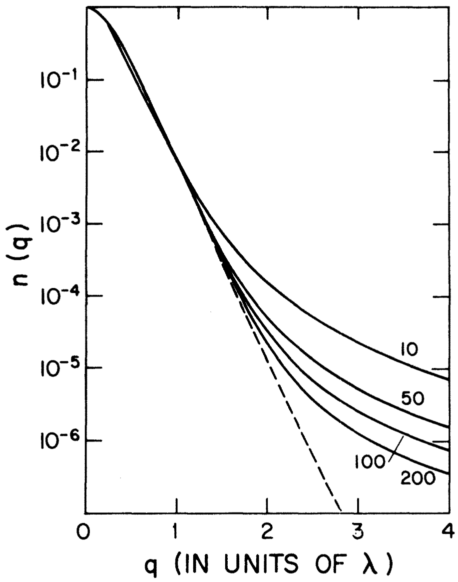

FIG. 5. The momentum distribution $n(q)$ for systems of 10, 50, 100, and 200 particles as a function of momentum q. The dashed line shows the Hartree result for $n(q)$.

n in the form (15) and finding its factors is complicated and for the general case (12) it is a formidable task. We have therefore turned to a numerical analysis of (12). This is a relatively straightforward task on a high speed computer although some caution and at least double precision is necessary since there is considerable cancellation among the terms. Our purpose is to see if $n(q)$ has a domain of exponential-like falloff in q, which grows with increasing N, and to compare that result with the Hartree result for n. The Hartree form is

$$n(q) = (\cosh \pi q/\lambda)^{-2}, \tag{17}$$

where $\lambda = 2\gamma N$. It is λ and not γ that determines the size of the system, since the forces do not saturate [see (3)]. It should be recalled that, for fixed λ, the form factor $F(q)$ has a domain of exponential falloff that grows with N.

Figure 5 shows $n(q)$ for $N = 10, 50, 100, 200$ as a function of q with the normalization $n(0) = 1$. The Hartree result (17) is also shown. In Fig. 6 we show the quantity $(q/\lambda)^4 n(q)$ which gives some indication of when the asymptotic regime $[n(q) \sim q^{-4}]$ has been attained. We see that for reasonably large N the exact Hartree results for $n(q)$ agree for $q \sim 0(\lambda)$ and that the asymptotic q^{-4} behavior sets in when q is about 3λ. Unlike the form factor for which the domain of exponential falloff increases proportionally[8] to N, the region of exponential behavior of $n(q)$ increases exceedingly slowly with N. But we do see that (with our normalization) the coefficient of the $1/q^4$ part of $n(q)$ is of order $1/N$. In Appendix C we show explicitly that Eq. (12) reduces to the Hartree result (17) for (q/λ) small and N large.

IV. CONCLUSION

Using Feynman graph methods, we have calculated the single particle momentum distribution $n(q)$ in the one-dimensional model of bosons interacting via δ function forces. The model is sufficiently sophisticated that the calculation is by no means trivial, but one can question whether it is rich enough to provide insight into the momentum distribution of a real system like the nucleus.

We are particularly interested in whether $n(q)$ has a domain of exponential falloff, for example, as provided by the Hartree approximation (17), before the asymptotic power law q^{-4} sets in, and whether that exponential domain grows with N. The diagrammatic representation of $n(q)$ given in Sec. II provides a mechanism for this. It is clear that for sufficiently large q the momentum flowing in at point 1 can all flow in on one line to some point $2, 3, \ldots, N$ and out from that point to $1'$. Since only two propagators will then contain q, we get q^{-4}. But is it also possible for $q/N-1$ to flow from point 1 to each of the $N-1$ points $2, 3, \ldots, N$ and back out. For q large but $q/N-1$ small the large number of ways of distributing the momenta could compensate for the additional propagators which must carry a momentum q.

