Dynamics of nuclear fluid. II. Normal sound, spin sound, isospin sound, and spin-isospin sound

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Starting with the time-dependent Hartree-Fock equation in density matrix form, we investigate the macroscopic description of the dynamics of the nuclear fluid involving the spin and isospin degrees of freedom, in conjunction with the presence of only central exchange interactions. The time-dependent Hartree-Fock equation can be cast into a set of conservation laws of the classical type coupling spin and isospin densities. With simple zero-range interactions, we obtain the normal modes of density propagations and the corresponding speeds of sound waves. In addition to the normal sound waves in which the total density varies with space and time, there are the spin sound waves in which the difference of the spin-up and the spin-down densities varies with space and time, the isospin sound waves in which the difference of neutron and proton densities varies with space and time, and finally, the spin-isospin sound waves in which the difference of the "parallel" spin and isospin densities and the "antiparallel" spin and isospin densities varies with space and time. It is found that for a zero-range interaction whose density dependence is of the type $t_3(1+\mathcal{P}^B)n\delta(\vec{r}_1-\vec{r}_2)/6$, the speeds of spin sound a_2, isospin sound a_3 and spin-isospin sound a_4 satisfy $a_2^2+a_3^2=2a_4^2$. With the parameters of Golin and Zamick and Vautherin and Brink, we have in addition $a_3>a_4>a_2$, the numerical values of a_2, a_3, and a_4 being in the range of 0.17 to 0.27c.

[NUCLEAR STRUCTURE Dynamics of nuclear fluid, normal sound wave, spin sound wave, isospin sound wave, and spin-isospin sound wave. Time-dependent Hartree-Fock theory. Estimate speeds of different sound waves.]

I. INTRODUCTION

In our previous publications on the dynamics of the nuclear fluid,[1] we discussed a macroscopic description and the equations governing the dynamics derived from the microscopic time-dependent Hartree-Fock (TDHF) theory. An independent approach to obtain these equations using the Wigner functions has been carried out by Koonin and Kerman.[2] Not surprisingly, these equations turn out to be the equation of continuity and an equation analogous to the Euler equation in fluid dynamics. In the other articles of this series, we consider the kinetic theory of quantum fluids based on the exact many-body theory and the conditions for a hydrodynamical description of a heavy-ion collision.[3]

For simplicity, the discussions of the dynamics of the nuclear fluid in our other investigations did not touch upon the spin and the isospin degrees of freedom. The presence of these degrees of freedom gives characteristics to the nuclear fluid to make it distinct from any other fluid. The dynamics of the nuclear fluid involving these degrees of freedom is the subject of investigation in the present paper.

That these degrees of freedom may give rise to peculiar phenomena is well known in nuclear physics. The isospin degree of freedom is exploited in the Goldhaber-Teller model[4] of giant dipole resonance. Spin density oscillations were discussed by Wild,[5] Glassgold, Heckrotte, and Watson,[6] Raphael, Uberall, and Werntz,[7] and Bohr and Mottelson.[8] All the previous studies either assume a fluid-dynamical model[5,7,8] from the beginning or exhibit oscillatorlike solutions in the spectrum.[6] We would like to examine how fluid-dynamical type equations governing these densities can be derived from the microscopic theory. One may wish to quantize these degrees of freedom with the appropriate boundary conditions as is done in Refs. 7 and 8.

Similar to I (Ref. 1), the starting point of our investigation is again the TDHF equations, including now the spin and isospin degrees of freedom. In Sec. II, we write down the TDHF equation in density matrix form. Limiting our attention to central exchange interactions, we reduce the TDHF equation in such a way that the exchange operators make no appearance. From the TDHF equation, we describe in Sec. IV how a simplified equation can be obtained when the spin and isospin degrees of freedom are constrained in a prescribed manner. Returning to the full TDHF equation in Sec. V, we cast it in the form of conservation laws of classical type which couple spin and isospin density fields and velocity fields. Before studying the propagation of small density pertur-

bations, we generalize in Sec. VI the TDHF equation to the case when the interaction is density dependent and we obtain an amended TDHF equation with an additional rearrangement term. With simple zero-range density-dependent interactions, the various normal modes of sound propagation can be separated, and the speeds of the sound waves evaluated in Secs. VII and VIII. Section IX concludes the present investigation.

II. TIME-DEPENDENT HARTREE-FOCK EQUATION WITH SPIN AND ISOSPIN DEGREES OF FREEDOM

We start with the many-body Schrödinger equation for a system of N particles

$$i\hbar\frac{\partial}{\partial t}\Psi(x_1,\ldots,x_N)=-\frac{\hbar^2}{2m}\sum_i^N\nabla_i^2\Psi(x_1,\ldots,x_N)+\frac{1}{2}\sum_{i,j}v(x_i,x_j)\Psi(x_1,\ldots,x_N)\ , \tag{2.1}$$

where x_i is the coordinate of the ith particle and $v(x_i,x_j)$ is the two-body interaction between the ith and the jth particle. The coordinate x_i includes the spatial coordinate $\vec{r}_i$, the spin coordinate ζ_i, and the isospin coordinate η_i:

$$x_i=\{\vec{r}_i,\zeta_i,\eta_i\}\ . \tag{2.2}$$

From Eq. (2.1), one obtains the equation for the N-body density matrix in the form

$$i\hbar\frac{\partial}{\partial t}\mathfrak{N}^{(N)}(x_1x_2\cdots x_N;x_1'x_2'\cdots x_N')=\left\{-\frac{\hbar^2}{2m}\sum_i^N(\nabla_{r_i}^2-\nabla_{r_i'}^2)+\frac{1}{2}\sum_{i,j}^N[v(x_i,x_j)-v(x_i',x_j')]\right\}\mathfrak{N}^{(N)}(x_1x_2\cdots x_N;x_1'x_2'\cdots x_N')\ , \tag{2.3}$$

where

$$\mathfrak{N}^{(N)}(x_1x_2\cdots x_N;x_1'x_2'\cdots x_N')=\Psi(x_1x_2\cdots x_N)\Psi^+(x_1'x_2'\cdots x_N')\ . \tag{2.4}$$

Upon integrating this equation successively over the particle coordinates, we obtain a hierarchy of quantum kinetic equations[9,10,1] for the reduced density matrices $\mathfrak{N}^{(s)}$ defined by

$$\mathfrak{N}^{(s)}(x_1\cdots x_s;x_1'\cdots x_s')=N(N-1)\cdots(N-s+1)\int dx_{s+1}\cdots dx_N\mathfrak{N}^{(N)}(x_1\cdots x_sx_{s+1}\cdots x_N;x_1'\cdots x_s'x_{s+1}\cdots x_N)\ , \tag{2.5}$$

where the integral $\int dx_i$ is an abbreviation for the trace over ζ_i and η_i and the integral over $\vec{r}_i$:

$$\int dx_i\cdots=\int d^3r_i\,\mathrm{tr}_{\zeta_i\eta_i}\cdots\ . \tag{2.6}$$

In particular, upon integrating over all the coordinates x_i but one, we get the equation for the one-body density matrix

$$i\hbar\frac{\partial}{\partial t}\mathfrak{N}^{(1)}(x_1,x_1')=-\frac{\hbar^2}{2m}(\nabla_{r_1}^2-\nabla_{r_1'}^2)\mathfrak{N}^{(1)}(x_1;x_1')+\int dx_2\lim_{x_2'\to x_2}[v(x_1,x_2)-v(x_1',x_2')]\mathfrak{N}^{(2)}(x_1x_2;x_1'x_2')\ . \tag{2.7}$$

The hierarchy of the quantum kinetic equation will not terminate unless simple assumptions are made concerning the many-body wave function. In the time-dependent Hartree-Fock theory, one assumes that the many-body wave function is in the form of a Slater determinant consisting of single-particle states $\Phi_\alpha(x)$. The one-body and the two-body density matrices are then

$$\mathfrak{N}^{(1)}(x;x')=\sum_\alpha\Phi_\alpha(x)\Phi_\alpha^+(x') \tag{2.8}$$

and

$$\mathfrak{N}^{(2)}(x_1x_2;x_1'x_2')=\sum_{\alpha\beta}[\Phi_\alpha(x_1)\Phi_\beta(x_2)\Phi_\alpha^+(x_1')\Phi_\beta^+(x_2')-\Phi_\alpha(x_1)\Phi_\beta(x_2)\Phi_\alpha^+(x_2')\Phi_\beta^+(x_1')]\ , \tag{2.9}$$

where the summation is carried out over the occupied states α and β. In this case, the hierarchy of the quantum kinetic equation terminates at the lowest level, namely, with Eq. (2.7). In consequence, it suffices to concern ourselves with the one- and two-body density matrices. For simplicity of notation, we can omit the superscript (1) for the one-body density matrix and denote it by

$\mathfrak{N}(x;x_1')$:

$$\mathfrak{N}(x_1;x_1') \equiv \mathfrak{N}^{(1)}(x_1;x_1') \tag{2.10}$$

and, in situations where confusion will not arise, we can even use the shorter term "density matrix" to stand for the one-body density matrix. Equation (2.7), with $\mathfrak{N}^{(1)}$ and $\mathfrak{N}^{(2)}$ given by Eqs. (2.8) and (2.9), is known as the time-dependent Hartree-Fock equation in density matrix form.

