Extra nodes and the phase shift of the scattering wave function for a nonlocal potential

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This paper examines the presence of extra nodes in the scattering wave function for a nonlocal potential. Extra nodes are known to result from the nonlocality of an effective potential which incorporates the Pauli principle. It is shown that an extra node is directly linked to the existence of a continuum bound state or a spurious state in the scattering spectrum. Thus the presence of extra nodes occurs in conjunction with zeros of the Fredholm determinants $D^{\pm}(k)$ and $D(k)$ associated with the integral equations for the physical and regular scattering solutions, respectively. The behavior of the nodes due to spurious states and continuum bound states is differentiated. Two possible definitions of the phase shift for a nonlocal potential are discussed in connection with this behavior. Both are consistent with the local limit. The definition of the phase shift as the negative of the phase of the Jost function $\mathfrak{L}^{+}(k)$ is suggested as preferable. This definition is shown to be in accord with the nodal behavior of the wave function and its interpretation in terms of an absolute value of the phase shift. Examples of potentials with a spurious state and of potentials with a continuum bound state are given. The nodal behavior of the wave function and the associated phase shift behavior are examined for each.

[NUCLEAR REACTIONS Scattering by a nonlocal potential, extra nodes, Fredholm determinants and their zeros, continuum bound states, spurious states, phase shifts.]

I. INTRODUCTION

The effective potential describing the interaction between composite particles is inherently nonlocal.[1-7] Consider, for example, α-α scattering. Because of the Pauli principle, the α-α relative wave function must exhibit effects of excluded radial configurations. Calculations[8-11] using the resonating group formalism,[12] as well as those[13] based on the method of Saito,[2] show that the radial wave functions in the S and D partial waves have nodes at small distances. These nodes are required by the Pauli principle but are "extra" in the sense that there are no bound states of the α-α system. Such nodes are incompatible with a local potential model of the effective interaction. This feature of α-α scattering is discussed in detail in the introduction of Ref. 14, where it is shown that if the α-α relative wave function is to be the solution of a radial equation with an effective potential, that potential must be nonlocal.

The manner in which the nonlocality of the effective potential incorporates the Pauli principle in the resonating group formulation of the scattering of composite particles has another important consequence. The wave function for the relative motion of the clusters obtained by solving the resonating group integrodifferential equation exhibits redundancies. The redundancies are solutions (known as redundant solutions[15,16]) of the integrodifferential equation which occur in addition to the scattering solution. They can be added in any arbitrary amount to the scattering solution without affecting its asymptotic form.[17] In this regard, Shakin and Weiss[18] have examined the existence of the extra nodes in the solution of the resonating group radial equation. They ascribe the nodes to the fact that the Pauli principle restricts the scattering wave function to a subspace which is orthogonal to the space spanned by the redundant solutions. Subspace-restricted scattering, known as orthogonality scattering, has been extensively investigated by Scheerbaum, Shakin, and Thaler[19] and by Shakin and Thaler.[20] The effective potential in subspace-restricted scattering is necessarily a nonlocal potential. Again one sees the connection between additional nodes in the scattering wave function and the nonlocality of the effective potential.

In this paper we employ Fredholm determinants[14,21] to discuss nonlocality. These determinants are essential in describing properties of the scattering wave function for a nonlocal potential. We demonstrate the connection between a single, simple zero of Fredholm determinants and the presence of an additional node in the radial wave function. As shown in Ref. 14, two different kinds of behavior, spurious states and continuum bound states, may result from zeros of Fredholm determinants associated with a nonlocal potential.

It will be demonstrated that the presence of additional nodes in the wave function is found with both. The manner in which the extra node behaves as a function of the energy depends upon the type of state under consideration.

The connection between a zero of the Fredholm determinant associated with one or more of the radial scattering integral equations for a nonlocal potential and the presence of an extra node in the radial wave function is established by examining the ambiguity which occurs in the definition of the scattering phase shift due to the zero of the Fredholm determinant. In this regard, we review in Sec. III the definition of the phase shift for a short-range local potential. For a local potential an ambiguity of $p\pi$, where p is an integer, exists in the absolute value of the zero-energy phase shift. This is due to the possibility of bound states for that potential. The ambiguity is resolved by the fact that each bound state of the potential results in an additional node of the radial wave function relative to the free-particle wave function. Thus an absolute phase shift for a local potential can be obtained by counting the number of additional nodes of the wave function, or by counting the number of bound states and applying Levinson's theorem.[22]

For a nonlocal potential, ambiguities in the phase shift exist in addition to that due to the presence of negative energy bound states. The ambiguity in the phase shift due to a continuum bound state has been discussed in detail by Bolsterli.[23] We find that a similar ambiguity exists with respect to the definition of the phase shift in the presence of a spurious state. We investigate modifications of Levinson's theorem which are necessary in light of this.

The analysis in this paper applies to nonlocal potentials which are symmetric and real. For simplicity, we consider the $l=0$ partial wave; similar considerations also apply for $l>0$. The results presented in Secs. II–V are illustrated by specific examples in Secs. VI and VII. The conclusions concerning necessary revisions of Levinson's theorem are given in Sec. VIII.

II. DEFINING EQUATIONS FOR FREDHOLM DETERMINANTS

For $l=0$ the radial equation for a nonlocal potential is

$$u(k,r)'' + k^2 u(k,r) = \int_0^\infty V(r,r')\,u(k,r')\,dr' . \tag{1}$$

In all cases, the potential $V(r,r')$ will be assumed to be symmetric and real. We impose on the potential the condition that there exists a $\beta>0$ such that

$$e^{\beta r}\int_0^\infty V(r,s)\,ds < \infty . \tag{2}$$

This condition is sufficient to insure the convergence of all integrals given in this paper. Equation (1) can be converted to an integral equation by the use of the appropriate Green's function. Several integral equations and their solutions must be considered in analyzing the phase shift.

The physical wave function $\psi^+(k,r)$ and its conjugate $\psi^-(k,r)$ are defined to be the solutions of the pair of integral equations

$$\psi^\pm(k,r) = \sin kr + \int_0^\infty\int_0^\infty G^\pm(k,r,r')\,V(r',s)\,\psi^\pm(k,s)\,ds\,dr' \tag{3}$$

with

$$G^\pm(k,r,r') = -k^{-1}e^{\pm ikr_>}\sin(kr_<) . \tag{4}$$

The Fredholm determinants associated with the kernels of Eqs. (3) are referred to as $D^\pm(k)$. The solutions of Eqs. (3) will be regular at the origin, and $\psi^+(k,r)$ will satisfy the boundary condition that, as $r\to\infty$,

$$\psi^+(k,r)\to\sin kr + T(k)\,e^{ikr} , \tag{5}$$

where $T(k)$, the s-wave T matrix, is given by

$$T(k) = -k^{-1}\int_0^\infty\int_0^\infty \sin kr\;V(r,s)\,\psi^+(k,s)\,ds\,dr . \tag{6}$$

The regular solution of Eq. (1) is defined to be the solution of the integral equation

$$\varphi(k,r) = k^{-1}\sin kr + \int_0^r\int_0^\infty G(k,r,r')\,V(r',s)\,\varphi(k,s)\,ds\,dr' \tag{7}$$

with

$$G(k,r,r') = k^{-1}\sin k(r-r') . \tag{8}$$

The Fredholm determinant associated with the kernel of Eq. (7) is $D(k)$. The regular solution $\varphi(k,r)$ is real and is governed by the boundary conditions

$$\varphi(k,0) = 0 \tag{9a}$$

and

$$\varphi(k,0)' = 1 . \tag{9b}$$

The Jost solutions $f^\pm(k,r)$ satisfy the integral equations

$$f^{\pm}(k,r) = e^{\pm ikr} - \int_r^\infty \int_0^\infty G(k,r,r')\, V(r',s) f^{\pm}(k,s)\, ds\, dr' . \tag{10}$$