In the case of the form factor $F(q)$ in the one-dimensional model, there is an intermediate domain of q that agrees with the Hartree form factor, which is exponential, and that domain does grow with N even after one factor of N is taken out to account for the fact that the system does not satu-

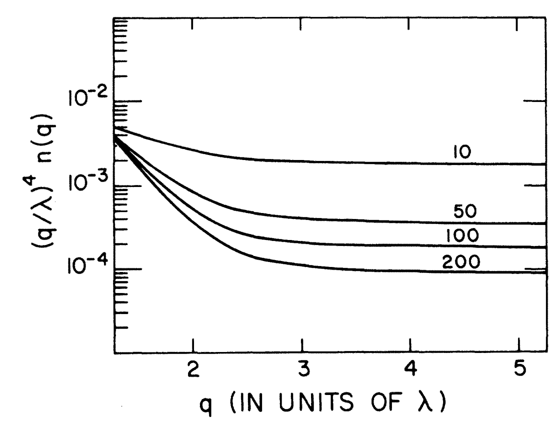

FIG. 6. The quantity $(q/\lambda)^4 n(q)$ for systems of 10, 50, 100, and 200 particles as a function of momentum q.

rate.[9] For $n(q)$ we saw in Sec. III that when one factor of N is removed, the exponential region of n does not grow linearly with N. One obvious conclusion is that, at least for the one-dimensional system, the Hartree result for the momentum distribution has a smaller domain of validity than for the form factor, which is the Fourier transform of the position density. This is not totally surprising since for the saturating infinite N system (nuclear matter) the Hartree result for the position density (uniform) is exact, while the $n(q)$ (zero temperature Fermi gas) clearly is not.

In a previous article[1] we interpreted the inclusive scattering data of Frankel *et al.*[2] in terms of a single scattering model. With this mechanism, a momentum distribution with a large region of exponential falloff is required in order to describe the data. The motivation for the present calculation was to see if such behavior of the momentum distribution could be accounted for by the cooperative effect of the many degrees of freedom in the N-body wave function. From this point of view our results are not encouraging. Our $n(q)$ starts to deviate from exponential falloff for momenta in the range relevant to the data of Ref. 2. Furthermore, $n(q)$ only falls three or four decades before the asymptotic power law takes over, while the Frankel data require the momentum distribution to drop many more orders of magnitude. It should be remembered however that realistic nucleon-nucleon forces and Fermi statistics would both lead to a more rapid asymptotic decrease of $n(q)$. Therefore a larger domain of exponential falloff of $n(q)$ cannot be ruled out in more realistic models.

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APPENDIX A: GENERAL N-LINE REDUCED GRAPH

Consider the integral I_N represented by the N-line reduced graph shown in Fig. 7(a). Each line i has mass γ_i. Let us distribute the momenta as shown in Fig. 7(b). The first loop is given by

$$\int \frac{dk_1}{(k_1^2+\gamma_1^2)[\,(k_2-k_1)^2+\gamma_2^2]} = \frac{\pi(\gamma_1+\gamma_2)}{\gamma_1\gamma_2[\,k_2^2+(\gamma_1+\gamma_2)^2]} \quad . \tag{A1}$$

Hence

$$I_N=\frac{\pi(\gamma_1+\gamma_2)}{\gamma_1\gamma_2} I_{N-1}(\gamma_2 \rightarrow \gamma_1+\gamma_2) \ , \tag{A2}$$

where the argument in I_{N-1} signifies that I_{N-1} is to be evaluated with γ_2 replaced by $\gamma_1+\gamma_2$. By induction we obtain

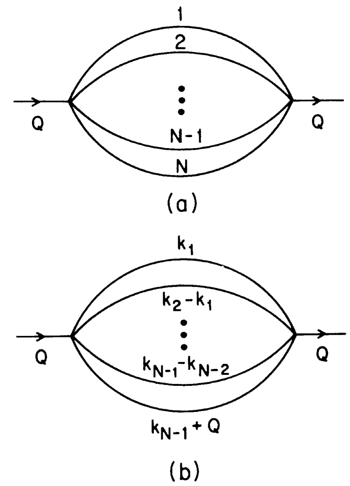

FIG. 7. (a) The N-line reduced graph. (b) Distribution of internal loop momenta.

