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Corrections to the Glauber model for intermediate energy proton-alpha elastic scattering*

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Corrections to the Glauber model for intermediate energy proton-α elastic scattering have been calculated at the lab energies $K_L = 0.58$, 1, and 2.1 GeV using the multiple diffraction expansion formalism. In this expansion, the Glauber amplitude appears as a leading term in a perturbation series. Leading corrections due to (i) wave spreading (noneikonal), (ii) internal excitation (recoil), as well as (iii) zero-point motion (different Galilean frames) effects are obtained in a simple model. Their structures are analyzed and their energy dependences are examined. The wave-spreading and internal-excitation corrections agree with those obtained by Wallace. The use of Wallace's relativistic kinematics changes the differential cross section significantly. A new zero-point motion correction is found to be important. A simple dependence of the results on the sign of ρ ($\equiv \mathrm{Re}f/\mathrm{Im}f$ of the NN scattering amplitude) is pointed out. The relative importance of these corrections decreases as energy increases, as expected. The significance of our results is briefly discussed.

[NUCLEAR REACTIONS ^{4}He(p,p), K_L=0.58, 1, 2.1 GeV; calculated $\sigma(K_L,\theta)$; wave spreading, internal excitation, and zero-point motion corrections to Glauber theory.]

I. INTRODUCTION

In the study of intermediate-energy projectile-nucleus scattering, Glauber's phenomenological multiple-diffraction (MD) theory[1] has long provided both qualitative insights and quantitative descriptions in a surprisingly simple model.[2-4] It has also been recognized that corrections to this phenomenological theory must be understood before reliable nuclear information can be extracted. These corrections have been widely studied. They include noneikonal (i.e., wave-spreading),[5-8] off-shell and overlapping-potential,[9] internal-excitation (or recoil),[10] and other inelastic effects.[11] Most of these studies suffer from a serious deficiency in that they lack a consistent formalism in which Glauber's phenomenology appears in a natural fashion.

Recently, such a consistent formalism has been proposed by Wong and Young[12] and by Wallace[13] by first introducing a projectile-nucleon pseudopotential. This pseudopotential is defined by the requirement that it gives the empirical projectile-nucleon scattering amplitude when used with Glauber's linearized propagator. In the many-body, multiple-scattering problem, it permits the construction of Glauber's MD amplitude as the leading term of a MD expansion which generates perturbative corrections in powers of k^{-1}, where k is the projectile-target relative momentum. Wallace[13] has also given numerical results for the leading wave-spreading (WS) and internal-excitation (INT) corrections for p-^{4}He elastic scattering at the lab energy $K_L = 1$ GeV. Corrections due to zero-point motion (ZPM) and wave-function antisymmetrization[12] have not been calculated in the context of the MD expansion.

In this paper, we study numerically the leading ZPM as well as the WS and INT corrections for p-^{4}He elastic scattering at $K_L = 0.58$, 1, and 2.1 GeV. Detailed analysis of the nature and energy dependence of these corrections will be made.

This paper is organized as follows. The MDE is reviewed in Sec. II. The model used in the study of p-^{4}He elastic scattering is described in Sec. III. Some salient features of the correction terms are also discussed. Section IV gives results of the calculations and discussions of various features that emerge. Section V contains brief concluding remarks.

II. MULTIPLE DIFFRACTION EXPANSION (MDE)

We first define our notations by briefly reviewing the MD expansion[12,13] for the elastic scattering amplitude between two ions A, B. The first term is the Glauber's empirical amplitude

$$F_G(q) = \frac{ik}{2\pi}\int e^{i\vec{q}\cdot\vec{b}}\langle AB|[1 - e^{i\chi(\vec{b};\vec{s})}]|AB\rangle d^2b \qquad (2.1)$$

for a momentum transfer $\vec{q} = \vec{k}_i - \vec{k}_f$. Here $k = |\vec{k}_i| = |\vec{k}_f|$ is the relative momentum and $\vec{b}$ is the impact vector on a plane (impact parameter plane) perpendicular to $\vec{k}_{av} = \frac{1}{2}(\vec{k}_i + \vec{k}_f)$, the average of the initial and final relative momenta. The vector $\vec{s}$ refers collectively to the projections of the projectile and target internal coordinates $\vec{x}_i, \vec{x}_k$ onto the impact parameter plane.

Glauber[1] shows that under conditions appropriate to high-energy scattering, the total phase shift function $\chi(\vec{b};\vec{s})$ may be approximated by a sum of elementary phase shift functions χ_{ik} between elementary scatterers i,k from ions A,B, respectively:

$$\chi(\vec{b};\vec{s})=\sum_{i=1}^{N_A}\sum_{k=1}^{N_B}\chi_{ik}(\vec{b}_{ik}). \tag{2.2}$$

Here N_A, N_B are the numbers of elementary scatterers in the ions, and χ_{ab} is defined in terms of the scattering amplitude $f_{ab}(q)$ between the elementary scatterers a and b:

$$\exp[i\chi_{ab}(\vec{b})]=1-\frac{1}{2\pi ik}\int e^{-i\vec{q}\cdot\vec{b}}f_{ab}(\vec{q})d^2q. \tag{2.3}$$

In order to derive correction terms to the Glauber amplitude, it is necessary to construct this amplitude in a logical fashion. This is achieved by first defining a pseudopotential v' between two elementary scatterers which will generate the exact two-body t matrix

$$t=v'+v'g_g t \tag{2.4}$$

when the Glauber linearized propagator

$$g_g=[\vec{u}_{ab}\cdot(\hbar\vec{k}_{ab(\mathrm{av})}-\vec{p})+i\eta]^{-1}, \tag{2.5}$$

is used. Here $\vec{u}_{ab}=\hat{k}_{ab(\mathrm{av})}\hbar k_{ab}/\mu_{ab}$, μ_{ab} being the reduced mass of the pair a,b, and $\vec{p}$ is the relative momentum operator.

We should note that knowledge of t on the energy shell may not be sufficient to determine a unique v' by Eq. (2.4). However, on-shell information alone can be used to define a unique local (but energy-dependent) pseudopotential

$$v'(q)=-2\pi\hbar u_{ab}\int_0^\infty J_0(qb)\chi_{ab}(b)\,b\,db. \tag{2.6}$$

Being local, this pseudopotential has the desirable property that when used in the ion-ion scattering problem with the many-body linearized propagator

$$G_g=[\vec{u}\cdot(\hbar\vec{k}_{\mathrm{av}}-\vec{P})+i\eta]^{-1}, \tag{2.7}$$

where

$$\vec{u}=\hat{k}_{\mathrm{av}}\hbar k/\mu, \quad \mu=M_AM_B/(M_A+M_B),$$

it will reproduce the Glauber amplitude (2.1) in the ion-ion center-of-mass frame.

The corresponding many-body T matrix is the MD approximation

$$T_{\mathrm{MD}}=V'+V'G_gT_{\mathrm{MD}}=V'\Omega_{\mathrm{MD}}, \quad V'=\sum_{ik}v'_{ik}. \tag{2.8}$$

It differs from the actual T matrix

$$T=V+VGT, \quad V=\sum_{ik}v_{ik}, \tag{2.9}$$

in both the potential and the propagator. The actual potential v reproduces the two-body t matrix when the actual propagator

$$g=(e-h_0+i\eta)^{-1}, \quad e=\frac{\hbar^2k_{ab}^2}{2\mu_{ab}}, \quad h_0=\frac{p^2}{2\mu_{ab}}, \tag{2.10}$$

is used. Consequently

$$v=v'-v'(g-g_g)v. \tag{2.11}$$

Finally, the actual propagator

$$G=(E-H_0+i\eta)^{-1}, \tag{2.12a}$$

where

$$E=\frac{\hbar^2k^2}{2\mu}+E_A^{\mathrm{INT}}+E_B^{\mathrm{INT}}, \quad H_0=K_{AB}+H_A^{\mathrm{INT}}+H_B^{\mathrm{INT}}, \tag{2.12b}$$

contains many-body internal Hamiltonians H_i^{INT} and their eigenvalues E_i^{INT}, as well as the relative kinetic energy K_{AB}.