The Fredholm determinant associated with the kernel of Eqs. (10) is denoted by $\Delta(k)$, and is equal to $D(k)$ for a symmetric potential.[21] The boundary conditions on $f^{\pm}(k,r)$ are imposed at ∞, and are

$$\lim_{r\to\infty} e^{\mp ikr} f^{\pm}(k,r) = 1 . \tag{11}$$

The Jost function $\mathfrak{L}^{+}(k)$ and its conjugate $\mathfrak{L}^{-}(k) \equiv \mathfrak{L}^{+}(k)^*$ are defined for a symmetric nonlocal potential by the behavior of the Jost solutions at the origin according to[21,24]

$$\mathfrak{L}^{\pm}(k) = f^{\pm}(k,0) . \tag{12}$$

For a *local* potential, the kernels of each of the integral equations defined previously, with the exception of those for $\psi^{\pm}(k,r)$, become Volterra kernels, with Fredholm determinants identically equal to unity.[25] Thus $D(k)$ and $\Delta(k)$ have no explicit role in the description of the scattering process. For a local potential, all phenomena associated with scattering are determined by the Fredholm determinants $D^{\pm}(k)$. It is well known that $D^{\pm}(k)$ and the Jost functions $\mathfrak{L}^{\pm}(k)$ are identically equal[25,26]:

$$\mathfrak{L}^{\pm}(k) = D^{\pm}(k) \quad [\text{local potential}] . \tag{13}$$

For a symmetric *nonlocal* potential, it has been shown[21,24,27] that $D^{\pm}(k)$ and $\mathfrak{L}^{\pm}(k)$ are not identically equal; rather, they are related by

$$\mathfrak{L}^{\pm}(k) = D^{\pm}(k)/D(k) \quad [\text{nonlocal potential}] . \tag{14}$$

The local potential result, Eq. (13), is a special case of Eq. (14) since $D(k) = 1$ for a local potential. The integral equations with which $D(k)$ and $\Delta(k)$ are associated have Fredholm rather than Volterra kernels when the potential is nonlocal. In general, the Fredholm determinants of integral equations which have Fredholm kernels may have zeros. Thus the Fredholm determinants $D(k)$ and $\Delta(k)$ for a nonlocal potential may have zeros for any real value of k. Furthermore, the Fredholm determinants $D^{\pm}(k)$ may also have zeros for any real $k \neq 0$ when the potential is nonlocal.

III. PHASE SHIFT FOR A SHORT-RANGE LOCAL POTENTIAL

The local potential $V(r)$ is a special case of the nonlocal potential, namely

$$V(r,r') = V(r)\, \delta(r - r') . \tag{15}$$

For a local potential, Eq. (1) assumes the form

$$u(k,r)'' + k^2 u(k,r) = V(r)\, u(k,r) . \tag{16}$$

The integral equations for the physical, regular, and Jost solutions of Eq. (16) follow from substituting Eq. (15) into Eqs. (3), (7), and (10), respectively.

Since $V(r)$ is taken to be a short-range potential, there will exist a region $r > R$ for which $V(r)$ can be neglected. In this region, the most general solution $u(k,r)$ of Eq. (16) has been shown to be[28]

$$u(k,r) = A \sin(kr + \delta), \quad r > R . \tag{17}$$

Thus for $r > R$, the regular solution $\varphi(k,r)$, which is real, must be of the form given in Eq. (17). The boundary condition (9a) establishes $\varphi(k,r)$ as regular at the origin. This fact is sufficient to determine δ by comparison of $\varphi(k,r)$ with the form given in Eq. (17) for $r > R$. The constant A is determined by the boundary condition (9b).

The phase shift δ can be interpreted geometrically[29] as the amount by which the radial wave function $\varphi(k,r)$ is "pulled in" or "pushed out" by the potential $V(r)$ relative to the free-particle solution $k^{-1}\sin kr$. That is, the phase shift for scattering from a potential can be defined uniquely by comparing the scattering solution in the presence of the potential with the free-particle solution. Comparison at very large values of r allows one to determine the phase shift within the range $\pm\frac{1}{2}\pi$. However, the absolute phase shift can be found by investigating the wave function for all r, and noting the increase or decrease in the number of nodes of the wave function relative to the free-particle solution. Inherent in this definition of the absolute value of the phase shift is the assumption

$$\lim_{k\to\infty} \delta = 0 . \tag{18}$$

Equating $T(k)$ in terms of the phase shift with the local limit of Eq. (6) gives

$$T(k) = e^{i\delta}\sin\delta = -k^{-1} \int_0^\infty \sin kr\, V(r)\, \psi^{+}(k,r)\, dr . \tag{19}$$

The solutions $\psi^{+}(k,r)$ and $\varphi(k,r)$ are related through the Jost function $\mathfrak{L}^{+}(k)$. This relationship, which is well known[30] and follows from the definitions given in Sec. II, is

$$\psi^{+}(k,r) = \frac{k\varphi(k,r)}{\mathfrak{L}^{+}(k)} . \tag{20}$$

Substituting this expression for $\psi^{+}(k,r)$ into Eq. (19) gives

$$T(k) = e^{i\delta}\sin\delta = -[\mathfrak{L}^{+}(k)]^{-1} \int_0^\infty \sin kr\, V(r)\, \varphi(k,r)\, dr . \tag{21}$$

In the limiting case that the potential goes to zero, the Jost function $\mathfrak{L}^+(k)$ must become unity and the phase shift δ must go to zero. Thus it follows from Eq. (21) that

$$\delta = -\text{phase of } \mathfrak{L}^+(k) \equiv \text{phase of } \mathfrak{L}^-(k) \tag{22}$$

and that

$$\sin\delta = -(|\mathfrak{L}^+(k)|)^{-1} \int_0^\infty \sin kr\, V(r)\, \varphi(k,r)\, dr\,. \tag{23}$$

An absolute value for the phase shift can be obtained either by comparing the scattering wave function with the free-particle wave function for all r or by starting with the condition of Eq. (18) and requiring that the phase shift be continuous from $k=0$ to $k=\infty$.

IV. PHASE SHIFT FOR A NONLOCAL POTENTIAL

In the case of a spurious state $[D(k)=0, D^\pm(k)\neq 0]$, it has been shown in Ref. 14 that the regular solution $\varphi(k,r)$ does not exist. Nevertheless, a real regular solution of Eq. (1) does exist at a spurious state. This solution, discussed in Ref. 14 and referred to there as $\Phi(k,r)$, is defined by the boundary condition $\Phi(k,0)' = D(k)$. Outside the range of the potential the solution $\Phi(k,r)$ must exhibit the asymptotic form given in Eq. (17). Also, since $D^+(k)\neq 0$, the solution $\psi^+(k,r)$ will exist at a spurious state. For a continuum bound state $[D(k)=0, D(k)^\pm = 0]$, it has been demonstrated in Ref. 14 that the real regular solution $\varphi(k,r)$ and the physical solution $\psi^+(k,r)$ both exist, but that each is arbitrary with respect to the addition of the solution of the associated homogeneous integral equation. In each case, however, the arbitrary part of the wave function is orthogonal to $\int_0^\infty \sin kr\, V(r,s) dr$.