III. EXPLICIT FORM OF THE TDHF EQUATION

We wish to write the TDHF form explicitly in terms of the different component of the two-body interactions such that the exchange operators make no appearance. As it can be readily seen, the algebraic reduction of the TDHF equation is already quite complicated even with only the central interaction. The spin-orbit and the tensor interactions further allow the coupling between the space and the spin degree of freedom in a nontrivial way. We shall consider only central exchange interactions represented by

$$v(x_1x_2) = v_W(\vec{r}_1,\vec{r}_2) + v_M(\vec{r}_1,\vec{r}_2)\mathcal{P}^M(x_1x_2) + v_B(\vec{r}_1,\vec{r}_2)\mathcal{P}^B(x_1x_2) + v_H(\vec{r}_1,\vec{r}_2)\mathcal{P}^H(x_1x_2), \tag{3.1}$$

where the subscript W stands for Wigner, M for Majorana, B for Bartlett, H for Heisenberg, and $\mathcal{P}^M$, $\mathcal{P}^B$, and $\mathcal{P}^H$ are the corresponding exchange operators. The TDHF equation can be written in terms of the one-body density matrix $\mathfrak{N}(x_1;x_1')$ and the spatial part of the different exchange potentials. This will be done first for a density-independent interaction. Any amendment due to the presence of density dependence will be considered in Sec. VI.

We choose our representation such that the (one-body) density matrix $\mathfrak{N}(r_1r_1';\zeta_1\zeta_1';\eta_1\eta_1')$ has only diagonal matrix elements in the ζ and η degrees of freedom. The simple exchange character of the potentials in Eq. (3.1) will result in diagonal matrix elements (in ζ and η degrees of freedom) in the TDHF equation. Such a simplification, of course, will not be obtained with the addition of spin-orbit and tensor interactions.

After some simplification, the TDHF Eq. (2.7) can be written explicitly in terms of the spatial coordinates and the density matrix as follows:

$$\begin{aligned}
i\hbar\frac{\partial}{\partial t}\mathfrak{N}(\vec{r}\vec{r}';\zeta\zeta';\eta\eta') = &-\frac{\hbar^2}{2m}(\nabla_r^2-\nabla_{r'}^2)\mathfrak{N}(\vec{r}\vec{r}';\zeta\zeta';\eta\eta')\\
&+\int d^3r_2[v_W(\vec{r},\vec{r}_2)-v_W(\vec{r}',\vec{r}_2)][\mathfrak{N}(\vec{r}\vec{r}';\zeta\zeta';\eta\eta')\,\mathrm{tr}_{\zeta_2\eta_2}\mathfrak{N}(\vec{r}_2\vec{r}_2;\zeta_2\zeta_2;\eta_2\eta_2)\\
&\qquad -\mathfrak{N}(\vec{r}\vec{r}_2;\zeta\zeta';\eta\eta')\mathfrak{N}(\vec{r}_2\vec{r}';\zeta\zeta';\eta\eta')]\\
&+\int d^3r_2[v_M(\vec{r},\vec{r}_2)\,\mathfrak{N}(\vec{r}_2\vec{r}';\zeta\zeta';\eta\eta')\,\mathrm{tr}_{\zeta_2\eta_2}\mathfrak{N}(\vec{r}\vec{r}_2;\zeta_2\zeta_2;\eta_2\eta_2)\\
&\qquad -v_M(\vec{r}',\vec{r}_2)\mathfrak{N}(\vec{r}\vec{r}_2;\zeta\zeta';\eta\eta')\,\mathrm{tr}_{\zeta_2\eta_2}\mathfrak{N}(\vec{r}_2\vec{r}';\zeta_2\zeta_2;\eta_2\eta_2)]\\
&-\int d^3r_2[v_M(\vec{r},\vec{r}_2)-v_M(\vec{r}',\vec{r}_2)]\mathfrak{N}(\vec{r}\vec{r}';\zeta\zeta';\eta\eta')\mathfrak{N}(\vec{r}_2\vec{r}_2;\zeta\zeta';\eta\eta')\\
&+\int d^3r_2[v_B(\vec{r},\vec{r}_2)-v_B(\vec{r}',\vec{r}_2)]\mathfrak{N}(\vec{r}\vec{r}';\zeta\zeta';\eta\eta')\,\mathrm{tr}_{\eta_2}\mathfrak{N}(\vec{r}_2\vec{r}_2;\zeta\zeta';\eta_2\eta_2)\\
&-\int d^3r_2[v_B(\vec{r},\vec{r}_2)\mathfrak{N}(\vec{r}_2\vec{r}';\zeta\zeta';\eta\eta')\,\mathrm{tr}_{\zeta_2}\mathfrak{N}(\vec{r}\vec{r}_2;\zeta\zeta';\eta\eta')\\
&\qquad -v_B(\vec{r}',\vec{r}_2)\mathfrak{N}(\vec{r}\vec{r}_2;\zeta\zeta';\eta\eta')\,\mathrm{tr}_{\zeta_2}\mathfrak{N}(\vec{r}_2\vec{r}';\zeta_2\zeta_2;\eta\eta')]\\
&+\int d^3r_2[v_H(\vec{r},\vec{r}_2)\mathfrak{N}(\vec{r}_2\vec{r}';\zeta\zeta';\eta\eta')\,\mathrm{tr}_{\eta_2}\mathfrak{N}(\vec{r}\vec{r}_2;\zeta\zeta';\eta_2\eta_2)\\
&\qquad -v_H(\vec{r}',\vec{r}_2)\mathfrak{N}(\vec{r}\vec{r}_2;\zeta\zeta';\eta\eta')\,\mathrm{tr}_{\eta_2}\mathfrak{N}(\vec{r}_2\vec{r}';\zeta\zeta';\eta_2\eta_2)]\\
&-\int d^3r_2[v_H(\vec{r},\vec{r}_2)-v_H(\vec{r}',\vec{r}_2)]\mathfrak{N}(\vec{r}\vec{r}';\zeta\zeta';\eta\eta')\,\mathrm{tr}_{\zeta_2}\mathfrak{N}(\vec{r}_2\vec{r}_2;\zeta_2\zeta_2;\eta\eta')\ .
\end{aligned} \tag{3.2}$$

In the stationary case, the density matrix is independent of time and one obtains the Hartree-Fock equation by setting the right-hand side of Eq. (3.2) to zero. This is the Hartree-Fock equation in density matrix form and is not the usual one we encounter in terms of an equation for the single-particle wave function.

IV. SIMPLIFIED TDHF EQUATION IN SPATIAL COORDINATES

Before we go on to investigate the fluid-dynamical form of the TDHF equation, we may wish to digress a little and write down a simplified TDHF equation, involving only spatial coordinates. We shall not use this simplified equation in the discussion to be followed in the next sections. The simplified TDHF can be of use in other problems where the spin and isospin degrees of freedom can be assumed to behave in a simple and prescribed way. In that case, it is not necessary to start with a TDHF equation of the complicated form as Eq. (3.2), but a much simplified equation whose interactions are now admixtures of the different exchange components and whose density matrix is only a function of the coordinates $\vec{r}$ and $\vec{r}'$.