$$I_N=\frac{\pi^{N-2}(\gamma_1+\gamma_2+\cdots\gamma_{N-1})}{\gamma_1\gamma_1\cdots\gamma_{N-1}}\, \mathcal{J} \ , \tag{A3}$$

where

$$\mathcal{J}=\int \frac{dk}{[\,(k+Q)^2+\gamma_N{}^2](k^2+\Gamma^2)} \ ,$$

with

$$\Gamma=\sum_{i=1}^{N-1}\gamma_i \ .$$

$\mathcal{J}$ is the same integral as (A1). We thus finally obtain

$$I_N=\frac{\pi^{N-1}\sum_{i=1}^{N}\gamma_i}{\gamma_1\gamma_2\cdots\gamma_N}\ \frac{1}{Q^2+\left(\sum_{i=1}^{N}\gamma_i\right)^2} \ . \tag{A4}$$

APPENDIX B: FACTOR GRAPHS

In calculating the residues at the poles of the momentum distribution we must evaluate[10] factors corresponding to graphs (which we call factor graphs) contracted out of the main graph when the reduced graph is constructed. The general factor graph $S_n(Q)$ has momentum Q flowing into a vertex that is connected by n lines of mass λ (or ρ) to n vertices. Out of each of these flows momentum Q/n. Each of these n vertices is connected to all the others by lines of mass μ. S_2 and S_3 are shown in Fig. 8. These graphs are generalizations of the form factor graphs (in that case $\mu=\lambda$) with the N-body form factor corresponding to S_{N+1}. As with the form factor, there are n different contractions of S_n giving n poles, but since (again as with the form factor) $S_n(Q)\sim Q^{-2n}$ for large Q, the general form of S_n must have the n poles multiplied together. Thus to evaluate S_n we need only find the residue at one

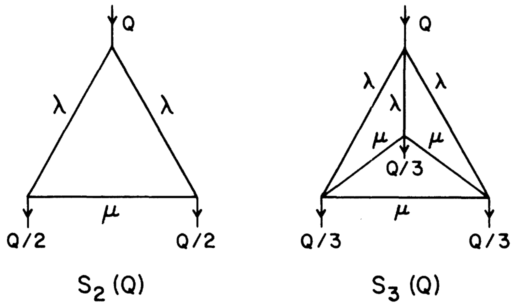

FIG. 8. The factor graphs $S_2(Q)$ and $S_3(Q)$.

pole to obtain the coefficient of that product form.

The mth contraction of $S_n(Q)$ will have $n-m$ lines of mass λ and $m(n-m)$ lines of mass μ through which flows a momentum $[(n-m)/n]Q$. Hence we can write

$$S_n(Q)=C\prod_{m=0}^{n-1}[Q^2+n^2(\lambda+m\mu)^2]^{-1}\,, \tag{B1}$$

where C must now be determined. The pole at $Q^2=-n^2\lambda^2$ $(m=0)$ corresponds to contracting all the μ lines. The residue of (B1) at this pole is given by

$$C\prod_{m=1}^{n-1}[-n^2\lambda^2+n^2(\lambda+m\mu)^2]^{-1}$$

$$=C(n^2\mu)^{-n+1}\left((n-1)!\prod_{m=1}^{n-1}(m\mu+2\lambda)\right)^{-1}. \tag{B2}$$

Direct examination of the contracted graph shows that it is an n-loop reduced graph with all lines of mass λ times a factor represented by n points each connected to all the others with a line of mass μ and with no momenta flowing through it. Call this last factor $L_n(\mu)$. Then using Appendix A, we obtain for the residue at the $Q^2=-n^2\lambda^2$ pole

$$n(\pi/\lambda)^{n-1}L_n(\mu)\,. \tag{B3}$$

To calculate $L_n(\mu)$ we use the same ideas as those fo $S_n(Q)$. Consider an n vertex graph with momenta Q flowing in at one vertex and $Q/(n-1)$ flowing out at all other vertices. Let each vertex be connected to all others by lines of mass μ. Call this $P_{n-1}(Q)$. Clearly

$$L_n(\mu)=P_{n-1}(0)\,. \tag{B4}$$

By the same arguments used for $S_n(Q)$

$$P_{n-1}(Q)=C\prod_{m=1}^{n-1}[Q^2+(n-1)^2m^2\mu^2]^{-1}\,. \tag{B5}$$

Now the pole at $Q^2=-(n-1)^2\mu^2$ has a residue given by

$$(\pi/\mu)^{n-2}(n-1)L_{n-1}(\mu)\,. \tag{B6}$$

Comparing this with direct evaluation of the residue in (B5), and using (B4) we obtain the recursion relation

$$L_n(\mu)=\left(\frac{\pi}{\mu}\right)^{n-2}\frac{n}{2[\mu(n-1)]^2}L_{n-1}(\mu)\,.$$