Keeping these considerations in mind, it is possible to show that for a nonlocal potential

$$T(k) = e^{i\delta}\sin\delta = -k^{-1}\int_0^\infty \int_0^\infty \sin kr\, V(r,s)\, \psi^+(k,s)\, ds\, dr\,. \tag{24}$$

Equation (24) is convenient for discussing the behavior of the phase shift in the neighborhood of a spurious state or a continuum bound state. It has been recognized for some time that because of the possibility of a zero of the Fredholm determinant $D^\pm(k)$ at a continuum bound state, an ambiguity exists with respect to the definition of the phase shift. Arguments regarding this difficulty have been crystallized by Bolsterli,[23] who pointed out the difference in behavior of the phase shift resulting from two possible definitions of δ at a continuum bound state. In formulating these arguments, he considered only zeros of $D^\pm(k)$. It has since been pointed out,[14] however, that the existence of a zero of $D^\pm(k)$ implies a simultaneous zero of $D(k)$. Furthermore, the possibility of the existence of a spurious state, characterized by a zero of $D(k)$ without a corresponding zero of $D^\pm(k)$, is now known. It is necessary, therefore, to reexamine Bolsterli's discussion in the light of this additional information.

In analogy with the local potential case, expressions for the phase shift follow from separating the right hand side of Eq. (24) into parts equal to $e^{i\delta}$ and $\sin\delta$. For a nonlocal potential there are two possible ways in which this separation can take place which are consistent with the local potential results given in Eqs. (22) and (23). These separations differ in the manner in which the Fredholm determinant $D(k)$ enters into the resulting expressions for $e^{i\delta}$ and $\sin\delta$. For a local potential the separation is unambiguous, since $D(k)\equiv 1$ in that case. For a nonlocal potential, however, because of the possibility of zeros of $D(k)$ different separations correspond to different definitions of the phase shift. Despite ambiguities when $D(k)=0$, for sufficiently large k the phase shift for a nonlocal potential is uniquely defined by the condition of Eq. (18) and the asymptotic condition

$$\lim_{k\to\infty} D(k) = 1\,. \tag{25}$$

As long as one is in the vicinity of (but not at) a zero of $D^\pm(k)$ or $D(k)$, Eq. (20) remains valid. In substituting this result into Eq. (24) and separating into a phase factor and $\sin\delta$, one can either set

$$\delta_D = -\text{phase of } D^+(k)\,, \tag{26}$$

$$\sin\delta_D = -\frac{D(k)}{|D^+(k)|}\int_0^\infty \int_0^\infty \sin kr\, V(r,s)\, \varphi(k,s)\, ds\, dr \tag{27}$$

or

$$\delta_{\mathfrak{L}} = -\text{phase of } \mathfrak{L}^+(k) \equiv -\text{phase of } \left(\frac{D^+(k)}{D(k)}\right), \tag{28}$$

$$\sin\delta_{\mathfrak{L}} = -(|\mathfrak{L}^+(k)|)^{-1}\int_0^\infty \int_0^\infty \sin kr\, V(r,s)\, \varphi(k,s)\, ds\, dr\,. \tag{29}$$

The pair of equations (26) and (27) and the pair (28) and (29) represent the two possible ways of treating the ambiguity of the phase due to $D(k)$. Using either the pair of equations (26) and (27) or the pair (28) and (29) to construct $e^{i\delta}\sin\delta$ results in the same expression for $T(k)$. The phase shifts δ_D and $\delta_{\mathfrak{L}}$ are identical for a local potential, and for a nonlocal potential for which there are no zeros of $D^\pm(k)$ or $D(k)$ for real k. However, if there is a continuum bound state or a spurious

state associated with the potential V, then δ_D and $\delta_{\mathfrak{L}}$ behave differently.

Consider a continuum bound state at the wave number k_c. The Fredholm determinant $D^+(k)$ must be zero at a continuum bound state. Its phase for $k=k_c-\epsilon$ differs by π from its phase for $k=k_c+\epsilon$ in the limit as $\epsilon\to 0$. On the other hand, if there is a spurious state at the wave number k_s, the Fredholm determinant $D^+(k)$ will be unaffected. The presence of this spurious state leaves the phase of $D^+(k)$ unchanged as k goes from $k=k_s-\epsilon$ to $k=k_s+\epsilon$. Thus the phase shift δ_D given by Eq. (26) will experience a discontinuity of π at a continuum bound state and will be continuous at a spurious state.

By contrast, the phase of $\mathfrak{L}^+(k)\equiv D^+(k)/D(k)$ will not change sign at a continuum bound state. That is, the phase of $\mathfrak{L}^+(k)$ for $k=k_c+\epsilon$ will be identical to the phase of $\mathfrak{L}^+(k)$ for $k=k_c-\epsilon$ in the limit as $\epsilon\to 0$. On the other hand, for a spurious state at $k=k_s$ the Jost function $\mathfrak{L}^+(k)$ must differ by π as k goes from $k=k_s-\epsilon$ to $k=k_s+\epsilon$. Thus the phase shift $\delta_{\mathfrak{L}}$ given by Eq. (28) will be continuous at a continuum bound state and will experience a discontinuity of π at a spurious state.

The net result of these considerations is that it is not possible to define in a straightforward manner a phase shift for a nonlocal potential which is continuous both at a spurious state and at a continuum bound state. The major difference between the behavior of the phase shift for a nonlocal potential as contrasted to that for a short-range local potential is this presence of a discontinuity in the phase shift. Wigner[31] and Lüders[32] have shown that the phase shift for a short-range local potential is continuous. These proofs must break down in the case of a nonlocal potential. Thus it is possible to reject the conjecture by Martin[33] that continuity can be extended to all nonlocal potentials.

V. RELATIONSHIP OF THE PHASE SHIFT TO THE EXTRA NODE

The choice of δ_D or $\delta_{\mathfrak{L}}$ as the phase shift for a nonlocal potential can be resolved by considering the existence of an extra node in the radial wave function. In Secs. VI and VII several examples of potentials with either a spurious state or a continuum bound state are given. Analytic expressions are obtained for the scattering solution for each of these potentials. The wave functions in each case show the existence and behavior of an extra node. The resolution given below of the phase shift ambiguity is consistent with all of these examples.

In terms of the examples, the phase shift $\delta_{\mathfrak{L}}$ appears to be the better choice. As discussed in the previous section, this phase shift is discontinuous by π at a spurious state and continuous at a continuum bound state. A discontinuity of π in the phase shift at wave number k_s implies an abrupt change in the wave function. This change is the equivalent of multiplication of the wave function by -1 as k goes from $k_s-\epsilon$ to $k_s+\epsilon$. This, in turn, implies that the regular solution $\varphi(k,r)$ suddenly changes from a function with a positive slope at the origin (at $k_s-\epsilon$) to a function with a negative slope at the origin (at $k_s+\epsilon$), or vice versa. This sudden change is not compatible, however, with the additional feature imposed on the regular solution $\varphi(k,r)$ that $\varphi(k,0)'=1$ both for $k=k_s+\epsilon$ and $k=k_s-\epsilon$. The resolution of this apparent discrepancy rests in the existence of an extra node in the wave function at wave numbers below k_s as compared with wave numbers above k_s. As k approaches k_s from below, this node recedes into the origin, and disappears. Thus below k_s the small additional oscillation between the origin and the extra node makes it possible to satisfy the condition $\varphi(k,0)'=1$. When the extra node disappears into the origin at wave number k_s, the wave function changes sign.

The behavior just described is exactly that exhibited at the spurious state in each of the examples in Sec. VI. Moreover, the examples in Sec. VII show that for a continuum bound state an essentially energy-independent node is present both above and below the continuum bound state wave number k_c. Thus at a continuum bound state no abrupt change in the scattering wave function occurs. This fact is consistent with the selection of $\delta_{\mathfrak{L}}$ as the definition of the phase shift.