We define a one-body spatial density matrix $\mathcal{R}(\vec{r},\vec{r}')$ which is the trace of the one-body density matrix $\mathcal{R}(\vec{r}\vec{r}';\zeta\zeta';\eta\eta')$ over the spin and isospin degrees of freedom:

$$\mathcal{R}(\vec{r},\vec{r}')=\mathrm{tr}_{\zeta\eta}\mathcal{R}(\vec{r}\vec{r}';\zeta\zeta';\eta\eta')=\sum_{\zeta,\eta}\mathcal{R}(\vec{r}\vec{r}';\zeta\zeta';\eta\eta)\ . \tag{4.1}$$

We assume that all the one-body density matrices are related to this spatial density matrix by the simple relation

$$\frac{\mathcal{R}(\vec{r}\vec{r}';\zeta\zeta;\eta\eta)}{\int d^3r\,\mathcal{R}(\vec{r}\vec{r};\zeta\zeta;\eta\eta)}=\frac{\sum_{\zeta\eta}\mathcal{R}(\vec{r}\vec{r}';\zeta\zeta;\eta\eta)}{\int d^3r\sum_{\zeta\eta}\mathcal{R}(\vec{r}\vec{r};\zeta\zeta;\eta\eta)}\ . \tag{4.2}$$

Those density matrices which are nondiagonal in ζ and η vanish in our representation and need not concern us here. Using the notation of $\frac{1}{2}$ for a (iso)spin-up variable and $-\frac{1}{2}$ for a (iso)spin-down variable and introducing U as the number of spin-up particles and D as the number of spin-down particles, we have

$$\mathcal{R}(\vec{r}\vec{r}';\tfrac{1}{2}\tfrac{1}{2};\tfrac{1}{2}\tfrac{1}{2})=(UN/A^2)\mathcal{R}(\vec{r},\vec{r}')\ , \tag{4.3}$$

$$\mathcal{R}(\vec{r}\vec{r}';\tfrac{1}{2}\tfrac{1}{2};-\tfrac{1}{2}-\tfrac{1}{2})=(UZ/A^2)\mathcal{R}(\vec{r},\vec{r}')\ . \tag{4.4}$$

$$\mathcal{R}(\vec{r}\vec{r}';-\tfrac{1}{2}-\tfrac{1}{2};\tfrac{1}{2}\tfrac{1}{2})=(DN/A^2)\mathcal{R}(\vec{r},\vec{r}')\ , \tag{4.5}$$

$$\mathcal{R}(\vec{r}\vec{r}';-\tfrac{1}{2}-\tfrac{1}{2};-\tfrac{1}{2}-\tfrac{1}{2})=(DZ/A^2)\mathcal{R}(\vec{r},\vec{r}')\ . \tag{4.6}$$

We obtain the simplified TDHF equation for our special case

$$\begin{aligned}i\hbar\frac{\partial}{\partial t}\mathcal{R}(\vec{r},\vec{r}')&=-\frac{\hbar^2}{2m}(\nabla_r{}^2-\nabla_{r'}{}^2)\mathcal{R}(\vec{r},\vec{r}')\\&+\int d^3r_2[v_{\mathrm{di}}(\vec{r},\vec{r}_2)-v_{\mathrm{di}}(\vec{r}',\vec{r}_2)]\mathcal{R}(\vec{r},\vec{r}')\mathcal{R}(\vec{r}_2,\vec{r}_2)\\&-\int d^3r_2[v_{\mathrm{ex}}(\vec{r},\vec{r}_2)-v_{\mathrm{ex}}(\vec{r}',\vec{r}_2)]\mathcal{R}(\vec{r},\vec{r}_2)\mathcal{R}(\vec{r}_2,\vec{r}')\ ,\end{aligned} \tag{4.7}$$

where

$$v_{\mathrm{di}}=v_{\mathrm{W}}-[(U^2+D^2)(N^2+Z^2)/A^4]v_{\mathrm{M}}+[(U^2+D^2)/A^2]v_{\mathrm{B}}-[(N^2+Z^2)/A^2]v_{\mathrm{H}} \tag{4.8}$$

and

$$v_{\mathrm{ex}}=[(U^2+D^2)(N^2+Z^2)/A^4]v_{\mathrm{W}}-v_{\mathrm{M}}+[(N^2+Z^2)/A^2]v_{\mathrm{B}}-[(U^2+D^2)/A^2]v_{\mathrm{H}}\ . \tag{4.9}$$

The subscript di stands for the effective potential associated with the "direct" part $\mathcal{R}(\vec{r},\vec{r}')\mathcal{R}(\vec{r}_2,\vec{r}_2)$ of the two-body spatial density matrix, while the subscript ex stands for the effective potential associated with the "exchange part" $\mathcal{R}(\vec{r},\vec{r}_2)\mathcal{R}(\vec{r}_2,\vec{r}')$ of the two-body spatial density matrix. In the particular case which is often encountered, the total spin of the system is zero and we have

$$U=D=\tfrac{1}{2}A\ . \tag{4.10}$$

Then, v_{di} and v_{ex} becomes

$$v_{\mathrm{di}}=v_{\mathrm{W}}-[(N^2+Z^2)/2A^2]v_{\mathrm{M}}+\tfrac{1}{2}v_{\mathrm{B}}-[(N^2+Z^2)/A^2]v_{\mathrm{H}} \tag{4.11}$$

and

$$v_{\mathrm{ex}}=[(N^2+Z^2)/2A^2]v_{\mathrm{W}}-v_{\mathrm{M}}+[(N^2+Z^2)/A^2]v_{\mathrm{B}}-\tfrac{1}{2}v_{\mathrm{H}}\ . \tag{4.12}$$

In nuclear fluid, it is also necessary to include the two-body Coulomb interaction on the right-hand side of Eqs. (4.8) and (4.9) or (4.11) and (4.12). As the neutrons and protons lose their identity in our averaging over the isospin space, the Coulomb two-body interaction must be modified accordingly.

One observes that in the present simple case in which the spin and isospin degrees of freedom are constrained to behave in a prescribed way, the TDHF equation is the same as in the case where there is no such freedom, with the exception that the effective interaction is now different for the direct term and the exchange term. It is interesting to note in passing that if one has the freedom, one can judiciously adjust the strength of the various exchange components such that v_{ex} can be chosen to be zero. In that case, the mathematics of solving the TDHF can be greatly simplified.

In the presence of interactions which depend on densities, Eq. (4.7) needs to be amended to take into account the density dependence. This will be discussed in Sec. VI.

V. FLUID DYNAMICAL EQUATIONS INVOLVING SPIN DENSITIES AND ISOSPIN DENSITIES

We return now to Eq. (3.2) to investigate the fluid-dynamical form of the TDHF equation. Following Hill and Wheeler[11] and Griffin and Kan,[12] we separate the intrinsic motion of the nucleons from the collective motion by writing the spatial part of the single-particle wave function in terms of its real intrinsic amplitude $\phi_\alpha(\vec{r},t)$, real phase factor for extrinsic motion $S_\alpha(\vec{r},t)$, and energy factor $\Omega_\alpha(t)$ which does not depend on positions:

$$\Phi_\alpha(\vec{r},t)=\phi_\alpha(\vec{r},t)e^{imS_\alpha(\vec{r},t)/\hbar-i\Omega_\alpha(t)}\chi_\alpha(\zeta)\nu_\alpha(\eta)\ , \tag{5.1}$$

where m is the nucleon mass, and χ and ν are the spin and isospin wave functions. The density matrix derived from a Slater determinant becomes

$$\mathfrak{N}(\vec{r}\vec{r}';\zeta\zeta';\eta\eta')=\sum_{\{\alpha\}}\phi_\alpha(\vec{r})\phi_\alpha(\vec{r}')e^{im[S_\alpha(\vec{r})-S_\alpha(\vec{r}')]/\hbar}\times\chi_\alpha(\zeta)\chi_\alpha^\dagger(\zeta')\nu_\alpha(\eta)\nu_\alpha^\dagger(\eta')\ , \tag{5.2}$$

where, and henceforth, for simplicity of notation, all the symbols indicating explicit time dependence are understood and the symbol $\{\alpha\}$ implies that summation over α is carried out over the set of occupied states $\{\alpha\}$. The fact that all the density matrices in Eq. (3.2) are diagonal in the spin and isospin spaces in our representations allows one to limit one's attention henceforth to the case where $\zeta=\zeta'$ and $\eta=\eta'$. One introduces notation

$$n(\zeta\eta,\vec{r})=\mathfrak{N}(\vec{r}\vec{r};\zeta\zeta;\eta\eta) \tag{5.3}$$

to represent the spatial density (in numbers per fm^3) for particles having spin coordinate ζ and isospin coordinate η.