We are now in a position to expand the many-body T matrix about T_{MD}:

$$T=T_{\mathrm{MD}}+T^{pp\prime}+T^{\mathrm{WS}\prime}+T^{\mathrm{INT}\prime}+R, \tag{2.13}$$

where

$$T^{pp\prime}=\Omega_{\mathrm{MD}}^\dagger(V-V')\Omega_{\mathrm{MD}} \tag{2.14}$$

$$T^{i\prime}=(\Omega_{\mathrm{MD}}^\dagger-1)\Delta H^i(\Omega_{\mathrm{MD}}-1), \quad i=\mathrm{WS},\mathrm{INT}; \tag{2.15}$$

$$\Delta H^{\mathrm{WS}}=\lambda G_g^{-1}+N_{AI}, \tag{2.16}$$

$$N_{AI}=(\vec{P}-\hbar\vec{k}_f)\cdot(\vec{P}-\hbar\vec{k}_i)/2\mu,$$

$$\lambda=1-[1-(q^2/4k^2)]^{1/2},$$

$$\Delta H^{\mathrm{INT}}=\Delta H_A^{\mathrm{INT}}+\Delta H_B^{\mathrm{INT}},$$

$$\Delta H_C^{\mathrm{INT}}=H_C^{\mathrm{INT}}-E_C^{\mathrm{INT}}, \quad C=A,B; \tag{2.17}$$

and R represents the remainder. Succeeding orders of perturbation involve additional powers of $\Omega_{\mathrm{MD}}-1$ or G_g. From Eq. (2.7) we see that this perturbative series is an expansion in powers of u^{-1} or k^{-1}.

The correction terms in Eq. (2.13) have different physical origins. The pseudopotential term $T^{pp\prime}$ corrects for the linearization of the two-body propagator g for each elementary potential. It contains two distinct effects: (i) a wave-spreading correction $T^{pp(\mathrm{WS})\prime}$, and (ii) a zero-point motion correction $T^{pp(\mathrm{ZPM})\prime}$ coming from the fact that the elementary potential v is momentum dependent. The terms $T^{\mathrm{WS}\prime}$ and $T^{\mathrm{INT}\prime}$ are ion-ion wave-spreading and internal-excitation (or recoil) corrections, respectively. These corrections to the Glauber scattering amplitude may be written in the form

$$F^{i\prime} = -\frac{\mu}{2\pi\hbar^2}\int d^2b e^{i\vec{q}\cdot\vec{b}}\langle AB|e^{i\chi(\vec{b};\vec{x})}f^{i\prime}(\vec{b};\vec{x})|AB\rangle\,, \tag{2.18}$$

where $\vec{x}$ denotes the internal coordinates. Explicit forms of $f^{i\prime}$, exclusive of terms proportional to λ, have been given in Ref. 12. The internal-excitation term can be simplified further, as shown by Wallace[13] in his Appendix B, to show a similarity with the wave-spreading term.

The λ terms appear in both $T^{pp\prime}$ and $T^{WS\prime}$. They are not included because of Wallace's conjecture[6] that they may cancel among themselves. Wallace[6] has given some indications of this conjectured cancellation in potential scattering. It is not known to what extent Wallace's conjecture is valid, especially in the present many-body context. This point should be investigated in the future.

We also find, in agreement with Wallace,[13] that it is useful to group three of the corrections [pp(WS), WS, and INT] together. In particular, one has for projectile-nucleus (ion B) scattering

$$f_W \equiv f^{WS\prime} + f^{pp(WS)\prime} + f^{INT\prime}$$
$$= -\frac{\hbar^2}{2\mu}\left\{\left[1-\frac{\mu}{\mu_{ab}}+\frac{\mu}{m}\left(1-\frac{1}{A}\right)\right]\sum_{i=1}^{A}\int_{-\infty}^{\infty}\vec{\nabla}_{0i}\chi_i^{(-)}\cdot\vec{\nabla}_{0i}\chi_i^{(+)}dz_0 + \left(1+0-\frac{\mu}{M}\right)\sum_i\sum_{j\neq i}\int_{-\infty}^{\infty}\vec{\nabla}_{0i}\chi_i^{(-)}\cdot\vec{\nabla}_{0j}\chi_j^{(+)}dz_0\right\}, \tag{2.19}$$

where m is the nucleon mass and $M = Am$ is the nucleus mass. Also $\vec{r}_0 = \vec{b}_0 + \hat{z}z_0$ is the projectile coordinate, $\vec{r}_{0i} = \vec{r}_0 - \vec{r}_i$, and

$$\chi = \sum_{j=1}^{A}\chi_j\,,$$
$$\chi_j^{(+)}(\vec{b},z) = -\frac{1}{\hbar u}\int_{-\infty}^{z} v'(\vec{b}+z'\hat{z})dz$$
$$= \chi_j(\vec{b}) - \chi_j^{(-)}(\vec{b},z)\,. \tag{2.20}$$

[The subscript W in Eq. (2.19) denotes equivalence with Wallace's result.] The three terms making up the coefficient in front of each integral correspond to the three terms on the left-hand side of Eq. (2.19). The first of these coefficients, $1-\mu/\mu_{ab}+\mu/M(1-1/A)$, vanishes identically so that there is no correction involving target nucleons singly. Therefore the leading corrections are many-body effects involving two target nucleons simultaneously. For a nucleon projectile, the internal excitation part of these many-body effects is exactly $-(A+1)^{-1}$ of the remaining, wave-spreading correction. Further examination of Eq. (2.19) shows that these corrections may be characterized as overlapping-potential effects for a pseudopotential of finite range, since contributions appear mostly when both nucleons i and j are close to the projectile. According to Appendix A, a leading term can be isolated which gives the contribution in the absence of eikonal distortion at any target nucleon. This leading term is roughly proportional to $\rho_1\sigma^2$, where ρ_1 is the nucleon density in the target nucleus. It is roughly independent of the range of the pseudopotential. This behavior appears to differ, at least formally, from that assumed by Gurvitz, Alexander, and Rinat,[14] in which the force range appears in a more crucial fashion. We also note that because of the ρ_1 dependence, these corrections appear to be volume effects which are likely to become more important for the heavier nuclei.

In contrast to f_W, the ZPM correction involves target nucleons singly:

$$f_{ZPM} = -\frac{\hbar^2}{2\mu_{ab}}\sum_{i=1}^{A}\int_{-\infty}^{\infty}\{[1-\exp(-i\chi_i^{(-)})]|\vec{\kappa}_i|^2[1-\exp(-i\chi_i^{(+)})]$$
$$+[1-\exp(-i\chi_i^{(-)})]\vec{\kappa}_i^*\cdot\vec{\nabla}_{0i}\chi_i^{(+)} - \vec{\nabla}_{0i}\chi_i^{(-)}\cdot\vec{\kappa}_i[1-\exp(-i\chi_i^{(+)})]\}dz_0\,, \tag{2.21}$$

where

$$\vec{\kappa}_j = -i\Phi_{INT}^{-1}\vec{\nabla}_{0j}\Phi_{INT} \tag{2.22}$$

is the momentum contributed by the internal wave function Φ_{INT}. Because of the momentum dependence of v from which this correction term arises, we expect it to be sensitive to high momentum components in the target wave function.

III. PROTON-^{4}He ELASTIC SCATTERING

The leading correction terms to the Glauber amplitude (2.1) are calculated for p-^{4}He elastic scattering using a model[3,4] containing the following ingredients:

(i) The nucleon-nucleon (NN) scattering amplitude is a spin-isospin averaged Gaussian function

$$f(q) = \frac{ik\sigma}{4\pi}(1 - i\rho)\exp(-\beta^2 q^2)\,. \tag{3.1}$$

Here σ is the total NN cross section, ρ is the ratio $\mathrm{Re}f/\mathrm{Im}f$ and is assumed to be independent of q.
(ii) A Gaussian pseudopotential

$$v'(r) = v_0 \exp(-\gamma^2 r^2) \tag{3.2}$$

is used, with parameters (v_0, γ) obtained by fitting $f(q)$ at $q = 0$ and q_1.
(iii) A product nuclear wave function

$$\Phi = \prod_{j=1}^{A} \phi(\vec{r}_j)\,, \tag{3.3}$$

with single Gaussian single-particle wave functions

$$\phi(\vec{r}) = \phi_{SG}(\vec{r}) = (\alpha^2/\pi)^{3/4}\exp(-\tfrac{1}{2}\alpha^2 r^2)\,, \tag{3.4}$$

is used for ^{4}He in the calculation of correction terms. Here $\alpha^2 = 0.535$ fm^{-2}.
(iv) Relativistic kinematics is used in calculating the momentum k.