The above discussion establishes the compatability of Eq. (29) with the definition of $\delta_{\mathfrak{L}}$ given in Eq. (28). At a spurious state the change in sign of $\varphi(k,s)$ changes the sign of the expression for $\sin\delta_{\mathfrak{L}}$. Since $\varphi(k,s)$ is continuous at a continuum bound state, Eq. (29) for $\sin\delta_{\mathfrak{L}}$ is unchanged in this case. On the other hand, the definition δ_D of the phase shift given in Eq. (26) implies a change of phase of π in the scattering wave function at a continuum bound state and no change in phase shift at a spurious state. The change in sign of $\varphi(k,s)$ in the integral in Eq. (27) compensates for the change in sign of $D(k)$ at a spurious state, resulting in a continuous phase shift. At a continuum bound state $\varphi(k,s)$ is continuous, and the change in sign of $\sin\delta_D$ comes from the change in sign of $D(k)$. Therefore Eq. (27) is compatible with Eq. (26).

We have investigated several cases, which are discussed in detail in the next two sections. For each case the configuration space wave function has been calculated, and the behavior of this wave function compared with the two possible definitions δ_D and $\delta_{\mathfrak{L}}$ of the phase shift. Starting from the

assumption that $\delta \to 0$ as $k \to \infty$, the phase shift $\delta_{\mathfrak{L}}$ best describes the character of the wave function in the neighborhood of spurious and continuum bound states. In each case considered the number of nodes of the scattering wave function changes discontinuously at a spurious state. For a continuum bound state the wave function is unchanged as the wave number moves from just below the wave number k_c of that state to just above.

In the examples which follow, the term phase shift and the symbol δ will always be used as meaning the phase shift $\delta_{\mathfrak{L}}$. That is

$$\delta(k) = -\text{phase}[\mathfrak{L}^+(k)] \equiv -\text{phase}[D^+(k)/D(k)]. \quad (30)$$

All of the potentials considered are of the type discussed in Ref. 34. In this reference a method is developed for obtaining in compact form analytic expressions for the configuration space wave functions and the phase shifts for a wide class of separable nonlocal potentials. Thus the wave functions, phase shifts, and Fredholm determinants quoted here are not the results of techniques involving numerical integration. In evaluating the analytic forms we have used $2m/\hbar^2 = 1/41.47$ $\text{MeV}^{-1}\text{fm}^{-2}$.

VI. SPURIOUS STATES

We have defined a spurious state as occurring at $k = k_s$ when $D(k_s)$ is zero, but $D^+(k_s)$ is not. Clearly if $D^+(k)$ is well behaved as k passes through k_s, the factor $D(k)$ in the denominator of Eq. (30) indicates that we can expect a change of phase of π as $D(k)$ goes through zero. We would thus expect a discontinuous change in the scattering wave function as k passes through k_s. To see that this takes place, we consider two simple examples in which a spurious state is known to occur.

A. One-term separable potential with a Yamaguchi form factor

Yamaguchi[35] has introduced a one-term separable nonlocal potential to describe nucleon-nucleon scattering. In configuration space his potential is of the form

$$V(r,r') = \lambda g(r) g(r'), \quad (31)$$

where

$$g(r) = e^{-\alpha r}. \quad (32)$$

Expressions for $D(k)$, $D^{\pm}(k)$, and $\varphi(k,r)$ for this potential are given in Ref. 14. No values of λ and α will make $D^{\pm}(k)$ zero. Therefore a continuum bound state cannot be associated with the Yamaguchi form factor. On the other hand, $D(k)$ can be zero for a wide range of values of λ and α. The

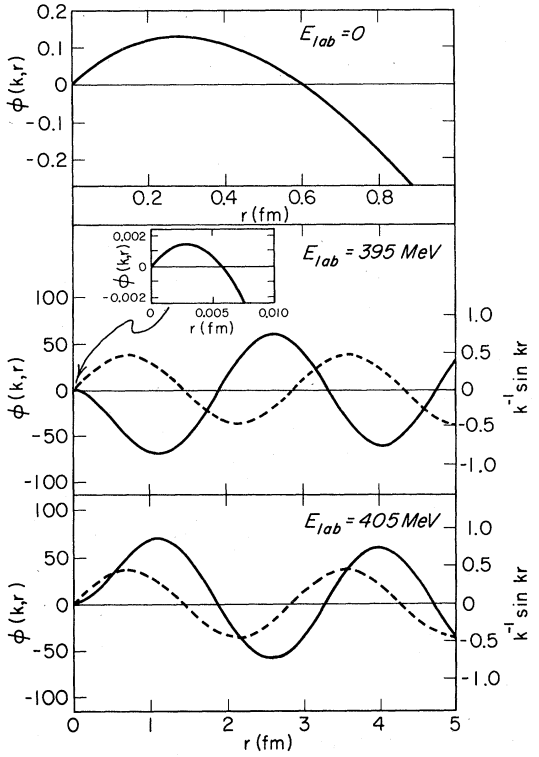

FIG. 1. Wave function $\varphi(k,r)$ at $E_{\text{lab}} = 0$, 395, and 405 MeV for a one-term separable potential with a Yamaguchi form factor with the parameters given in the text. This potential yields a spurious state at 400 MeV. The broken lines show the free-particle wave function.

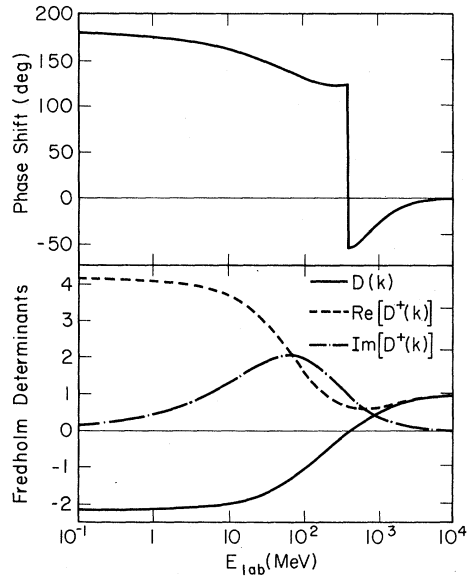

FIG. 2. Phase shift and Fredholm determinants for a one-term separable potential with a Yamaguchi form factor with parameters given in the text. This potential yields a spurious state at 400 MeV.

values of λ and α used by Yamaguchi do not generate a spurious state at any energy. However, if $\lambda > 2\alpha^3$ a spurious state will occur.

If we take

$$\lambda = 21.219 \text{ fm}^{-3}$$

and

$$\alpha = 1.5 \text{ fm}^{-1},$$

then $D(k)$ will be zero at $E_{\text{lab}} = 400$ MeV. Figure 1 shows the wave function $\varphi(k,r)$ at 395 and 405 MeV, along with the free-particle wave function for comparison. Note that at 395 MeV the wave function has an extra oscillation at very small r, as shown in the inset. At 405 MeV, this oscillation has disappeared. Calculations show that the extra oscillation vanishes at exactly 400 MeV and persists at all energies below 400 MeV. To emphasize this point the zero-energy wave function has been included in Fig. 1.

Figure 2 exhibits the phase shift as a function of the energy for this potential, showing the discontinuity of π at 400 MeV. Also given in Fig. 2 is the Fredholm determinant $D(k)$, showing its zero at 400 MeV. The $\text{Re}D^+(k)$ and $\text{Im}D^+(k)$ have been included to demonstrate that there is not a simultaneous zero of $D^+(k)$ at the zero of $D(k)$.