From this and direct evaluation of $L_3(\mu)$ we obtain

$$L_n(\mu)=\left(\frac{\pi}{\mu}\right)^{\frac{1}{2}(n-2)(n-1)}\frac{n}{(n-1)!}\,\frac{1}{(2\mu^2)^{n-1}}\,. \tag{B7}$$

Using (B2), (B3), and (B7)

$$S_n(Q)=\frac{n^{2n}}{(2\lambda)^n}\left(\frac{\pi}{\mu}\right)^{\frac{1}{2}n(n-1)}\prod_{i=0}^{n-1}\frac{(i\mu+2\lambda)}{Q^2+n^2(\lambda+i\mu)^2}\,. \tag{B8}$$

APPENDIX C

In this Appendix we show that for $q^2\lesssim\lambda 2/4$, and $N\gg 1$, the exact form for $n(q)$ (12) is well approximated by the Hartree result (17). In terms of λ, we can write for (12)

$$n(q)=C\sum_{s=1}^{N/2}\left(\frac{(N-2s+1)^2\{[N-1+2(N-s)(s-1)]/N^2\}^2}{\{(2q/\lambda)^2+(1/N^2)[N-1+2(N-s)(s-1)]^2\}^2}+\frac{1}{2}\,\frac{2(N-s)(s-1)+N-1-(N-2s+1)^2}{(2q/\lambda)^2+(1/N^2)[N-1+2(N-s)(s-1)]^2}\right), \tag{C1}$$

where C is a constant containing factors of N, λ, etc., but independent of q and of s. The quantity in the denominator of (C1) can be written

$$(1/N^2)[N-1+2(N-s)(s-1)]^2$$

$$=[2s-1+(2s-2s^2-1)/N]^2\,. \tag{C2}$$

As s increases from 1 to $\frac{1}{2}N$ this quantity grows from $(1-1/N)^2$ to $(\frac{1}{2}N-1/N)^2$. Hence if $2q/\lambda$ is of order 1, and N large, only terms in the sum with $s\sim 1$ will have significant dependence on $2q/\lambda$. We therefore can obtain the q dependence of $n(q)$ for N large, $2q/\lambda\sim 1$, by expanding (C1) (including extending the upper limit on the sum to infinity) for $s\ll N$. To first order we get

$$n(q)=CN^2\sum_{s=1}^{\infty}\left(\frac{(2s-1)^2}{[(2q/\lambda)^2+(2s-1)^2]^2}-\frac{1}{2}\,\frac{1}{(2q/\lambda)^2+(2s-1)^2}\right), \tag{C3}$$

$$=\frac{CN^2}{2}\sum_{s=1}^{\infty}\frac{(2s-1)^2-(2q/\lambda)^2}{[(2q/\lambda)^2+(2s-1)^2]^2}\,, \tag{C4}$$

$$=(\cosh\pi q/\lambda)^{-2}\,. \tag{C5}$$

In the last step we have used $n(0)=1$, and a standard form for $\sec^2 x$ in terms of simple fractions,[11] with $\sec^2 ix=\cosh^{-2}x$.

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[7]See, for example, R. J. Eden, P. V. Landshoff, D. I. Olive, and J. C. Polkinghorne, *The Analytic S-Matrix* (Cambridge U.P., Cambridge, 1966).
[8]We expect that for large N the power law asymptotic behavior of $F(q)$ will set in only when $q > N\lambda$. See Ref. 4.
[9]To account for the decreasing size of the system as N increases, we use $\lambda = 2Ng$ for the momentum scale.
[10]The methods used here to evaluate these factors can also be used to evaluate many of the integrals encountered in Ref. 6, for example, the wave function normalization.
[11]See, for example, I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals Series and Products* (Academic, New York, 1965), p. 36, Eq. 1.422.2.