Explicit expressions for the correction terms can now be obtained. Substituting Eqs. (3.2)–(3.4) into Eqs. (2.18), (2.19), and (2.21), we find

$$F_W = -\exp(q^2/4A\alpha^2)(v_0/\hbar u)^2 2^5 \frac{\alpha^8}{\gamma}\sqrt{\tfrac{1}{2}\pi}$$
$$\times\left\{\left[1 - \frac{\mu}{\mu_{ab}} + \frac{\mu}{m}\left(1 - \frac{1}{A}\right)\right](I_1 - 4\gamma^2 I_2) + \left(1 + 0 - \frac{\mu}{M}\right)\left[\frac{3\alpha}{(\alpha^2+\gamma^2)^{1/2}} I_3 + \frac{12\gamma^2}{\alpha}(\alpha^2+\gamma^2)^{1/2} I_4\right]\right\}, \tag{3.5}$$

$$F_{ZPM} = \exp(q^2/4A\alpha^2)6i\frac{\mu}{\mu_{ab}}\left(\frac{v_0}{\hbar u}\right)\alpha^{12} I_{ZPM}\,. \tag{3.6}$$

Here

$$I_i = I_i(q^2) = \int_0^\infty b_0 db_0 J_0(qb_0)\exp(-4\alpha^2 b_0^2) R_i(b_0)\,, \tag{3.7}$$

where

$$R_1 = H_{00}{}^3 H_{20}\,, \quad R_2 = H_{00}{}^3 H_{22}\,, \tag{3.8}$$
$$R_3 = H_{00}{}^2 H_{10}{}^2\,, \quad R_4 = H_{00}{}^2 A_{11}{}^2\,,$$

$$R_{ZPM} = \frac{2}{\alpha^2} H_{00}{}^3 H_{10}^P + 3H_{00}{}^3 H_{12}^P + 6H_{00}{}^2 A_{01} A_{11}^P + H_{00}{}^2 H_{02} H_{10}^P - 2H_{00} A_{01}{}^2 H_{10}^P + \frac{8\gamma^2}{\alpha^2}(H_{00}{}^3 H_{12}^Q + H_{00}{}^2 A_{01} A_{11}^Q)\,. \tag{3.9}$$

The functions H_{mn}, A_{mn}, $H_{mn}^{P(Q)}$, $A_{mn}^{P(Q)}$ of b_0, and analytic expressions for the integrals I_i are given in the Appendix A. We would like to point out here the following features of the above results. (i) Each term in R_i is a product of four H or A factors, with each target nucleon contributing one factor. (ii) The subscript m in H_{mn}^x or A_{mn}^x denotes the number of times that the target nucleon is involved actively in the perturbation responsible for the correction term. If $m = 0$, the target nucleon is "passive"; it then contributes only a wave-distortion effect through the Glauber function $\exp(i\chi)$. Thus I_1, I_2, and I_{ZPM} involve a single active nucleon, while I_3 and I_4 involve two active nucleons. (iii) The subscript n in H_{mn} denotes whether the effect is longitudinal ($n = 0$) or transverse ($n = 2$) in the Breit frame (c.m. frame with $\hat{z} = \hat{k}_{av}$). A_{mn} ($n = 1$ always) is always transverse. The longitudinal (or transverse) effect comes from a derivative of $\chi_i^{(\pm)}$ with respect to z (or b). Thus I_1 and I_3 are longitudinal, while I_2 and I_4 are transverse. (iv) For proton-nucleus scattering $\mu/\mu_{ab} = 2A/(A+1)$, $\mu/m = A/(A+1)$, and $\mu/M = 1/(A+1)$.

Equation (3.5) agrees with the results of Wallace[13] when two minor misprints in his Table I are corrected, as discussed in the Appendix A. Also our internal-excitation, or recoil, correction is called a Fermi-motion effect by Wallace. The real zero-point (or Fermi) motion correction of Eq. (3.6) is a new result of this paper.

Finally, the Glauber scattering amplitude itself will be calculated with the nuclear density function (for ^{4}He)

$$\rho_A(\vec{r}_1, \vec{r}_2, \vec{r}_3, \vec{r}_4) = N\delta(\vec{r}_1 + \vec{r}_2 + \vec{r}_3 + \vec{r}_4)\prod_{j=1}^{4}\rho_1(\vec{r}_j)\,, \tag{3.10}$$

where

$$\rho_1(\vec{r}) = \exp(-\alpha_1{}^2 r^2) - D\exp(-\alpha_2{}^2 r^2)\,, \tag{3.11}$$

and N is a normalization constant. The resulting Glauber amplitude for p-^{4}He elastic scattering is

$$F_G(q) = \sum_{n=1}^{4} F_G^{(n)}(q)\,, \tag{3.12}$$
$$F_G^{(n)}(q) = \tfrac{1}{2}ik(-1)^{n+1}\binom{A}{n}\left[\frac{\sigma(1-i\rho)\alpha_1{}^2}{2\pi(1+4\alpha_1{}^2\beta^2)}\right]^n W_n(q)\,,$$

where the function $W_n(q)$ is independent of the parameter ρ. Its explicit expression is given in Appendix B.

TABLE I. Two nucleon parameters.

Quantity \ E_L (GeV)	0.58	1	2.1
k (GeV/c)	0.871	1.169	1.761
σ (fm^2)	3.9	4.4	4.4
ρ	0.43	-0.3	-0.35
β^2 [(GeV/c)$^{-2}$]	2.15	2.725	3.125
Reference	15	15, 16	17
v_0 (GeV)	$0.425-0.014i$	$0.437-0.070i$	$0.396-0.102i$
γ (fm^{-1})	$-1.748+0.346i$	$1.612+0.312i$	$1.530+0.270i$

IV. RESULTS AND DISCUSSION

Calculations are made at three proton laboratory energies $K_L=0.58$, 1, 2.1 GeV using nonrelativistic kinematics. Gaussian pseudopotentials v' of Eq. (3.2) are fitted to Gaussian NN amplitudes $f(q)$ of Eq. (3.1) at $q^2=0$ and $q^2=q_1{}^2=0.3$ (GeV/c)^2. This fitting procedure ensures a good approximation for $q^2 \lesssim 0.4$ (GeV/c)^2, which covers the most important range for single scattering. Parameters defining $f(q)$ and v' at different energies are given in Table I. We note that variations of the matching point $q_1{}^2$ give rise to less than 2% variations in the elastic p-^{4}He differential cross section at the secondary maximum [$q^2 \approx 0.35$ (GeV/c)^2].

Figure 1 shows the moduli and phases of the corrections to the p-^{4}He elastic scattering amplitudes. For the moduli, the solid curves denote the total correction $F_1=F_W+F_{ZPM}$, while the broken curves give the results F_W only. The phases shown are those of F_1. They differ significantly from the phases in the absence of the ZPM term. (The latter phases, which are not shown, have the opposite sign at $q^2=0$ and pass through zero at the minimum [$q^2\approx 0.1$ (GeV/c)^2] of the corresponding amplitude curve.) We thus see that the ZPM term represents an important contribution for all q^2 values.

Two features of Fig. 1 deserve mention. First, the moduli are roughly energy independent. Roughly speaking, this is so because F_1 is of order $k^{-1}F_G$ and is therefore roughly proportional to the total NN cross section σ, and σ is roughly the same for all three energies (see Table I). Secondly, the phases pass zero near $q^2=0.4$ (GeV/c)^2 and are roughly the same for the cases $K_L=1$ and 2.1 GeV. It turns out that the phase depends primarily on the parameter ρ of the NN scattering amplitude, and it is an odd function of ρ. [To see the latter point, we note that for the Glauber amplitude (3.12), ReF_G is odd in ρ, while ImF_G is even in ρ. The correction F_1 is proportional to $F_G{}^2$ as far as the ρ dependence is concerned. Hence ReF_1 is even in ρ, while ImF_1 is odd.] Thus the energy dependence of the phases of F_1 shown in Fig. 1 is a direct consequence of the energy dependence of ρ.