Figure 1 shows that above 400 MeV the wave function is "pushed out" relative to that of a free-particle wave function, indicating a negative phase shift. Below 400 MeV, the wave function is "pulled in" relative to the free-particle wave function, indicating a positive phase shift. Below the energy at which the spurious state occurs the wave function has an extra node. As the energy of the particle becomes larger, the node moves closer to the origin. As the energy passes through E_s, the extra node disappears. Since we have imposed the boundary condition that the slope of the wave function be positive at the origin, the wave function must change sign when the node disappears. This results in a π change of phase at this energy.

B. Mongan potential

Another widely used potential is that suggested by Mongan.[36] The Mongan case IV potential is defined in configuration space by

$$V(r,r') = \lambda_1 e^{-\alpha_1(r+r')} + \lambda_2 e^{-\alpha_2(r+r')} \tag{33}$$

with

$$\lambda_1 = 3454.8 \text{ fm}^{-3}, \quad \lambda_2 = -28.293 \text{ fm}^{-3},$$

$$\alpha_1 = 6.157 \text{ fm}^{-1}, \quad \alpha_2 = 1.786 \text{ fm}^{-1}.$$

Expressions for $D(k)$, $D^{\pm}(k)$, and $\varphi(k,r)$ for this potential are given in Ref. 14. The potential is known to have a zero of $D(k)$ at a laboratory en-

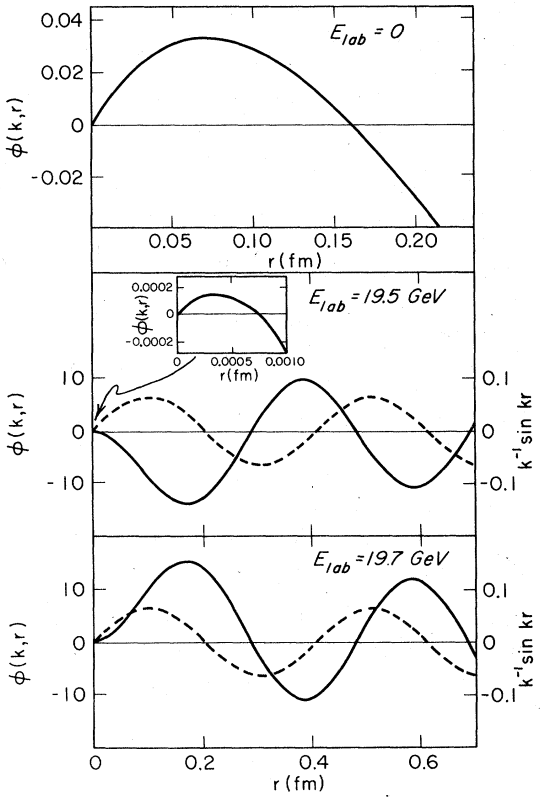

FIG. 3. Wave function $\varphi(k,r)$ at $E_{\text{lab}} = 0$, 19.5, and 19.7 GeV for the Mongan case IV potential. This potential yields a spurious state at 19.6 GeV. The broken lines show the free-particle wave function.

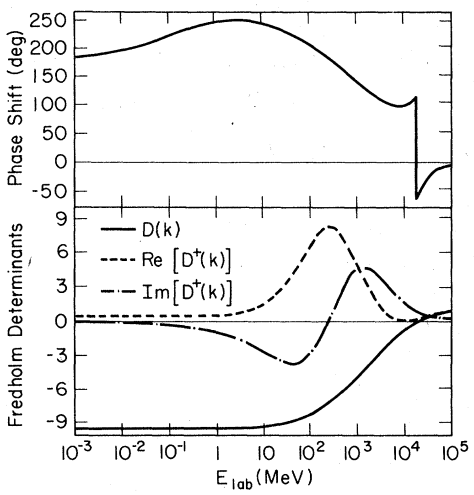

FIG. 4. Phase shift and Fredholm determinants for the Mongan case IV potential. This potential yields a spurious state at 19.6 GeV.

ergy of 19.6 GeV.

Figure 3 gives the scattering and free-particle wave functions at 19.5 and 19.7 GeV. Again, as the inset shows, below 19.6 GeV there is an extra node in the wave function not present above that energy. The zero-energy wave function included in Fig. 3 shows this extra node present at that energy. The existence of this extra node at zero energy has been pointed out previously by Arnold and MacKellar.[37]

Figure 4 gives the phase shift, $D(k)$, $\mathrm{Re}D^+(k)$, and $\mathrm{Im}D^+(k)$ for this potential. Calculations in the neighborhood of 19.6 GeV show that the change of phase of π in the wave function corresponds exactly to the zero of $D(k)$. Note that neither $\mathrm{Re}D^+(k)$ nor $\mathrm{Im}D^+(k)$ is zero at this energy.

VII. CONTINUUM BOUND STATES

A continuum bound state will occur at $k=k_c$ if and only if both $D^+(k_c)$ and $D(k_c)$ are zero. To investigate the behavior of the wave function and the phase shift when a continuum bound state occurs, it is useful to consider several potentials which can produce a continuum bound state.

A. One-term Tabakin potential

Tabakin[38] has suggested a one-term nucleon-nucleon separable potential which can be written in configuration space in the form of Eq. (31) with $g(r)$ given by

$$g(r)=(A_1\cos\alpha_1 r+A_2\sin\alpha_1 r)e^{-\alpha_1 r}+A_3 e^{-\alpha_2 r}. \tag{34}$$

For the $^1S_0$ state the parameters are

$$\lambda=-400.85\ \mathrm{fm}^{-3},$$
$$A_1=-1.0248,\quad A_2=0.0773,\quad A_3=0.0248,$$
$$\alpha_1=2.863\ \mathrm{fm}^{-1},\quad \alpha_2=1.085\ \mathrm{fm}^{-1}$$

and for the $^3S_1$ state

$$\lambda=-104.61\ \mathrm{fm}^{-3},$$
$$A_1=-1.8579,\quad A_2=-1.3877,\quad A_3=0.8579,$$
$$\alpha_1=2.863\ \mathrm{fm}^{-1},\quad \alpha_2=2.360\ \mathrm{fm}^{-1}.$$

In both cases the potential has a continuum bound state at 240 MeV. The deuteron binding energy for the $^3S_1$ potential is 2.2357 MeV, while the $^1S_0$ potential has no bound state. For this potential

$$\begin{aligned}D(k)\equiv\Delta(k)&=1+\frac{\lambda A_1^2(2\alpha_1^2-3k^2)}{8\alpha_1(4\alpha_1^4+k^4)}-\frac{\lambda A_2^2(2\alpha_1^2+k^2)}{8\alpha_1(4\alpha_1^4+k^4)}-\frac{\lambda A_3^2}{2\alpha_2(\alpha_2^2+k^2)}-\frac{\lambda A_1A_2(2\alpha_1^2+k^2)}{4\alpha_1(4\alpha_1^4+k^4)}\\&-\frac{\lambda A_1A_3}{(2\alpha_1^2+2\alpha_1\alpha_2+\alpha_2^2)}\left[\frac{\alpha_1+\alpha_2}{\alpha_2^2+k^2}+\frac{\alpha_1k^2+\alpha_2k^2-2\alpha_1^3}{4\alpha_1^4+k^4}\right]\\&-\frac{\lambda A_2A_3}{(2\alpha_1^2+2\alpha_1\alpha_2+\alpha_2^2)}\left[\frac{\alpha_1}{\alpha_2^2+k^2}+\frac{\alpha_1k^2+2\alpha_1^2\alpha_2+2\alpha_1^3}{4\alpha_1^4+k^4}\right]\end{aligned} \tag{35}$$

and

$$D^{\pm}(k)=D(k)+C\left[\frac{A_1\alpha_1(2\alpha_1^2+k^2)}{4\alpha_1^4+k^4}+\frac{A_2\alpha_1(2\alpha_1^2-k^2)}{4\alpha_1^4+k^4}+\frac{A_3\alpha_2}{\alpha_2^2+k^2}\right]\pm ikC\left[\frac{A_1k^2+2A_2\alpha_1^2}{4\alpha_1^4+k^4}+\frac{A_3}{\alpha_2^2+k^2}\right], \tag{36}$$

where

$$C=-\lambda\left[\frac{A_1k^2+2A_2\alpha_1^2}{4\alpha_1^4+k^4}+\frac{A_3}{\alpha_2^2+k^2}\right]. \tag{37}$$