It is clear that the TDHF contains more information about the dynamics of the system than there is in the equation of motion of the classical type where the density field is only a function of one coordinate and not two coordinates. The passage to conservation laws is made possible if one restricts oneself to the limit where $\vec{r}\to\vec{r}'$. One obtains, after some simplification,

$$\frac{\partial}{\partial t}n(\zeta\eta)+\nabla\cdot[n(\zeta\eta)u(\zeta\eta)]=\frac{2}{\hbar}\,\mathrm{Im}\Big[\int d^3r_2 v_M(\vec{r},\vec{r}_2)\mathfrak{N}(\vec{r}_2\vec{r};\zeta\zeta,\eta\eta)\,\mathrm{tr}_{\zeta_2\eta_2}\mathfrak{N}(\vec{r}\vec{r}_2;\zeta_2\zeta_2;\eta_2\eta_2)$$
$$-\int d^3r_2 v_B(\vec{r},\vec{r}_2)\mathfrak{N}(\vec{r}_2\vec{r};\zeta\zeta;\eta\eta)\,\mathrm{tr}_{\zeta_2}\mathfrak{N}(\vec{r}\vec{r}_2;\zeta_2\zeta_2;\eta\eta)$$
$$+\int d^3r_2 v_H(\vec{r},\vec{r}_2)\mathfrak{N}(\vec{r}_2\vec{r};\zeta\zeta;\eta\eta)\,\mathrm{tr}_{\eta_2}\mathfrak{N}(\vec{r}\vec{r}_2;\zeta\zeta;\eta_2\eta_2)\Big]\ . \tag{5.4}$$

As one observes, this is an equation of continuity for the density $n(\zeta\eta)$. The presence of the exchange potentials, which change one type of particle into another and thus alter the densities at different positions, gives rise to the source terms in the equation of continuity.

We note also in passing that by summing Eq. (5.4) over ζ and η, we obtain, as expected, the equation of continuity for the total density

$$\frac{\partial}{\partial t}\sum_{\zeta\eta}n(\zeta\eta)+\nabla\cdot\Big[\sum_{\zeta\eta}n(\zeta\eta)u(\zeta\eta)\Big]=0\ . \tag{5.5}$$

Following Ref. 1, we again obtain an equation for the variation of the probability current by applying the operator $(\nabla_r-\nabla_{r'})$ to the TDHF Eq. (3.2) and then taking the limit as $\vec{r}'\to\vec{r}$:

$$m\frac{\partial}{\partial t}n(\zeta\eta)u_i(\zeta\eta)+\sum_{j=1}^{3}\nabla_j[mn(\zeta\eta)u_i(\zeta\eta)u_j(\zeta\eta)+p_{ij}^{(t)}(\zeta\eta)+p_{ij}^{(q)}(\zeta\eta)]$$
$$=-n(\zeta\eta)\int d^3r_2\mathfrak{N}(\vec{r}_2,\vec{r}_2)\nabla v_W(\vec{r},\vec{r}_2)+\int d^3r_2|\mathfrak{N}(\vec{r}\vec{r}_2;\zeta\zeta;\eta\eta)|^2\nabla v_W(\vec{r},\vec{r}_2)$$
$$-\mathrm{Re}\int d^3r_2\{[\mathfrak{N}(\vec{r}_2\vec{r};\zeta\zeta;\eta\eta)\mathfrak{N}(\vec{r},\vec{r}_2)\nabla v_M(\vec{r},\vec{r}_2)+v_M(\vec{r},\vec{r}_2)[\mathfrak{N}(\vec{r}_2r;\zeta\zeta;\eta\eta)\nabla\mathfrak{N}(\vec{r},\vec{r}_2)-\mathfrak{N}(\vec{r},\vec{r}_2)\nabla\mathfrak{N}(\vec{r}_2\vec{r};\zeta\zeta;\eta\eta)]\}$$
$$+n(\zeta\eta)\int d^3r_2 n(\zeta\eta,\vec{r}_2)\nabla v_M(\vec{r},\vec{r}_2)-n(\zeta\eta)\int d^3r_2\,\mathrm{tr}_{\eta_2}\mathfrak{N}(\vec{r}_2\vec{r}_2;\zeta\zeta;\eta_2\eta_2)\nabla v_B(\vec{r},\vec{r}_2)$$
$$+\mathrm{Re}\int d^3r_2\{\mathfrak{N}(\vec{r}_2\vec{r};\zeta\zeta;\eta\eta)\,\mathrm{tr}_{\zeta_2}\mathfrak{N}(\vec{r}\vec{r}_2;\zeta_2\zeta_2;\eta\eta)\nabla v_B(\vec{r},\vec{r}_2)$$
$$+v_B(\vec{r},\vec{r}_2)[\mathfrak{N}(\vec{r}_2\vec{r};\zeta\zeta;\eta\eta)\,\mathrm{tr}_{\zeta_2}\nabla\mathfrak{N}(\vec{r}\vec{r}_2;\zeta_2\zeta_2;\eta\eta)-\mathrm{tr}_{\zeta_2}\mathfrak{N}(\vec{r}\vec{r}_2;\zeta_2\zeta_2;\eta\eta)\nabla\mathfrak{N}(\vec{r}_2\vec{r};\zeta\zeta;\eta\eta)]\}$$
$$-\mathrm{Re}\int d^3r_2\{\mathfrak{N}(\vec{r}_2\vec{r};\zeta\zeta;\eta\eta)\,\mathrm{tr}_{\eta_2}\mathfrak{N}(\vec{r}\vec{r}_2;\zeta\zeta;\eta\eta)\nabla v_H(\vec{r},\vec{r}_2)$$
$$+v_H(\vec{r},\vec{r}_2)[\mathfrak{N}(\vec{r}_2\vec{r};\zeta\zeta;\eta\eta)\,\mathrm{tr}_{\eta_2}\nabla\mathfrak{N}(\vec{r}\vec{r}_2;\zeta\zeta;\eta_2\eta_2)-\mathrm{tr}_{\eta_2}\mathfrak{N}(\vec{r}\vec{r}_2;\zeta\zeta;\eta_2\eta_2)\nabla\mathfrak{N}(\vec{r}_2\vec{r};\zeta\zeta;\eta\eta)]\}$$
$$+n(\zeta\eta)\int d^3r_2\,\mathrm{tr}_{\zeta_2}\mathfrak{N}(\vec{r}_2\vec{r}_2;\zeta_2\zeta_2;\eta\eta)\nabla v_H(\vec{r},\vec{r}_2)\ , \tag{5.6}$$

where[2,13,18]

$$p_{ij}^{(t)}(\zeta\eta) = m\sum_{\alpha}\phi_\alpha{}^2[\nabla_i S_\alpha - u_i(\zeta\eta)][\nabla_j S_\alpha - u_j(\zeta\eta)]\chi_\alpha(\zeta)\chi_\alpha^\dagger(\zeta)\nu_\alpha(\eta)\nu_\alpha^\dagger(\eta) \tag{5.7}$$

and

$$p_{ij}^{(q)}(\zeta\eta) = +\frac{\hbar^2}{4m}\nabla_i\nabla_j n(\zeta\eta) - \frac{\hbar^2}{m}\sum_{\alpha}\phi_\alpha\nabla_i\nabla_j\phi_\alpha\,\chi_\alpha(\zeta)\chi_\alpha^\dagger(\zeta)\nu_\alpha(\eta)\nu_\alpha^\dagger(\eta)\ . \tag{5.8}$$

This is the equation which follows directly from the TDHF equation, but written in a form analogous to the Euler equation in fluid dynamics. So far, we have not made any approximation. We have only introduced the stress tensors by combining the appropriate terms.