Also plotted in Fig. 1 are the full Glauber amplitude $|F_G^{SG}|$ and its double scattering part $|F_G^{(2)SG}|$. These are calculated at 0.58 GeV with a single Gaussian density for ^{4}He, i.e., by using Eq. (3.11) with $\alpha_1{}^2=\alpha^2=0.535$ fm^{-2}, and $D=0$. We see that in the ab-

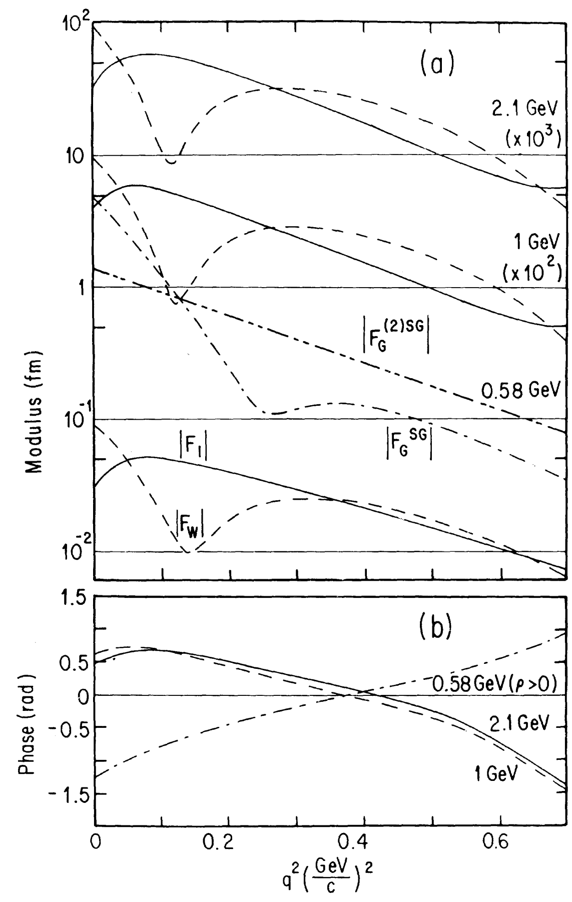

FIG. 1. (a) Moduli of the scattering amplitudes F_1 (total correction: full curves), F_W (wave-spreading plus internal-excitation correction: broken curves), F_G^{SG} (Glauber amplitude with a single-Gaussian density function for ^{4}He: dash-dot curve), and $F_G^{2(SG)}$ (the double-scattering part of F_G^{SG}: dash-double-dot curve). (b) Phases of F_1 at 2.1 GeV (full curve), 1 GeV (broken curve), and 0.58 GeV (dash-dot curve).

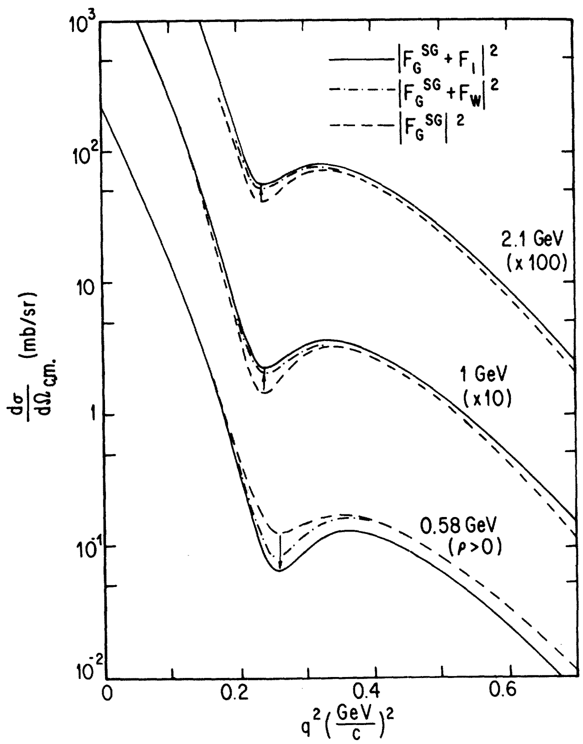

FIG. 2. Differential cross sections for p-^{4}He elastic scattering at 0.58, 1, and 2.1 GeV calculated with a single-Gaussian density function for ^{4}He.

sence of the ZPM correction $|F_W/F_G^{(2)SG}|$ has a complicated q^2 dependence. With its inclusion, we find $|F_1/F_G^{(2)SG}| \approx 0.1$ at 0.58 GeV. It is not clear, however, if this relationship is not somewhat accidental, since F_1 may be more sensitive to the choice of density function for ^{4}He than $F_G^{(2)}$.

We next calculate $d\sigma/d\Omega$ after adding these correction terms to the Glauber amplitude F_G^{SG} calculated with the single-Gaussian density. The results are shown in Fig. 2. The Glauber results are represented by broken curves. The dash-dot curves include F_W, while the solid curves contain the full correction F_1. We see that the corrections decrease in importance as energy increases, in agreement with the k^{-1} behavior of the perturbation expansion for the scattering amplitude. The corrections fill up the first minimum for $K_L = 1$ and 2.1 GeV for which $\rho<0$, and deepen the first minimum for $K_L = 0.58$ GeV for which $\rho>0$. The precise manner in which the dependence on the sign of ρ occurs will be discussed later. Finally, we note that the ZPM effect is particularly important at the lower energies and for the larger q^2 values.

These results cannot be compared with Wallace's[13] because he uses a double-Gaussian (DG) density and a special relativistic kinematics.

The effect of the DG density alone is illustrated in Fig. 3. We used the DG density of Bassel and Wilkin[4] [corresponding to Eq. (3.11) with $\alpha_1^2 = 0.579$ fm^{-2}, $\alpha_2^2 = 2.459$ fm^{-2}, and $D = 0.858$], which gives a nice fit to the e-^{4}He elastic form factor. We see that the diffraction pattern is compressed and the second maximum is raised (by a factor of almost 2), as expected. The effects of the correction terms appear roughly the same as before. Of course, the treatment is now somewhat inconsistent, since correction terms are calculated with a single-Gaussian wave function. We believe that F_W will not be changed significantly in a better treatment, but F_{ZPM} may be increased owing to the presence of high-momentum components in the wave function. The total effect on $d\sigma/d\Omega$ will probably be qualitatively the same. (The fourth set of curves in Fig. 3 corresponds to $\rho = -0.43$. They will be discussed later.)

Finally, we examine the effect of Wallace's relativistic kinematics,[13] which is not a conventional

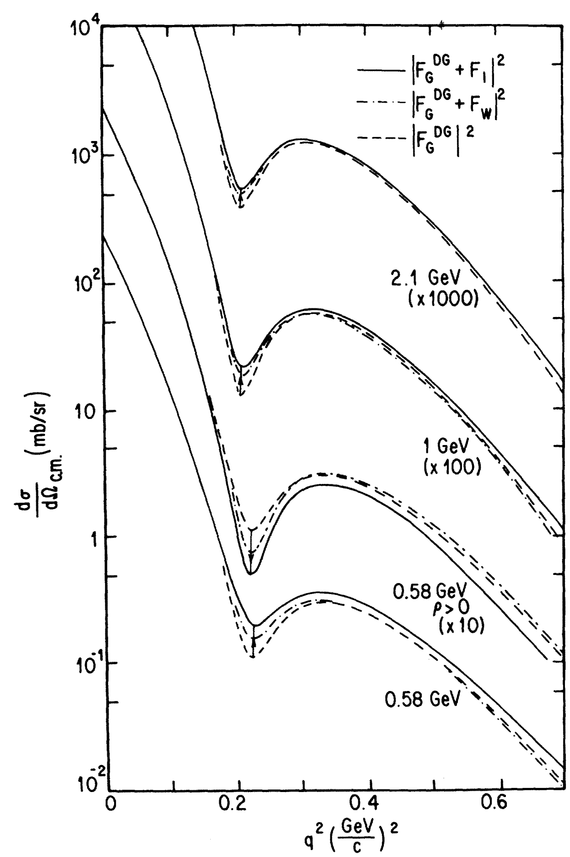

FIG. 3. Differential cross sections for p-^{4}He elastic scattering at 0.58, 1, and 2.1 GeV calculated with a double-Gaussian density function for ^{4}He. The lowest set of curves are for 0.58 GeV, but calculated with $\rho = -0.43$.