The regular solution $\varphi(k,r)$ for this potential is

$$\begin{aligned}\varphi(k,r)&=k^{-1}\sin kr+\frac{C}{kD(k)}\left[\frac{A_1\alpha_1(2\alpha_1^2+k^2)}{4\alpha_1^4+k^4}+\frac{A_2\alpha_1(2\alpha_1^2-k^2)}{4\alpha_1^4+k^4}+\frac{A_3\alpha_2}{\alpha_2^2+k^2}\right]\sin kr\\&-\frac{C}{D(k)}\left[\frac{A_3}{\alpha_2^2+k^2}+\frac{A_1k^2+2A_2\alpha_1^2}{4\alpha_1^4+k^4}\right]\cos kr\\&+\frac{C}{D(k)}\left[\frac{A_1k^2+2A_2\alpha_1^2}{4\alpha_1^4+k^4}\cos\alpha_1r+\frac{A_2k^2-2A_1\alpha_1^2}{4\alpha_1^4+k^4}\sin\alpha_1r\right]e^{-\alpha_1r}+\frac{C}{D(k)}\left[\frac{A_3}{\alpha_2^2+k^2}\right]e^{-\alpha_2r}.\end{aligned} \tag{38}$$

Figure 5 gives the wave function for the $^1S_0$ potential at 235 and 245 MeV, with the free-particle wave function included for comparison. The wave function calculated for energies between 235 and 245 MeV shows no discontinuity.

As required for the potential to have a continuum bound state at 240 MeV, both $D^+(k)$ and $D(k)$ are zero at that energy. Thus the ratio $D^+(k)/D(k)$ and,

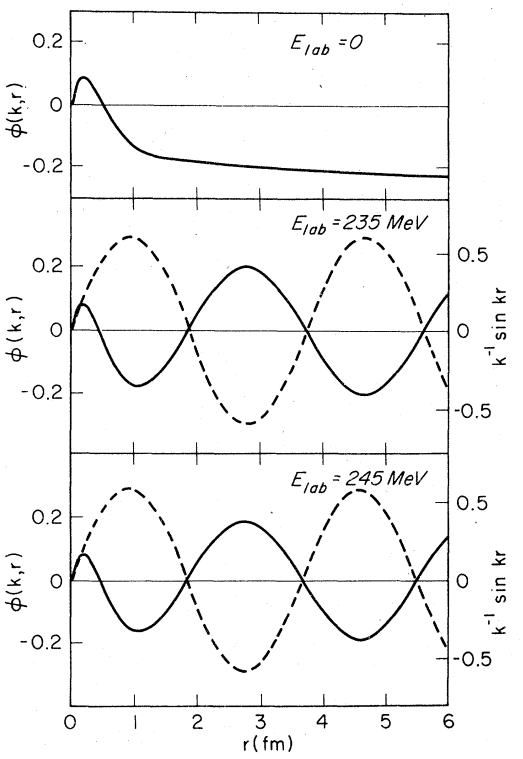

FIG. 5. Wave function $\varphi(k,r)$ at $E_{\rm lab}=0$, 235, and 245 MeV for the one-term Tabakin $^1S_0$ potential. This potential yields a continuum bound state at 240 MeV. The broken lines show the free-particle wave function.

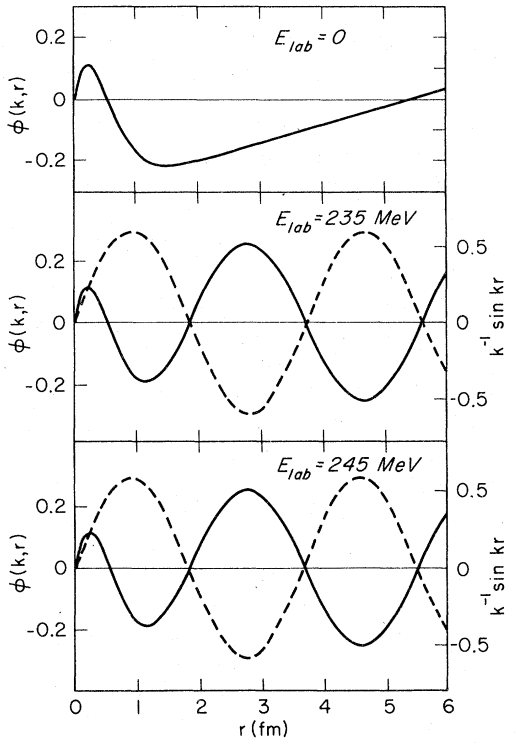

FIG. 7. Wave function $\varphi(k,r)$ at $E_{\rm lab}=0$, 235, and 245 MeV for the one-term Tabakin $^3S_1$ potential. This potential yields a continuum bound state at 240 MeV. It also has a negative-energy bound state. The broken lines show the free-particle wave function.

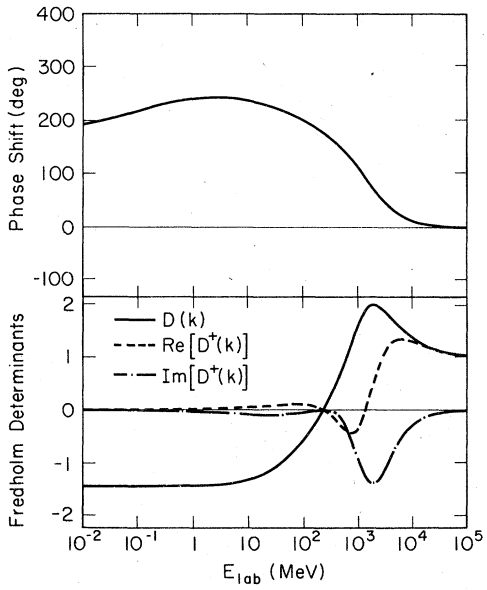

FIG. 6. Phase shift and Fredholm determinants for the one-term Tabakin $^1S_0$ potential. This potential yields a continuum bound state at 240 MeV.

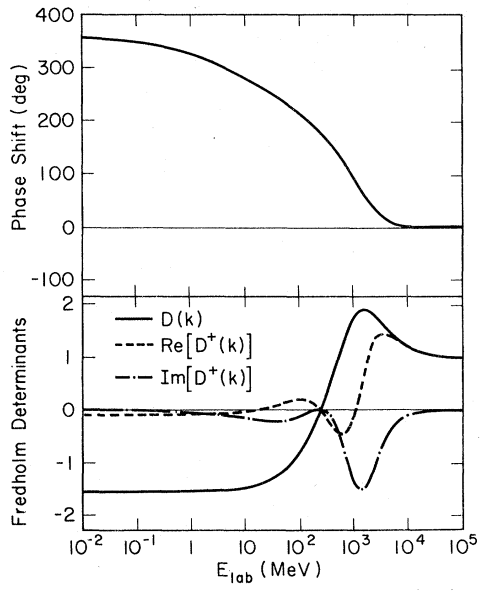

FIG. 8. Phase shift and Fredholm determinants for the one-term Tabakin $^3S_1$ potential. This potential yields a continuum bound state at 240 MeV. It also has a negative-energy bound state.

therefore, the phase shift from this definition, become undefined. Figure 5 shows that at 240 MeV the scattering wave function is π out of phase with the free-particle solution. We conclude that, at 240 MeV, the phase shift is exactly π. The zero-energy wave function also has been given in Fig. 5. We find that the shape of this wave function near the origin and the position of the first node are almost unchanged as the energy is increased from zero to E_c.