VI. TDHF EQUATIONS FOR DENSITY-DEPENDENT FORCES

Our discussions in the preceding sections have been concerned with density-independent interactions. In many physical applications such as in the use of the Skyrme interaction introduced by Brink and Vautherin[14,15] and Golin and Zamick,[16,17] it is often desirable to consider forces which are dependent on the density. We seek to obtain the TDHF by a variational principle

$$\delta\langle\Psi, (H - i\hbar\frac{\partial}{\partial t})\Psi\rangle = 0\ , \tag{6.1}$$

where Ψ is taken to be a single Slater determinant and

$$H = -\frac{\hbar^2}{2m}\sum_{\{\alpha\}}\nabla_\alpha{}^2 + \frac{1}{2}\sum_{\alpha\beta}v(|\vec{r}_\alpha - \vec{r}_\beta|, n(\tfrac{1}{2}|\vec{r}_\alpha + \vec{r}_\beta|), \zeta_\alpha, \zeta_\beta, \eta_\alpha, \eta_\beta)\ . \tag{6.2}$$

We specialize to the case where the density-dependent component of the interaction has a zero range and we obtain the familiar TDHF equation in the form

$$i\hbar\frac{\partial}{\partial t}\Phi_\alpha(x_1) = -\frac{\hbar^2}{2m}\nabla^2\Phi_\alpha(x_1) + \int dx_2\sum_\beta\Phi_\beta^\dagger(x_2)v(x_1,x_2)\{\Phi_\alpha(x_1)\Phi_\beta(x_2) - \Phi_\alpha(x_2)\Phi_\beta(x_1)\} + \Delta(\vec{r}_1)\Phi_\alpha(x_1) \tag{6.3}$$

where the additional rearrangement potential $\Delta(r)$ is given by

$$\Delta(\vec{r}_1) = \Big[\sum_{\zeta_3\eta_3}\sum_{\zeta_2\eta_2}\int d^3r_2\lim_{\substack{x_2'\to x_2\\ x_3'\to x_3}}\frac{1}{2}\frac{\partial v(x_2,x_3)}{\partial n}\mathfrak{N}^{(2)}(x_2x_3;x_2'x_3')\Big]_{\vec{r}_3=\vec{r}_1}\ . \tag{6.4}$$

The TDHF equation in density matrix form is therefore amended to be

$$i\hbar\frac{\partial}{\partial t}\mathfrak{N}^{(1)}(x_1;x_1') = -\frac{\hbar^2}{2m}(\nabla_{r_1}{}^2 - \nabla_{r_1'}{}^2)\mathfrak{N}^{(1)}(x_1;x_1')$$
$$+\int dx_2\lim_{x_2'\to x_2}\{v(x_1,x_2) - v(x_1',x_2')\}\mathfrak{N}^{(2)}(x_1x_2;x_1'x_2') + [\Delta(\vec{r}_1) - \Delta(\vec{r}_1')]\mathfrak{N}^{(1)}(x_1;x_1')\ , \tag{6.5}$$

which differs from the usual TDHF Eq. (2.7) only by the presence of the last rearrangement term.

How does this additional rearrangement term affect the dynamical equation? We can again follow the procedures outlined in the last section by taking the appropriate limits. Upon taking the limit $\vec{r}_1' \to \vec{r}_1$, we find

$$\lim_{\vec{r}_1'\to\vec{r}_1}[\Delta(\vec{r}_1) - \Delta(\vec{r}_1')]\mathfrak{N}(x_1,x_1') = 0\ . \tag{6.6}$$

Thus, from the procedures leading to the equation of continuity (5.4), it can be concluded that the equation of continuity is unaffected by the density dependence of a zero-range interaction. It is clear that such simple results are obtained because of the zero range of the density-dependent force; when the range of the force is nonzero, one expects a source term in the equation of continuity due to the exchange of particles at different locations.

The presence of the rearrangement potential however gives rise to additional force fields in the equation of motion. Following our previous procedures, we take the limit $\vec{r}_1' \to \vec{r}_1$ after applying the operator $\nabla_{r_1} - \nabla_{r_1'}$ onto the last term of Eq. (6.5). We obtain

$$\lim_{\vec{r}_1'\to\vec{r}_1}(\nabla_{r_1} - \nabla_{r_1'})\{[\Delta(\vec{r}_1) - \Delta(\vec{r}_1')]\mathfrak{N}(x_1,x_1')\}$$
$$= 2[\nabla_{r_1}\Delta(\vec{r}_1)]\mathfrak{N}(\vec{r}_1\vec{r}_1';\zeta_1\zeta_1';\eta_1\eta_1')\ . \tag{6.7}$$

Therefore, it is now necessary to add a force density

$$\vec{F}_{\Delta}(\vec{r}) = -n(\zeta\eta)[\nabla(\vec{r})] \tag{6.8}$$

to the right-hand side of the equation of motion (5.6) to take into account the additional force field arising from the density dependence of the interaction.

From Eq. (6.5), it is clear that in the presence of a density-dependent interaction and when the densities are constrained to vary according to Eq. (4.2), the simplified TDHF equation in spatial coordinates, Eq. (4.7), needs to be amended to read

$$i\hbar\frac{\partial}{\partial t}\mathfrak{N}(\vec{r},\vec{r}') = -\frac{\hbar^2}{2m}(\nabla_r^2 - \nabla_{r'}^2)\mathfrak{N}(\vec{r},\vec{r}') + \int d^3r_2[v_{\rm di}(\vec{r},\vec{r}_2) - v_{\rm di}(\vec{r}',\vec{r}_2)]\mathfrak{N}(\vec{r},\vec{r}')\mathfrak{N}(\vec{r}_2,\vec{r}_2)$$
$$- \int d^3r_2[v_{\rm ex}(\vec{r},\vec{r}_2) - v_{\rm ex}(\vec{r}',\vec{r}_2)]\mathfrak{N}(\vec{r},\vec{r}_2)\mathfrak{N}(\vec{r}_2,\vec{r}') + [\Delta(\vec{r}) - \Delta(\vec{r}')]\mathfrak{N}(\vec{r},\vec{r}') \ . \tag{6.9}$$

VII. FLUID DYNAMICAL EQUATION WITH ZERO-RANGE FORCES

The complicated form of Eq. (5.6) makes it difficult to perform a simple analysis unless the spatial form of the potential can be very simple. We shall specialize now to a short-range δ-function force to write the two-body potential in the form

$$v(x_1x_2) = [\mathrm{W} + \mathrm{M}\mathcal{P}^{\mathrm{M}}(x_1x_2) + \mathrm{B}\mathcal{P}^{\mathrm{B}}(x_1x_2) + \mathrm{H}\mathcal{P}^{\mathrm{H}}(x_1x_2)]\delta(\vec{r}_1 - \vec{r}_2) + \tfrac{1}{6}t_3[1 + \mathcal{P}^{\mathrm{B}}(x_1x_2)]n\delta(\vec{r}_1 - \vec{r}_2) \ , \tag{7.1}$$

where W, M, B, H, and t_3 are constants and the density-dependent term t_3 is introduced to simulate the three-body interaction and to give rise to saturation of nuclear matter as is introduced by Vautherin and Brink and others.[14,15] The density n without the labels of ζ and η is the total density

$$n = \sum_{\zeta,\eta} n(\zeta\eta) \ , \tag{7.2}$$

to be evaluated at the point $\frac{1}{2}(\vec{r}_1 + \vec{r}_2)$.