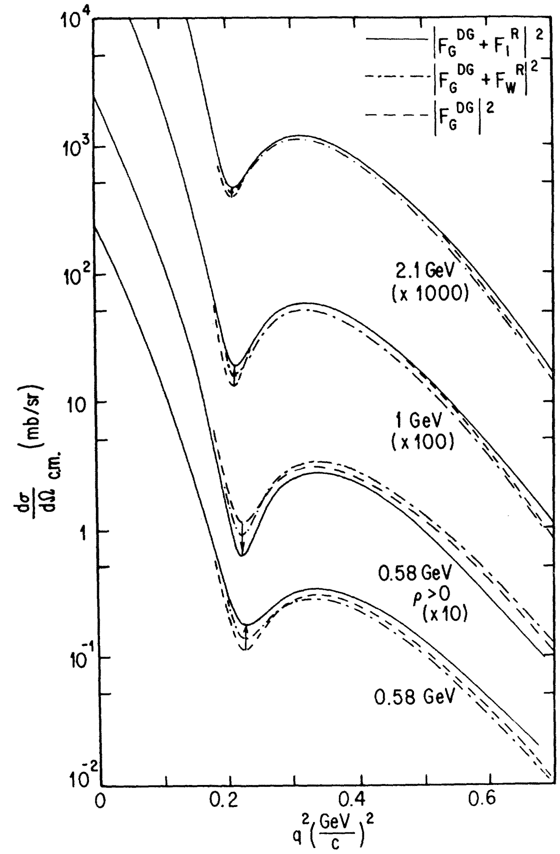

FIG. 4. Same as Fig. 3, but with the inclusion of Wallace's relativistic kinematics.

one. Wallace's prescription is (i) to replace the reduced masses μ_{ab} and μ by the relativistic analogs

$$\epsilon_2 = E_L(m/\sqrt{s_2}), \quad \epsilon = E_L(M/\sqrt{s}), \tag{4.1}$$

where

$$\begin{aligned} s_2 &= (m_0{}^2 + m^2 + 2mE_L/c^2)c^4, \\ s &= (m_0{}^2 + M^2 + 2ME_L/c^2)c^4, \\ E_L &= K_L + m_0c^2, \end{aligned} \tag{4.2}$$

and m_0 is the projectile rest mass, and (ii) to add to the correction F_i a relativistic factor $(m_0+M)c^2/\sqrt{s}$. The major change is in F_W^R (the superscript denoting the relativistic result), where the cancellation between WS and INT terms in the first two terms of Eq. (3.5) is no longer complete, as pointed out by Wallace.[13] The resulting cross sections, shown in Fig. 4, look significantly different, being lower than the nonrelativistic results at and beyond the first minimum.

The detailed effect of F_i on $d\sigma/d\Omega$ depends, of course, on both their moduli and phases. It is therefore interesting to show these explicitly. We give in Fig. 5 the results for the Glauber amplitudes F_G with single-Gaussian (SG) or double-Gaussian (DG) density, the total corrections F_1 (nonrelativistic) and F_1^R (relativistic), and F_G+F_1. Also shown are the components of F_1: the sum of two-body and many-body contributions to WS and INT corrections and the ZPM correction. We see that Wallace's relativistic kinematics reduces the modulus and changes the phase of F_W. Both features cause reduction in $d\sigma/d\Omega$.

Figure 5 also illustrates the dependence of the effect of F_1 on the sign of ρ. We first note from Figs. 3 and 4 that the first minimum is filled in

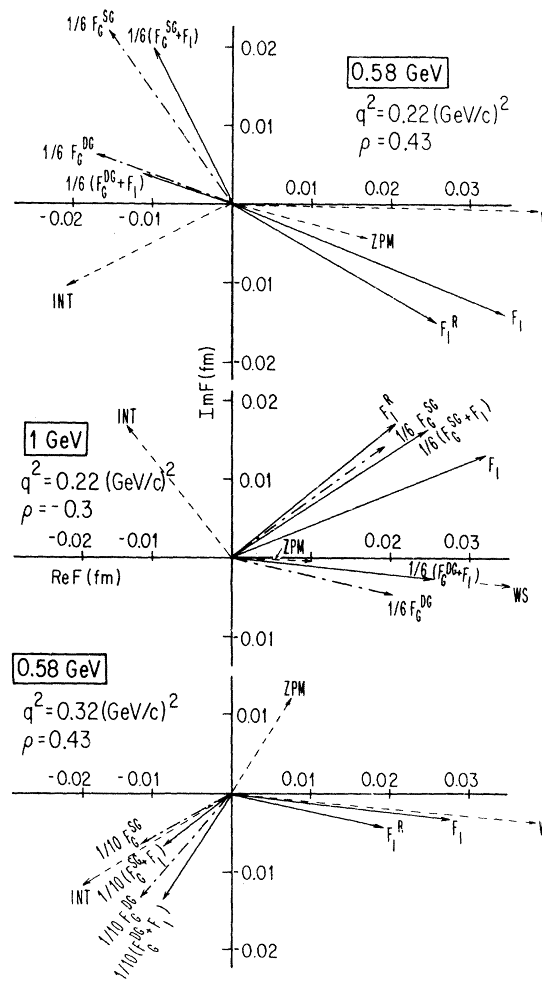

FIG. 5. Complex vector diagrams for the scattering amplitudes F_1 (total nonrelativistic correction), its component WS (wave-spreading), INT (internal-excitation), and ZPM (zero-point motion) contributions, F_1^R (total correction with Wallace's relativistic kinematics), and the Glauber amplitudes F_G^{SG} (for a single-Gaussian density function) and F_G^{DG} (for a double-Gaussian density function) at (a) the first minimum of $d\sigma/d\Omega$ at 0.58 GeV, (b) the first minimum at 1 GeV, and (c) the second maximum at 0.58 GeV.

when $\rho<0$, but deepened when $\rho>0$. (The actual sign of ρ is not well determined at 0.58 GeV, according to Gurvitz *et al.*[14] We therefore show the results for $\rho=-0.43$ also in Figs. 3 and 4.) Figure 5(a) shows that at the first minimum for 0.58 GeV, there is destructive interference between F_G (in the second quadrant) and F_1 (in the fourth quadrant) when $\rho=0.43$ is used. If ρ changes sign, $\mathrm{Re}F_G$ and $\mathrm{Im}F_1$ change sign, while $\mathrm{Im}F_G$ and $\mathrm{Re}F_1$ are unchanged. Both vectors then move into the first quadrant, causing constructive interference and a filling in of the minimum in the cross section. A rather similar constructive interference at the first minimum for 1 GeV is illustrated in Fig. 4(b). Figure 4(c) describes a rather interesting situation at (and also beyond) the second maximum, here at 0.58 GeV. Here we see that although F_{ZPM} is shorter than F_W (WS + INT), the former has a greater effect on the cross section. This is because the latter tends to change the phase rather than the length of F.