The phase shift as defined in Eq. (30) is continuous at a continuum bound state. Had the definition δ_D given in Eq. (26) been used, a discontinuity of π would occur in the phase shift at the position of the continuum bound state. Figure 5 indicates that the definition of Eq. (30) corresponds to the behavior of the wave function, and is to be preferred on that basis. The phase shift for the one-term Tabakin $^1S_0$ potential is shown in Fig. 6 along with $D(k)$, $\mathrm{Re}D^+(k)$, and $\mathrm{Im}D^+(k)$ as functions of E_{lab}. In particular, the phase shift for this potential does not go through zero, but starts with a value of π at zero energy and approaches zero asymptotically as k approaches infinity.

The wave function for the one-term Tabakin $^3S_1$ potential is given in Fig. 7 for zero, 235, and 245 MeV. At energies near the continuum bound state the $^1S_0$ and $^3S_1$ wave functions are quite similar. The $^3S_1$ wave function at zero energy, however, differs from the $^1S_0$ wave function in that because of the negative-energy bound state it exhibits an additional node. There is no change in the wave function as the energy passes through E_c.

The phase shift resulting from this potential, as well as $D(k)$, $\mathrm{Re}D^+(k)$, and $\mathrm{Im}D^+(k)$ are given in Fig. 8. In the $^3S_1$ case the phase shift starts off at 2π at zero energy and approaches zero asymptotically for large k. Calculations of the wave function over the energy range show no discontinuity as a function of energy.

B. Beregi potential

Beregi[39] has suggested a separable nonlocal potential of the form of Eq. (38) which reproduces the characteristic features of the Tabakin $^3S_1$ potential in that it has a continuum bound state at 259.3 MeV and a bound state at −2.225 MeV. Beregi's potential function $g(r)$ is of the form

$$g(r) = e^{-\alpha_1 r} - a e^{-\alpha_2 r}. \tag{39}$$

The parameters for the potential are

$$\lambda = -302.73\ \mathrm{fm}^{-3},$$

$$\alpha_1 = 2.67\ \mathrm{fm}^{-1},$$

$$\alpha_2 = 5.34\ \mathrm{fm}^{-1},$$

$$a = 3.0854.$$

For this potential

$$D(k) \equiv \Delta(k) = 1 - \frac{\lambda}{2\alpha_1(\alpha_1^2+k^2)} + \frac{\lambda a}{(\alpha_1+\alpha_2)}\left(\frac{1}{\alpha_1^2+k^2} + \frac{1}{\alpha_2^2+k^2}\right) - \frac{\lambda a^2}{2\alpha_2(\alpha_2^2+k^2)} \tag{40}$$

and

$$D^{\pm}(k) = D(k) + \frac{\lambda\alpha_1}{(\alpha_1^2+k^2)^2} - \frac{\lambda a(\alpha_1+\alpha_2)}{(\alpha_1^2+k^2)(\alpha_2^2+k^2)} + \frac{\lambda a^2\alpha_2}{(\alpha_2^2+k^2)^2} \pm ik\left[\frac{\lambda}{(\alpha_1^2+k^2)^2} - \frac{2\lambda a}{(\alpha_1^2+k^2)(\alpha_2^2+k^2)} + \frac{\lambda a^2}{(\alpha_2^2+k^2)^2}\right]. \tag{41}$$

The regular solution $\varphi(k,r)$ for this potential is

$$\varphi(k,r) = k^{-1}\sin kr + \frac{1}{kD(k)}\left(\frac{C_1\alpha_1}{\alpha_1^2+k^2} + \frac{C_2\alpha_2}{\alpha_2^2+k^2}\right)\sin kr - \frac{1}{D(k)}\left(\frac{C_1}{\alpha_1^2+k^2} + \frac{C_2}{\alpha_2^2+k^2}\right)\cos kr + \frac{1}{D(k)}\left(\frac{C_1}{\alpha_1^2+k^2}\right)e^{-\alpha_1 r} + \frac{1}{D(k)}\left(\frac{C_2}{\alpha_2^2+k^2}\right)e^{-\alpha_2 r}, \tag{42}$$

where

$$C_1 = \frac{\lambda}{\alpha_1^2+k^2} - \frac{\lambda a}{\alpha_2^2+k^2} \tag{43}$$

and

$$C_2 = -\frac{\lambda a}{\alpha_1^2+k^2} + \frac{\lambda a^2}{\alpha_2^2+k^2}. \tag{44}$$

We have calculated the wave function for this potential in the neighborhood of 259.3 MeV and find no change in the wave function as k passes through k_s. The wave function at 254 and 264 MeV and the free-particle wave function are given in Fig. 9. Also shown is the wave function at zero energy for this potential.

As was the case with the Tabakin potential, the wave function near the origin is very much the same at energies below E_c. At energies above E_c for which the potential continues to have an important effect on the wave function, the behavior near the origin is unchanged. The phase shift and Fredholm determinants for this potential are given

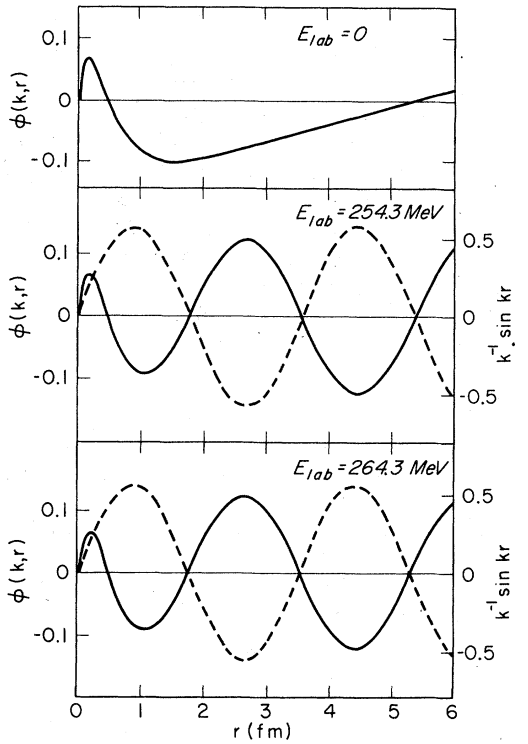

FIG. 9. Wave function $\varphi(k,r)$ at $E_{\text{lab}}=0$, 254.3, and 264.3 MeV for the Beregi potential. This potential yields a continuum bound state at 259.3 MeV. It also has a negative-energy bound state. The broken lines show the free-particle wave function.

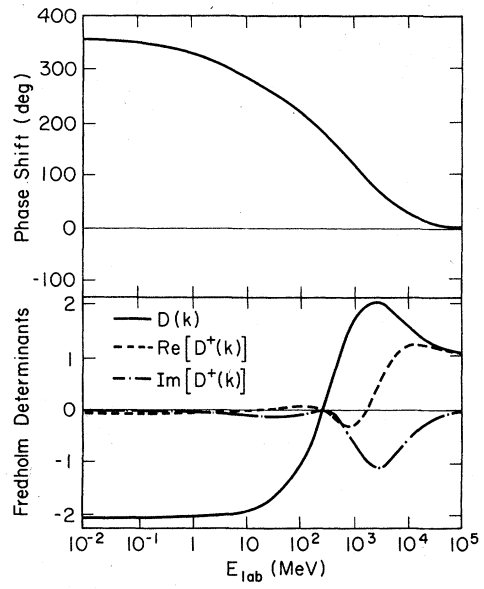

FIG. 10. Phase shift and Fredholm determinants for the Beregi potential. This potential yields a continuum bound state at 259.3 MeV. It also has a negative-energy bound state.

in Fig. 10. Note that, as for the Tabakin potential, the phase shift for this potential is π at the energy of the continuum bound state.