With a zero-range interaction of the type as in Eq. (7.1), the fluid-dynamical equations can be greatly simplified. The right-hand side of the equation of continuity [Eq. (5.4)] vanishes because density matrix elements diagonal in the spatial coordinates have no imaginary parts. One has

$$\frac{\partial}{\partial t}n(\zeta\eta) + \nabla\cdot[n(\zeta\eta)u(\zeta\eta)] = 0 \ . \tag{7.3}$$

The right-hand side of the dynamical Eq. (5.6) can also be integrated to yield

$$m\frac{\partial}{\partial t}n(\zeta\eta)u_i(\zeta\eta) + \sum_j \nabla_j[mn(\zeta\eta)u_i(\zeta\eta)u_j(\zeta\eta) + p_{ij}^{(t)}(\zeta\eta) + p_{ij}^{(q)}(\zeta\eta)]$$
$$= -n(\zeta\eta)\nabla_i\{(\mathrm{W} + \tfrac{1}{6}t_3n + \mathrm{M})[n - n(\zeta\eta)] + (\mathrm{B} + \tfrac{1}{6}t_3n + \mathrm{H})[\mathrm{tr}_{\eta_2}n(\zeta\eta_2) - \mathrm{tr}_{\zeta_2}n(\zeta_2\eta)]\} + F_{\Delta} \ . \tag{7.4}$$

We have, on the other hand,

$$\Delta(\vec{r}) = \tfrac{1}{12}t_3\sum_{\zeta_3\eta_3}[n(\zeta_3\eta_3)\,\mathrm{tr}_{\zeta_2}n(\zeta_2\bar{\eta}_3) + n(\zeta_3\eta_3)n(\zeta_3\bar{\eta}_3)] \ , \tag{7.5}$$

where $\bar{\eta}_3$ is the complement of η, namely

$$\bar{\eta}_3 = -\eta_3 \ . \tag{7.6}$$

The additional force field is therefore

$$\vec{F}_{\Delta} = -n(\zeta\eta)\nabla\{\tfrac{1}{12}t_3\,\mathrm{tr}_{\zeta_3\eta_3}[n(\zeta_3\eta_3)\,\mathrm{tr}_{\zeta_2}n(\zeta_2\bar{\eta}_3) + n(\zeta_3\eta_3)n(\zeta_3\bar{\eta}_3)]\} \ . \tag{7.7}$$

Equations (7.3) and (7.4) are the equations governing the dynamics of the nuclear fluid. As both ζ and η can assume two different values, one has four equations of continuity and four (vector) equations of motion. The equations of continuity are all uncoupled, whereas the equation of motion couples different densities for the different spin and isospin coordinates.

We shall now attempt to obtain the normal modes of the different density variations by making the linearization approximation for small deviations from equilibrium for which the second-order

terms in u can be neglected. We shall also consider temperatures not far from $T=0$, so that $p_{ij}^{(t)}$ can be neglected. For the quantum pressure, we assume a Thomas-Fermi model to write

$$p_{ij}^{(q)}(\zeta\eta) = n^2(\zeta\eta)\frac{\partial[\tau(\zeta\eta)/n(\zeta\eta)]}{\partial n(\zeta\eta)}\delta_{ij} \ , \tag{7.8}$$

where

$$\tau(\zeta\eta) = \frac{3\hbar^2}{10m}(6\pi^2)^{2/3}[n(\zeta\eta)]^{5/3} \ , \tag{7.9}$$

and a correction term of the form $\hbar^2\nabla^2 n(\zeta\eta)/12m$ obtained recently[18] for the quantum stress tensor has been neglected. We further assume that the equilibrium density is given by

$$\begin{aligned} n^{(0)}(\tfrac{1}{2}\,\tfrac{1}{2}) &= n^{(0)}(\tfrac{1}{2}\,-\tfrac{1}{2}) = n^{(0)}(-\tfrac{1}{2}\,\tfrac{1}{2}) \\ &= n^{(0)}(-\tfrac{1}{2}\,-\tfrac{1}{2}) = \tfrac{1}{4}n^{(0)} \end{aligned} \tag{7.10}$$

and thus $n^{(0)}(\zeta\eta)\,\partial^2\tau^{(0)}(\zeta\eta)/\partial n^{(0)}(\zeta\eta)^2$ is independent of ζ and η.

For small deviations from equilibrium, we can write

$$n(\zeta\eta) = n^{(0)}(\zeta\eta) + \delta n(\zeta\eta) \ . \tag{7.11}$$

Equations (7.3) and (7.4) lead to

$$\begin{aligned} m\frac{\partial^2}{\partial t^2}\delta n(\zeta\eta) - n^{(0)}(\zeta\eta)\frac{\partial^2\tau^{(0)}(\zeta\eta)}{\partial n^{(0)}(\zeta\eta)^2}\nabla^2\delta n(\zeta\eta) - n^{(0)}(\zeta\eta) \\ \times\{(\mathrm{W}+\mathrm{M}+\tfrac{1}{6}t_3 n^{(0)})\nabla^2[\delta n - \delta n(\zeta\eta)] + \tfrac{1}{4}t_3 n^{(0)}\nabla^2\delta n \\ + (\mathrm{B}+\mathrm{H}+\tfrac{1}{6}t_3 n^{(0)})\nabla^2[\delta n(\zeta\tfrac{1}{2}) + \delta n(\zeta\,-\tfrac{1}{2}) \\ - \delta n(\tfrac{1}{2}\eta) - \delta n(-\tfrac{1}{2}\eta)]\} = 0 \ . \end{aligned} \tag{7.12}$$

In order to separate the normal modes, we shall write out Eq. (7.12) explicitly. To simplify the notations, we introduce the symbols

$$\begin{aligned} &n_1 = n(\tfrac{1}{2}\,\tfrac{1}{2}), \quad n_2 = n(\tfrac{1}{2}\,-\tfrac{1}{2}), \\ &n_3 = n(-\tfrac{1}{2}\,\tfrac{1}{2}), \quad n_4 = n(-\tfrac{1}{2}\,-\tfrac{1}{2}), \\ &c = \tfrac{1}{4}n^{(0)}(\mathrm{W}+\mathrm{M}+\tfrac{1}{6}t_3 n^{(0)}) \ , \\ &d = \tfrac{1}{4}n^{(0)}(\mathrm{B}+\mathrm{H}+\tfrac{1}{6}t_3 n^{(0)}) \ , \\ &f = \tfrac{1}{16}t_3(n^{(0)})^2 \ , \end{aligned} \tag{7.13}$$

and

$$g = n^{(0)}(\zeta\eta)\,\partial^2\tau^{(0)}(\zeta\eta)/\partial n^{(0)}(\zeta\eta)^2 \ .$$

Then Eq. (7.12) can be written, in full, as follows

$$m\frac{\partial^2}{\partial t^2}\delta n_1 - (g+f)\nabla^2\delta n_1 - (f+c+d)\nabla^2\delta n_2 - (f+c-d)\nabla^2\delta n_3 - (f+c)\nabla^2\delta n_4 = 0 \ , \tag{7.14}$$

$$m\frac{\partial^2}{\partial t^2}\delta n_2 - (f+c+d)\nabla^2\delta n_1 - (g+f)\nabla^2\delta n_2 - (f+c)\nabla^2\delta n_3 - (f+c-d)\nabla^2\delta n_4 = 0 \ , \tag{7.15}$$

$$m\frac{\partial^2}{\partial t^2}\delta n_3 - (f+c-d)\nabla^2\delta n_1 - (f+c)\nabla^2\delta n_2 - (g+f)\nabla^2\delta n_3 - (f+c+d)\nabla^2\delta n_4 = 0 \ , \tag{7.16}$$

$$m\frac{\partial^2}{\partial t^2}\delta n_4 - (f+c)\nabla^2\delta n_1 - (f+c-d)\nabla^2\delta n_2 - (f+c+d)\nabla^2\delta n_3 - (g+f)\nabla^2\delta n_4 = 0 \ . \tag{7.17}$$

From Eqs. (7.14)–(7.17), we get all the normal modes of sound propagation for the nuclear fluid under investigation. By adding Eqs. (7.14)–(7.17) together, we get

$$m\frac{\partial^2}{\partial t^2}\delta(n_1+n_2+n_3+n_4) - (g+4f+3c)\nabla^2\delta(n_1+n_2+n_3+n_4) = 0 \ . \tag{7.18}$$

We recognize the sum of the n's as the total density

$$n = n_1 + n_2 + n_3 + n_4 \tag{7.19}$$

and thus Eq. (7.18) is the equation of propagation of normal sound in which all the different components of the density are constrained to vary together. The speed of normal sound a_1 can be readily obtained from (7.18):

$$a_1^2 = (g+4f+3c)/m \ . \tag{7.20}$$

We get a different mode of sound propagation by adding the first two equations and subtracting the last two equations in the set (7.14)–(7.17). We obtain

$$m\frac{\partial^2}{\partial t^2}\delta[(n_1+n_2) - (n_3+n_4)] - (g-c+2d)\nabla^2\delta[(n_1+n_2) - (n_3+n_4)] = 0 \ . \tag{7.21}$$