Our results for $|F_G^{DG}+F_W^R|^2$ (dash-dot curves in Fig. 4) are in rough agreement with those of Wallace.[13]

It is interesting to compare our model with the multiple-scattering (MS) model of Bleszynski and Jaroszewicz.[8] The latter authors assume that the t matrix is a function of the three momentum transfer. Their model contains no recoil, rescattering, or ZPM corrections. However, wave-spreading effects are included exactly, since the (nonrelativistic) propagator is not linearized. (A similar, but more complete, MS model has been studied earlier by Lykasov and Tarasov,[18] who include also recoil and charge-exchange corrections.) We would like to mention the following differences between these models: (i) There is no correction in the MS model of Ref. 8 involving only one target nucleon and corresponding to the first term in Eq. (3.5). (ii) The MS correction to the double-scattering Glauber amplitude has roughly the same phase, but a much greater modulus (≈2.5 times greater for q^2 at and beyond the first minimum) than the leading MD correction, i.e., the second term of Eq. (3.5). The bigger MS correction again tends to change the phase rather than the modulus of the Glauber amplitude, except near the interference minima where the situation is more complex. (iii) The filling in of the minima of the cross section at 1 GeV is much greater in the MS model for the same nonrelativistic kinematics and the same NN scattering amplitude. This arises from the bigger MS correction mentioned before. (iv) The MDE model contains recoil, certain rescattering, and ZPM effects. Of these, the ZPM effect is a special feature; it originates in a difference in assumption concerning the NN t matrix.

The above preliminary comparison between these models suggests that even in the first minimum/second maximum region there are significant model-dependent differences which have to be clarified. In addition, one should also determine the extent to which the important relativistic kinematical corrections can be made uniquely. A detailed study of these questions should be rewarding.

All the MD correction terms considered here depend to a certain extent on the assumed model of NN interaction. The form of this interaction is dictated more by convenience (so that Glauber's empirical approximation is obtained as the leading term), than by fundamental physical considerations. Nevertheless, we expect the present procedure to be quite reliable at high energies provided that the system is not too far off shell. It is not clear, however, to what extent this expectation remains valid when the wave function contains significant high-momentum components.

V. CONCLUDING REMARKS

We have analyzed the leading MDE corrections to the Glauber MD approximation in a simple model of p-^{4}He elastic scattering at energies 0.58, 1, and 2.1 GeV. Our results may be summarized as follows.

(i) Our formula for the wave-spreading plus internal-excitation correction F_W agrees with that of Wallace.[13] Its inclusion causes the first minimum in $d\sigma/d\Omega$ to fill up when ρ ($=\mathrm{Re}F/\mathrm{Im}f$ of the NN scattering amplitude) <0, and to deepen when $\rho>0$.

(ii) The use of Wallace's relativistic kinematical corrections shortens the modulus of F_W and changes its phase. Both changes are such as to reduce the effect of F_W on $d\sigma/d\Omega$. Thus the choice and inclusion of correct relativistic kinematics are important considerations, even at 0.58 GeV.

(iii) A zero-point motion (ZPM) correction not included by Wallace is found to have important effects on $d\sigma/d\Omega$ because of its effectiveness in changing the modulus of the scattering amplitudes.

(iv) The total correction $F_1=F_W+F_{\mathrm{ZPM}}$ is roughly proportional to the Glauber double-scattering amplitude $F_G^{(2)}$ for a range of momentum transfers. Also $|F_1/F_G^{(2)}|\approx 0.1$ at 0.58 GeV. Since $F_G^{(2)}$ increases roughly linearly with k in the energy range studied (as a result of the NN total cross section σ being roughly energy independent), we find that F_1, which is of order k^{-1} relative to $F_G^{(2)}$, is roughly the same for all these energies. These results are obtained with an oscillator wave function for ^{4}He. If the nuclear wave function contains more high-momentum components, the resulting $F_G^{(2)}$

is expected to remain roughly the same, while F_1 may change because of the momentum dependence of its ZPM contributions.

(v) $\mathrm{Re}F_1$ and $\mathrm{Im}\,F_G$ are even in $\rho = \mathrm{Re}f/\mathrm{Im}f$ of the projectile-nucleon scattering amplitude f, while $\mathrm{Im}\,F_1$ and $\mathrm{Re}F_G$ are odd in ρ. This property accounts for the deepening of the first minimum of $d\sigma/d\Omega$ at 0.58 GeV in $\rho > 0$, and its being filled in at 1 and 2.1 GeV at which $\rho < 0$.

A few words of caution are in order in describing the effect of the correction terms on $d\sigma/d\Omega$. The effect depends on the phase as well as the modulus of the Glauber amplitude F_G. In a more realistic calculation, F_G should include spin-dependent,[19] Coulomb,[20] and other inelastic[4,11,18] contributions also. Consequently, the effect on $d\sigma/d\Omega$ might change. It is therefore interesting to note that a recent calculation of p-^{4}He elastic scattering by Auger, Gillespie, and Lombard[21] has included both spin-dependent and Coulomb contributions in F_G and an approximation to F_W. They find that the effect of the correction term on $d\sigma/d\Omega$ is roughly the same with or without spin and Coulomb contributions. In general, it appears more useful to use F_i, rather than its effect on $d\sigma/d\Omega$, in discussing the importance of correction terms.

We should also note that although the ratio $F_1/F_G^{(2)}$ appears reassuringly small even after the second maximum in $d\sigma/d\Omega$, we have no information on the behavior of higher-order corrections in the MDE. Consequently, the expected convergence of the MDE in this range of q^2 values has not actually been demonstrated. A particularly interesting question in this connection concerns the extent to which the partial cancellation between wave-spreading and internal-excitation effects might be destroyed in higher orders, thus leading to significant corrections, especially at the larger momentum transfers.

Finally, it should be emphasized that the question of the uncertainty in off-shell extrapolation cannot be studied in the present MDE, which represents one particular off-shell extrapolation procedure. Recent studies of related questions in different formalisms can be found in Ref. 22.

Thus there are still some unanswered questions which may affect the reliability of nuclear information extracted from differential cross sections for elastic scattering. The present study does show that the leading corrections to the Glauber theory can be calculated relatively easily and that these corrections decrease in importance as energy increases. These results suggest that the MD expansion considered here is a useful formalism for the analysis of projectile-nucleus scattering over an important range of momentum transfers.

We thank Professor S. J. Wallace for a conversation which clarifies the structure of the internal-excitation correction.

APPENDIX A. EXPLICIT FORMULAS FOR CORRECTION TERMS

We give here certain calculational details and the final formulas for the correction terms to the Glauber amplitude for p-^{4}He scattering. The calculation employs the Gaussian pseudopotential (3.2) and the oscillator wave function (3.4). For the scattering amplitude F_W of Eq. (2.19), integration over target nucleon coordinates yields a product of functions H_{mn} or A_{mn}, one for each target nucleon. Here

$$H_{mn} = \int_0^\infty J_0(i2\alpha^2 b b_0)\exp[-\alpha^2 b^2 + i\chi(b) - m\gamma^2 b^2]\, b^n\, b\,db\,, \tag{A1}$$

while A_{mn} ($n=1$ only) is given by the same formula, but with the Bessel function J_1 instead of J_0. The phase shift function

$$e^{i\chi(b)} = 1 - \sigma_1 e^{-\omega^2 b^2}\,, \quad \sigma_1 = \frac{\sigma(1-i\rho)}{8\pi\beta^2}\,, \quad \omega^2 = (4\beta^2)^{-1} \tag{A2}$$

is that due to a single target nucleon. The b_0 integration in Eq. (3.7) can now be performed. The results are

$$I_1 = \tfrac{1}{8}\,\alpha^{2-2A}\sum_{n=0}^{A-1}\binom{A-1}{n}(-\sigma_1 d_0\alpha^2)^n[a_2\Lambda_n\exp(-\Lambda_n q^2) - \sigma_1 d_2\Delta_n\exp(-\Delta_n q^2)]\,, \tag{A3a}$$

$$I_2 = \tfrac{1}{8}\,\alpha^{2-2A}\sum_{n=0}^{A-1}\binom{A-1}{n}(-\sigma_1 d_0\alpha^2)^n\{a_2{}^2\Lambda_n[1 + 4a_2\alpha^4\Lambda_n(1-\Lambda_n q^2)]\exp(-\Lambda_n q^2) - \sigma_1 d_2{}^2\Delta_n[1 + 4d_2\alpha^4\Delta_n(1-\Delta_n q^2)]\exp(-\Delta_n q^2)\}\,, \tag{A3b}$$

$$I_3 = \tfrac{1}{8}\,\alpha^{4-2A}\sum_{m=0}^{A-2}\sum_{n=0}^{A-2}\binom{A-2}{n}\binom{A-2}{m}a_1^{A-2-m}(-\sigma_1 d_0\alpha^2)^n(-\sigma_1 d_1)^m D_{mn}\exp(-D_{mn}q^2)\,, \tag{A3c}$$

$$I_4=-\tfrac{1}{2}\,\alpha^{8-2A}\sum_{m=0}^{A-2}\sum_{n=0}^{A-2}\binom{A-2}{n}\binom{A-2}{m}a_1^{2(A-2-m)}(-\sigma_1 d_0\alpha^2)^n(-\sigma_1 d_1^2)^m D_{mn}{}^2(1-D_{mn}q^2)\exp[-D_{mn}q^2]\,. \tag{A3d}$$