C. Mongan-like potential with a continuum bound state

As noted earlier, the Mongan potential using the parameters quoted in Sec. VI B exhibits a spurious state but does not have a continuum bound state. We now discuss a potential of the same form, but with parameters for which a continuum bound state, but no spurious state, appears in the spectrum. The method of cancellation of the Green's function[40] makes it convenient to pick potential parameters in such a way that we may select the energy at which the continuum bound state occurs. The following values for the parameters of the potential given in Eq. (33) result in a continuum bound state at 400 MeV:

$$\lambda_1 = 105.876\ \text{fm}^{-3},$$
$$\lambda_2 = -499.752\ \text{fm}^{-3},$$
$$\alpha_1 = 2.0\ \text{fm}^{-1},$$
$$\alpha_2 = 4.0\ \text{fm}^{-1}.$$

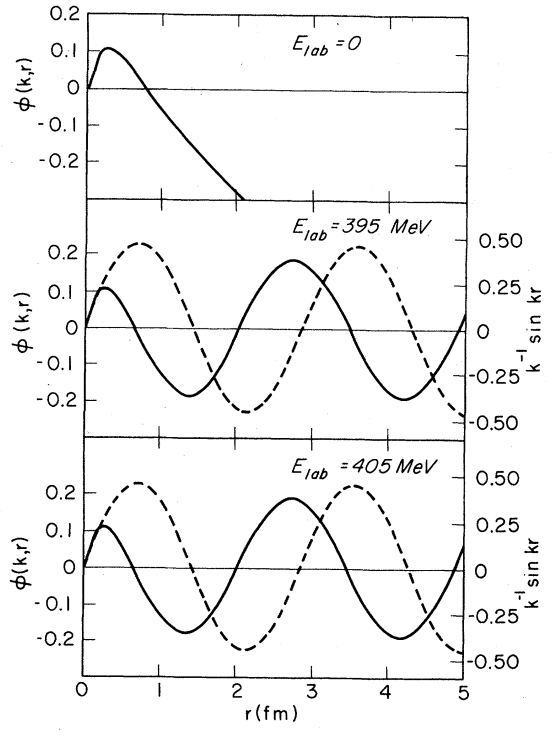

FIG. 11. Wave function $\varphi(k,r)$ at $E_{\text{lab}}=0$, 395, and 405 MeV for a two-term separable potential of the Mongan case IV form with parameters given in the text. This potential yields a continuum bound state at 400 MeV. The broken lines show the free-particle wave function.

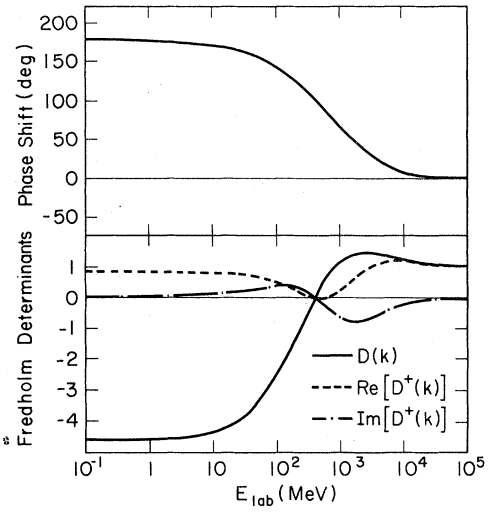

FIG. 12. Phase shift and Fredholm determinants for a two-term separable potential of the Mongan case IV form with parameters given in the text. This potential yields a continuum bound state at 400 MeV.

The wave function at 395 and 405 MeV is given in Fig. 11 along with the free-particle wave function. Also given in Fig. 11 is the zero-energy wave function. The phase shift and Fredholm determinants are given in Fig. 12. For this potential the undetermined ratio $D^+(k_c)/D(k_c)$ does not, under the application of L'hospital's rule, turn out to be zero, as is the case with the Tabakin and Beregi potentials. Thus the phase shift is not precisely π at that energy at which the continuum bound state occurs. Calculations show that the wave function and phase shift are continuous as k passes through k_c.

VIII. LEVINSON'S THEOREM FOR A NONLOCAL POTENTIAL

Levinson[41,22] has shown that for a short-range local potential the phase shift at zero energy $\delta(0)$ and the phase shift at infinite energy $\delta(\infty)$ are related by the condition

$$\delta(0) - \delta(\infty) = N\pi \quad \text{[local potential]}, \tag{45}$$

where N is the number of bound states of that potential. The possibility of states of zero binding energy is excluded in the present discussion. For a local potential a state of zero binding energy was shown by Levinson to require an additional $\frac{1}{2}\pi$ on the right hand side of Eq. (45). The number N is also equal to the number of nodes exhibited by the zero-energy wave function, excluding the node at the origin. Martin[33] and Mills and Reading[42] have pointed out that for the case of a nonlocal potential this relationship must be modified. In particular, they conclude that for each continuum bound state an additional π must be added to the right hand side of Eq. (45).

The results of the calculations of the phase shifts and of the zero-energy wave functions presented in the present paper suggest that a further modification of Eq. (45) is necessary for the case of a nonlocal potential. From these results we conclude that Levinson's theorem for the nonlocal potentials discussed here must assume the form

$$\delta(0) - \delta(\infty) = (N + N')\pi \quad \text{[nonlocal potential]}, \tag{46}$$

where N is, again, the number of negative-energy bound states of the potential and N' is the number of zeros of the Fredholm determinant $D(k)$ between $k=0$ and $k=\infty$. This result is in agreement with the conclusions of Martin and of Mills and Reading, and represents an extension of their results to include the π change of phase which we have shown to be necessary at each spurious state.

IX. SUMMARY AND CONCLUSION

In this paper we have discussed the fact that the wave function for a nonlocal potential can have an extra node within the range of the potential even in the absence of a negative-energy bound state. This behavior of the wave function is not possible for a short-range local potential. The extra node occurs whenever there is a zero of the Fredholm determinant $D(k)$ for real positive k. The nodes are found either with a spurious state $[D(k_s)=0,\ D^{\pm}(k_s)\neq 0]$ or with a continuum bound state $[D(k_c)=0, D^{\pm}(k_c)=0]$. Nodes due to a spurious state are present only for wave numbers below the wave number k_s of the spurious state. Above k_s the extra node is not present. Furthermore, the radial position of the extra node is at its maximum value when $k=0$ and decreases until the node reaches the origin when $k=k_s$.

For a continuum bound state the extra node is present essentially without change both above and below k_c. That is, the position of the extra node is relatively stable as a function of increasing wave number. It is important to realize, however, that in a numerical calculation apparent energy independence of an extra node does not guarantee that the node is due to a continuum bound state. For example, a spurious state at an energy very much higher than energies at which a calculation is performed will also result in an almost energy-independent node.

Another important result given in this paper is the superiority of the definition of the phase shift $\delta_{\mathcal{L}}$ over the phase shift δ_D. This definition has been shown to have the advantage of being directly related to the nodal behavior of the radial wave function. It is also open to unambiguous interpretation with respect to an absolute value of the phase shift as a function of k.

Finally, we have generalized Levinson's theorem to include the possibility of spurious states. The desirability of a more complete investigation of Levinson's theorem as it applies to a nonlocal potential is clear and this subject is currently under study.

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