We note that the quantity in the square brackets

$$(n_1+n_2) - (n_3+n_4) = n(\tfrac{1}{2}\,\tfrac{1}{2}) + n(\tfrac{1}{2}\,-\tfrac{1}{2}) - [n(-\tfrac{1}{2}\,\tfrac{1}{2}) + n(-\tfrac{1}{2}\,-\tfrac{1}{2})] \tag{7.22}$$

is the difference of the total spin-up density and the total spin-down density. Thus, we recognize Eq. (7.21) as describing the propagation of sound waves in which the difference of spin-up and spin-down densities varies in both space and time (spin sound wave). The speed of propagation of spin sound wave is

$$a_2^2 = (g - c + 2d)/m \ . \tag{7.23}$$

We can form a different sum of the n's by adding the first and the third equation to subtract from the second and the fourth equation in the set (7.14)–(7.17). We obtain

$$m\frac{\partial^2}{\partial t^2}\delta[n_1 + n_3) - (n_2 + n_4)] - (g - c - 2d)\nabla^2\delta[(n_1 + n_3) - (n_2 + n_4)] = 0 \ . \tag{7.24}$$

We note that the quantity in the square brackets

$$(n_1 + n_3) - (n_2 + n_4) = n(\tfrac{1}{2}\tfrac{1}{2}) + n(-\tfrac{1}{2}\tfrac{1}{2}) - [n(\tfrac{1}{2} - \tfrac{1}{2}) + n(-\tfrac{1}{2} - \tfrac{1}{2})] \tag{7.25}$$

is the difference of the total neutron density and the total proton density. Thus, we recognize Eq. (7.27) as an equation of sound propagation in which the difference of neutron and proton density changes with position and time (isospin sound waves). The speed of propagation of isospin sound is

$$a_3^2 = (g - c - 2d)/m \ . \tag{7.26}$$

Finally, we can form a normal mode of oscillation by adding the first and the last and subtracting the second and the third of the equations in the set (7.14)–(7.17). The resultant equation becomes

$$m\frac{\partial^2}{\partial t^2}\delta[(n_1 + n_4) - (n_2 + n_3)] - (g - c)\nabla^2\delta[(n_1 + n_4) - (n_2 + n_3)] = 0 \ . \tag{7.27}$$

The quantity in the square brackets

$$(n_1 + n_4) - (n_2 + n_3) = n(\tfrac{1}{2}\tfrac{1}{2}) + n(-\tfrac{1}{2} - \tfrac{1}{2}) - [n(\tfrac{1}{2} - \tfrac{1}{2}) + n(-\tfrac{1}{2}\tfrac{1}{2})] \tag{7.28}$$

is the difference for densities when both spin and isospin are "parallel" to each other and when they are "antiparallel." Equation (7.30) is the equation of propagation of such a density wave which we shall call spin-isospin sound waves. The speed of such a sound wave is

$$a_4^2 = (g - c)/m \ . \tag{7.29}$$

It is easy to see how the speed of normal sound is related to the nuclear incompressibility. With the interaction given by Eq. (7.1), the energy per nucleon in infinite nuclear matter is given by

$$\mathcal{W} = \frac{3}{10}\frac{\hbar^2}{m}k_f^2 + \tfrac{3}{8}(W + M + \tfrac{1}{6}t_3 n)n \ , \tag{7.30}$$

where k_f is the Fermi momentum and is related to the density by

$$k_f = (1.5\pi^2 n)^{1/3} \ . \tag{7.31}$$

From Eq. (7.30) the nuclear incompressibility is given by

$$K = 9n^2\frac{\partial \mathcal{W}}{\partial n} = -\frac{3}{5}\frac{\hbar^2}{m}k_f^2 + \tfrac{9}{8}t_3 n^2 \ . \tag{7.32}$$

One can show that for nuclear matter at equilibrium, we have

$$g + 4f + 3c = \tfrac{1}{9}K \ . \tag{7.33}$$

Thus, the speed of normal sound is related to the nuclear incompressibility K by

$$a_1^2 = K/9m \ , \tag{7.34}$$

as is expected.[6]

By comparing Eqs. (7.23), (7.26), and (7.29) with the expressions for the symmetry energies[19] s_i, one can show that the speeds and the symmetry energies are related to the symmetry energies by

$$a_i^2 = 2s_i/m \quad (i = 2,\ 3,\ \text{and}\ 4) \ . \tag{7.35}$$

It is easy to see that the spin sound, isospin sound, and spin-isospin sound are governed by the simple algebraic relation

$$a_2^2 + a_3^2 = 2a_4^2 \ . \tag{7.36}$$

With the addition of a zero-range momentum-dependent interaction, the algebraic relation (7.39) between a_2, a_3, and a_4 is nonetheless preserved.[19]

With the parameters of Zamick and Golin,[16,17] Vautherin and Brink,[14] and Negele and Vautherin,[15] the quantity d is negative and thus we have, in addition, the inequality

$$a_2 < a_4 < a_3 \ . \tag{7.37}$$

This is to say, an isospin sound propagates faster than a spin-isospin sound which in turn propagates faster than a spin sound.

Given the simple zero-range force and the Thomas-Fermi approximation for the stress tensor, the results in this section could have been obtained in a much "cleaner" way.[19] The evaluation of the speeds of various sound waves is, however, not our only objective. Knowing the fluid-dynamical equations (5.4) and (5.6), one knows exactly the modifications needed for different situations. One can, for example, consider interactions which have both a zero-range and a long-range component, as is discussed in Ref. 1. One can also have a different approximation to the quantum stress tensor in situations where the single-parti-

cle wave functions are constrained to vary coherently in a special way as in some special cases of the random phase approximation. The latter consideration, in line with the elastic response of nuclei suggested theoretically by Bertsch,[20] will be the subject of a forthcoming publication.

VIII. SPEEDS OF VARIOUS SOUND WAVES

We seek an effective density-dependent interaction in the form of Eq. (7.1). The density-dependent interactions introduced by Golin and Zamick[16] and Zamick[17] contain many of the terms of Eq. (7.1) and have a zero range. They are well suited for our purpose. However, since the parameters are fitted by using data from spin- and isospin-saturated nuclei, the Bartlett and Heisenberg components are not determined. An adequate description can still be possible if, in conjunction with the zero-range interactions of Zamick *et al.*, one uses the Bartlett and Heisenberg components as determined by other workers employing data from isospin-unsaturated nuclei. The density-dependent interaction introduced by Vautherin and Brink[14] and Negele and Vautherin[15] for Hartree-Fock calculations has terms of the form (7.1), with additional momentum-dependent components. It seems to be a suitable force for our purpose when the parameters are fitted to known nuclear data with the stipulation that the momentum-dependent part of the force is chosen to be relatively unimportant. In that case, there can still be an adequate description of a realistic nuclear fluid if one neglects the momentum-dependent part of the interaction of Vautherin and Brink to bring the interaction to the form of Eq. (7.1). The importance of the momentum-dependent part of the interaction can be assessed by treating the momentum dependence as arising from an expansion of a finite-range interaction with radial dependence $e^{-\gamma|\vec{r}_1-\vec{r}_2|^2}$ and by examining the range of such an equivalent finite-range potential. For various Skyrme interactions, Davies and Satchler[21] found that $\gamma = 2.241$ fm^{-2} for Skyrme I, 0.9988 fm^{-2} for Skyrme II, and 1.637 fm^{-2} for the interaction of Negele and Vautherin (at nuclear matter density). As the neglect of the momentum-dependent part of the interaction can still be appropriate if the range $1/\sqrt{\gamma}$ is small, we find that the Skyrme I interaction, having an effective range $(1/\sqrt{\gamma})$ of 0.67 fm, is the one closest to a zero-range force, while the other sets of similar interactions cannot be used in the present context. Accordingly, in using the interactions of Golin and Zamick,[16,17] the strength of the Bartlett component is adopted from the Skyrme I interaction of Vautherin and Brink.[14] The set of parameters used is listed in Table I. Shown there also are the incompressibility and symmetry energies obtained for these parameters.