Here

$$\begin{aligned}&a_m=(\alpha^2+m\gamma^2)^{-1}\,,\quad d_m=(\alpha^2+m\gamma^2+\omega^2)^{-1}\,,\\&\lambda_m=m\gamma^2\alpha^2(\alpha^2+m\gamma^2)^{-1}\,,\\&\delta_m=\alpha^2(m\gamma^2+\omega^2)(\alpha^2+m\gamma^2+\omega^2)^{-1}\,,\\&\Lambda_n=\tfrac{1}{4}\,(\lambda_2+n\delta_0)^{-1}\,,\quad \Delta_n=\tfrac{1}{4}\,(\delta_2+n\delta_0)^{-1}\,,\\&D_{mn}=\tfrac{1}{4}\,[n\delta_0+m\delta_1+(A-2-m)\lambda_1]^{-1}\,.\end{aligned} \tag{A4}$$

We note that setting $\sigma_1=0$ in I_i gives a Born approximation with no distortion at any target nucleon, according to Eq. (A2). The sum in I_i then collapses to a single term $(m=0, n=0)$. For example,

$$\begin{aligned}&I_3\approx 2^{-5}(a_1/\alpha^2)^{A-2}\exp[-q^2/4(A-2)\lambda_1]/(A-2)\lambda_1\,,\\&I_4\approx-2^{-5}\alpha^4(a_1{}^2/\alpha^2)^{A-2}\left[1-\frac{q^2}{4(A-2)\lambda_1}\right]\\&\qquad\times\exp[-q^2/4(A-2)\lambda_1]/(A-2)^2\lambda_1{}^2\end{aligned} \tag{A5}$$

are obtained. Substitution of Eq. (A5) into Eq. (3.5) then yields (for $A=4$)

$$\begin{aligned}F_W(q^2)&=F_W(q^2=0)(1-q^2/4\lambda_1)\\&\quad\times\exp\left[-\frac{q^2}{16}\left(\frac{2}{\lambda_1}-\frac{1}{\alpha^2}\right)\right],\end{aligned} \tag{A6}$$

$$F_W(0)=\tfrac{6}{5}\,(\tfrac{1}{2}\,\pi)^{1/2}\left(\frac{v_0}{\hbar u\gamma^3}\right)^2\alpha^3\left(1+\frac{\alpha^2}{\gamma^2}\right)^{-3/2}.$$

Since

$$\frac{v_0}{\hbar u\gamma^3}=-\tfrac{1}{2}\,i\pi^{-3/2}(1-i\rho)\sigma\sum_{n=0}^{\infty}\frac{\sigma_1^n}{(n+1)^2} \tag{A7}$$

and

$$\alpha^3=(\tfrac{1}{2}\,\pi)^{1/2}\pi\bar\rho_1\,, \tag{A8}$$

where $\bar\rho_1$ is the average nucleon density in ^{4}He (≈ 0.11 fm^{-3}), we may write

$$F_W(0)\approx C\,\frac{\bar\rho_1\sigma^2}{4\pi}\left(1+\frac{\alpha^2}{\gamma^2}\right)^{-1/2}. \tag{A9}$$

Here

$$C=\tfrac{1}{2}(1-i\rho)^2\left[\sum_{n=0}^{\infty}\frac{\sigma_1^n}{(n+1)^2}\right]^2 \tag{A10}$$

is a dimensionless complex constant. Since $|C|$ is of order unity (actually $|C|=0.957$, 1.075, and 1.08 at 0.58, 1, and 2.1 GeV, respectively), we see from Eq. (A9) and Table I that $F_W(q^2=0)\approx 0.1$ fm. This is in rough agreement with the exact results shown in Fig. 1.

Our results for F_W agree with Wallace's when the following misprints (or approximations) are corrected. In Wallace's Eq. (34), the factor $[2\pi(B+\beta)]^{-1/2}$ should read $[2\pi(B+\gamma)]^{-1/2}$. In his Table 1, the constant γ_2 involves a factor $(\beta+\gamma/2)^{-1}$, not $(\beta+2\gamma)^{-1}$, while the constant c_2 is proportional to γ^{-1}, not $\gamma_1{}^{-1}$.

The calculation of the zero-point motion (ZPM) term is much more complicated because the z_0 integration cannot be done analytically. Integration over the target wave function yields Eq. (3.6) in which the integral I_{ZPM} depends on the target-nucleon functions

$$H_{mn}^{P[Q]}=\int_0^\infty J_0(i2\alpha^2bb_0)\begin{Bmatrix}P(b)\\Q(b)\end{Bmatrix}\exp[-\alpha^2b^2+i\chi(b)-m\gamma^2b^2]\,b^n\,bdb \tag{A11}$$

and $A_{mn}^{P(Q)}$, which differ from Eq. (A11) only in the replace of J_0 by J_1. The functions P and Q are defined by

$$P(b)=2\int_0^\infty z\exp(-\gamma^2z^2)[\exp(-i\chi_i^{(+)})-\exp(-i\chi_i^{(-)})]\,dz\,, \tag{A12a}$$

$$Q(b)=2\int_0^\infty z\exp(-\gamma^2z^2)[\exp(-i\chi_i^{(+)})-\exp(-i\chi_i^{(-)})+i(\exp(-i\chi_i^{(+)})\chi_i^{(-)}-\exp(-i\chi_i^{(-)})\chi_i^{(+)})]\,dz\,, \tag{A12b}$$

where

$$\chi_i^{(\pm)}=(-1)\frac{v_0}{\hbar u}\,\frac{\sqrt{\pi}}{2\gamma}\,\exp(-\gamma^2b^2)(1\pm\operatorname{erf}\gamma z)\,. \tag{A13}$$

The z integration for P, Q cannot be done analytically unless one can expand $\exp[-i\chi_i^{(\pm)}]\approx 1-i\chi_i^{(\pm)}$, in which case

$$P(b)\approx i\,\frac{v_0}{\hbar u}\,\frac{\sqrt{\pi}}{\gamma}\,\exp(-\gamma^2b^2)\left(\frac{1}{\sqrt{2}}\;\frac{1}{\gamma^2}\right)\approx\frac{Q(b)}{2}\,. \tag{A14}$$

This approximation is not valid for the intermediate energy range under study. However, an extremely good approximation can be found using the above approximation as a guide.

We find that the approximations

$$P(b) \approx i\,\frac{v_0}{\hbar u}\,\frac{\sqrt{\pi}}{\gamma}\,\exp(-\gamma^2 b^2)\left(\frac{1}{\sqrt{2}}\;\frac{1}{\gamma^2}\right)[1 - Z_1\exp(-\xi_1{}^2 b^2)]\,,$$
$$Q(b) \approx 2i\,\frac{v_0}{\hbar u}\,\frac{\sqrt{\pi}}{\gamma}\,\exp(-\gamma^2 b^2)\left(\frac{1}{\sqrt{2}}\;\frac{1}{\gamma^2}\right)[1 - 0.5\,Z_1\exp(-\xi_1{}^2 b^2) + Z_2\exp(-\xi_2{}^2 b^2)]\,, \tag{A15}$$

work to a few percent. Here Z_1, Z_2, ξ_1, and ξ_2 are complex constants. The values used are $Z_1 = 0.8394 + 1.0119i$ $(0.8219 - 1.3589i; 0.5443 - 1.4270i)$, $Z_2 = 0.1447$ $(0.2031; 0.1771)$, $\xi_1{}^2 = 2.1796 - 3.0820i$ $(2.1126 + 2.9096i; 2.1434 + 2.4294i)$, and $\xi_2{}^2 = 5.2$ $(5.85, 6.043)$ for $K_L = 0.58$ (1, 2.1) GeV.

With these approximations, the calculation can again be done analytically. The result for I_{ZPM} of Eqs. (3.7) and (3.9) is

$$\begin{aligned} I_{\text{ZPM}} = i(v_0/\hbar u)(\tfrac{1}{2}\pi)^{1/2}\gamma^{-3}\Big\{ & \frac{2}{\alpha^2}(I_1 - Z_1 I_1') + I_6 - Z_1 I_6' - 2(I_7 - Z_1 I_7') \\ & + \left(3 + 16\frac{\gamma^2}{\alpha^2}\right) I_2 - Z_1\left(3 + 8\frac{\gamma^2}{\alpha^2}\right) I_2' + 16\frac{\gamma^2}{\alpha^2} Z_2 I_2'' \\ & + \left(6 + 16\frac{\gamma^2}{\alpha^2}\right) I_5 - Z_1\left(6 + 8\frac{\gamma^2}{\alpha^2}\right) I_5' + 16\frac{\gamma^2}{\alpha^2} Z_2 I_2'' \Big\}. \end{aligned} \tag{A16}$$

Three new types of integrals ($I_{5,6,7}$) appear here. We list them for the sake of completeness:

$$\begin{aligned} I_5 &= \int_0^\infty J_0(qb_0)\exp(-4\alpha^2 b_0{}^2) H_{00}{}^2 A_{01} A_{21}\, b_0\, db_0 \\ &= -\tfrac{1}{2}\alpha^{8-2A}\sum_{n=0}^{A-2}\binom{A-2}{n}(-\sigma_1 d_0\alpha^2)^n\{(a_0 a_2\Lambda_n)^2(1 - \Lambda_n q^2)\exp(-\Lambda_n q^2) \\ &\qquad + (-\sigma_1 d_0{}^2)(a_2\Lambda_{n+1})^2(1 - \Lambda_{n+1}q^2)\exp(-\Lambda_{n+1}q^2) \\ &\qquad + (-\sigma_1 d_2{}^2)(a_0\Delta_n)^2(1 - \Delta_n q^2)\exp(-\Delta_n q^2) \\ &\qquad + (-\sigma_1 d_0{}^2)(-\sigma_1 d_2{}^2)\Delta_{n+1}^2(1 - \Delta_{n+1}q^2)\exp(-\Delta_{n+1}q^2)\}\,, \end{aligned} \tag{A3e}$$

$$\begin{aligned} I_6 &= \int_0^\infty J_0(qb_0)\exp(-4\alpha^2 b_0{}^2) H_{00}{}^2 H_{02} H_{20}\, b_0\, db_0 \\ &= \tfrac{1}{8}\{a_0{}^4 a_2\Lambda_0[1 + 4a_0\alpha^4\Lambda_0(1 - \Lambda_0 q^2)]\exp(-\Lambda_0 q^2) \\ &\quad + (-\sigma_1 d_0)a_0{}^2 a_2\Lambda_1[(2a_0 + d_0) + 4(2a_0{}^2 + d_0{}^2)\alpha^4\Lambda_1(1 - \Lambda_1 q^2)]\exp(-\Lambda_1 q^2) \\ &\quad + (-\sigma_1 d_0)^2 a_0 a_2\Lambda_2[(a_0 + 2d_0) + 4(a_0{}^2 + 2d_0{}^2)\alpha^4\Lambda_2(1 - \Lambda_2 q^2)]\exp(-\Lambda_2 q^2) \\ &\quad + (-\sigma_1 d_0)^3 d_0 a_2\Lambda_3[1 + 4d_0\alpha^4\Lambda_3(1 - \Lambda_3 q^2)]\exp(-\Lambda_3 q^2) \\ &\quad + 4 \text{ similar terms corresponding to the replacements } a_2 \to -\sigma_1 d_2,\ \Lambda_n \to \Delta_n\}\,, \end{aligned} \tag{A3f}$$

$$\begin{aligned} I_7 &= \int_0^\infty J_0(qb_0)\exp(-4\alpha^2 b_0{}^2) H_{00} A_{01}{}^2 H_{20}\, b_0\, db_0 \\ &= -\tfrac{1}{2}\alpha^4\sum_{n=0}^{A-1} c_n[a_2\Lambda_n{}^2(1 - \Lambda_n q^2)\exp(-\Lambda_n q^2) + (-\sigma_1 d_2)\Delta_n{}^2(1 - \Delta_n q^2)\exp(-\Delta_n q^2)]\,, \end{aligned} \tag{A3g}$$

where

$$c_0 = a_0{}^5, \quad c_1 = (-\sigma_1 d_0)a_0{}^3(a_0 + 2d_0), \quad c_2 = (-\sigma_1 d_0)(-\sigma_1 d_0{}^2)a_0(2a_0 + d_0), \quad c_3 = (-\sigma_1 d_0)(-\sigma_1 d_0{}^2)^2\,. \tag{A17}$$

We also need I_i' and I_i''. These differ from the I_i of Eqs. (A3) only in the replacement of the constants $(a_2, d_2, \lambda_2, \delta_2, \Lambda_n, \Delta_n)$ by their primed and double-primed analogs. The latter are defined by Eq. (A4) with the replacement of $2\gamma^2$ by $2\gamma^2 + \xi_1{}^2$ or by $2\gamma^2 + \xi_2{}^2$, respectively.

APPENDIX B. GLAUBER AMPLITUDE WITH DOUBLE-GAUSSIAN DENSITY

The function $W_n(q)$ which defines the Glauber MD amplitude (3.12) for the double-Gaussian nuclear density Eq. (3.11) is

$$W_n(q) = N(4\pi)^{-3/2}\left(\frac{\pi}{\alpha_1^2}\right)^{(3/2)A}\sum_{l=0}^{A-n}\sum_{m=0}^{n}\binom{A-n}{l}\binom{n}{m}(-1)^{l+m}\left(\frac{D\alpha_1^3}{\alpha_2^3}\right)^l\left[\frac{D(1+4\alpha_1^2\beta^2)\alpha_1}{(1+4\alpha_2^2\beta^2)\alpha_2}\right]^m$$

$$\times\lambda_{A,l+m}^{-1/2}(\lambda_{A-n,l}+\beta^2\mu_{n,m})^{-1}\nu_{l,m}^{(n)-1}\exp[-q^2/4\nu_{l,m}^{(n)}]. \tag{B1}$$

Here

$$\lambda_{p,q} = \frac{p}{4\alpha_1^2} + q\left(\frac{1}{4\alpha_2^2} - \frac{1}{4\alpha_1^2}\right),$$

$$\mu_{p,q} = \frac{p}{1+4\alpha_1^2\beta^2} + q\left(\frac{1}{1+4\alpha_2^2\beta^2} - \frac{1}{1+4\alpha_1^2\beta^2}\right), \tag{B2}$$

$$\nu_{l,m}^{(n)} = \frac{1}{4\beta^2}\left[n - \frac{\lambda_{A-n,l}\,\mu_{n,m}}{\lambda_{A-n,l}+\beta^2\mu_{n,m}}\right].$$

The normalization constant N, which also appears in Eq. (3.10), is given by

$$N = (4\pi)^{3/2}\left[\left(\frac{\pi}{\alpha_1^2\alpha_2^2}\right)^{(3/2)A}\sum_{m=0}^{A=4}\binom{A}{m}\alpha_2^{3(A-m)}(-D\alpha_1^3)^m(\lambda_{A,m})^{-3/2}\right]^{-1}. \tag{B3}$$

These formulas reduce to the well-known results for a single-Gaussian density by setting $D=0$ (and $\alpha_1=\alpha$):

$$W_n(q) = \frac{1}{n}\left(\frac{1+4\alpha^2\beta^2}{\alpha^2}\right)\exp\left[-\frac{q^2}{4\alpha^2}\left(\frac{1+4\alpha^2\beta^2}{n} - \frac{1}{A}\right)\right]. \tag{B4}$$

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