With these sets of interaction parameters, we obtain the speeds of various sound waves. They are listed in Table II. For the set of zero-range (or nearly zero-range) interactions we have investigated, the numerical value of the normal sound is about $0.18c$, while the numerical values of a_2, a_3, and a_4 are in the range of $0.17c$ to $0.27c$. Comparing the numerical values of the sound speeds obtained here with those obtained from the full Skyrme interactions which includes momentum dependence,[19] one finds that the present results are only slightly different from those obtained with Skyrme I, II, III, and VI. They differ significantly from those of Skyrme IV and V, where the density dependence and the Bartlett component are quite different from the other Skyrme interactions.

IX. DISCUSSION

It is worth noting that we have chosen the density dependence of the two-body potential to have a form given by the last term of Eq. (7.1). Such a

TABLE I. Parameters for the zero-range interaction Eq. (7.1) with M = H = 0. The quantities $n^{(0)}$ and K are the equilibrium density and nuclear incompressibility obtained with the sets of parameters. The quantities s_2, s_3, and s_4 are the spin, isospin, and spin-isospin symmetry energies (Ref. 19).

	Golin and Zamick I (Ref. 16)	Golin and Zamick II (Ref. 16)	Zamick (Ref. 17)	Skyrme I[a] (Ref. 14)
W (MeV fm^{-3})	−1045.754	−1077.083	−996.89	−1057.3
B (MeV fm^{-3})	−592.09	−592.09	−592.09	−592.09
t_3 (MeV fm^6)	19 189	18 133	16 259.209	14 463.5
$n^{(0)}$ (fm^{-3})	0.1213	0.1352	0.1361	0.1554
K (MeV)	280.7	333.4	299.3	286.2
s_2 (MeV)	13.99	16.07	14.12	16.85
s_3 (MeV)	26.38	28.48	29.31	33.75
s_4 (MeV)	20.18	22.28	21.72	25.30

[a]The momentum-dependent part of the interaction has been neglected.

TABLE II. Speeds of various sound waves in infinite nuclear matter in units of c calculated with different interactions.

	Golin and Zamick I (Ref. 16)	Golin and Zamick II (Ref. 16)	Zamick (Ref. 17)	Skyrme I[a] (Ref. 14)
Normal sound	0.1821	0.1986	0.1883	0.1840
Spin sound	0.1726	0.1850	0.1734	0.1895
Isospin sound	0.2370	0.2463	0.2499	0.2681
Spin-isospin sound	0.2073	0.2178	0.2151	0.2322

[a]The momentum-dependent part of the interaction has been neglected.

term is equivalent to the three-body contact potential in static Hartree-Fock theory[14] where parameters were determined from nuclear properties of many nuclei. In extrapolating to nuclear systems not in equilibrium, there are many different choices for such a term in the interaction. For example, one can take the view that the three-body contact potential is the more fundamental form of the interaction, in spite of its mathematical simplicity. One can then either consider the equivalent two-body interaction appropriate for the problem in question by evaluating the matrix element of the three-body contact potential for the particular transition in question,[21] or one can incorporate the three-body contact potential fully in the dynamics.[22] One finds, however, that in the linear response of the system subject to a nuclear force with such a three-body potential term, many sound waves turn out to be unstable. Such unrealistic instabilities have been traced to the specific parametrization of the density dependence of the interaction.[22,23]

We have, however, taken the view that density dependence such as given by the last term of Eq. (7.1), in conjunction with the static Hartree-Fock theory, has been found to give good account of many properties of finite nuclei. In extending from a static Hartree-Fock theory to a time-dependent Hartree-Fock theory for nonequilibrium dynamics, one makes use of the same mean field for single-particle states in the dynamical case as in the static case. Interactions such as given by Eq. (7.1) are therefore well adopted for the time-dependent Hartree-Fock theory, and it is indeed used for the zero-range part of the two-body interaction in a TDHF calculation.[24] Since our equations of sound have their origin from the TDHF theory, it is appropriate to use a density dependence in the form of Eq. (7.1). The sound speeds are found to be real and the various sound waves are now stable, as they should be.

From the equations of sound propagation, one may wish to investigate density oscillations appropriate for various boundary conditions.[19] For example, in the isospin sound waves, there is the hydrodynamical model of Steinwedel and Jensen,[25] the quantized form of which gives rise to the giant dipole oscillation. Other types of density oscillation can be worked out in a similar way, as is done by Überall and his collaborators[7] and Bohr and Mottelson.[8]

In the absence of noncentral forces and zero-range interactions, all four different modes can be well separated. However, when the ranges of the exchange forces are nonzero, complication arises as the equation of continuity has additional source terms. In terms of the eigenmodes of density oscillations, these exchange forces couple the various modes together and lead to a mixture of various eigenmodes. The mixing of these eigenmodes, due to the exchange forces, is, however, not expected to be very large, as evidenced by the presence of giant dipole states for which the widths are usually much smaller than the eigenenergies.

The mixing due to spin-orbit and tensor interactions is probably much stronger than that due to the range of the exchange interactions. It affects mainly the spin sound and the normal sound which are now coupled by the presence of such interactions. It will be of interest to investigate the degree of admixture due to these forces and to study whether a pure normal sound wave and a pure spin wave can remain a meaningful concept in the presence of these interactions.

The speed of various sound propagation allows a simple discussion on the dynamics of many nuclear phenomena. We shall discuss one such example concerning the equilibration of neutron to proton ratio in a heavy-ion reaction. The reaction time for a collision of two nuclei with radii R_1 and R_2 and a relative velocity u at the moment of contact is approximately

$$\tau_r \sim 2(R_1 + R_2)[1 - (l/l_g)^2]^{1/2}/u \ , \qquad (9.1)$$

where l is the angular momentum in the center-of-mass system and l_g the grazing angular momentum. One expects that in a deep-inelastic collision event, this is also approximately the time for the reaction to be completed. On the other hand, the time for a sound wave of the ith type to propagate through

from one end of a nucleus to the other is

$$\tau_i \sim 2(R_1+R_2)/a_i, \quad i=1,\ldots,4 \ . \tag{9.2}$$

Thus,

$$\tau_r/\tau_i = a_i[1-(l/l_g)^2]^{1/2}/u \ . \tag{9.3}$$

To be more specific, we can consider the collision of ^{40}Ca on a ^{64}Ni target at a laboratory energy of 280 MeV for which experimental results are available.[26] We have at the moment of contact

$$u \sim 0.08c \tag{9.4}$$

and thus

$$\tau_r/\tau_i \simeq 3.2[1-(l/l_g)^2]^{1/2}$$

where we have taken a_i to be $0.25c$. One observes that during the collision the number of times a sound wave is propagated through the nucleus depends on the angular momentum. For head-on collisions, the various sound waves pass through the system about three times and are probably able to bring the various densities close to equilibrium. For other values of l/l_g, the number of times the sound wave can pass through the nucleus during encounter diminishes slowly and thus there will be less chance of equilibrium for the various densities. Consider, for example, the isospin sound. Initially, the ^{64}Ni nucleus has an N/Z ratio of 1.28 and the ^{40}Ca nucleus a ratio of 1.00. For the combined system in isospin equilibrium, the ratio of neutron density to proton density is $(36+20)/(28+20)=1.17$. Thus, at the moment of contact, the neutron-proton density difference in ^{40}Ca nuclei is smaller, while that of the ^{64}Ni is larger than the equilibrium neutron-proton density difference. Then subsequently there occurs a propagation of the isospin sound from ^{64}Ni to ^{40}Ca and back. Thus, in a deep-inelastic scattering with 280 MeV, ^{40}Ca beam, the isospin sound travels through the nuclei a few times. However, because of the small number of times the isospin waves can pass through, the equilibration process cannot be complete. Furthermore, the reaction time diminishes with angular momentum; there are those collisions with large values of impact parameter for which equilibration of the isospin densities is impossible. Such complications indicate that the observed resultant nucleus has the $\bar{N}/Z$ value close to the equilibrium value but the spread of N/Z should be large, as is indeed the case.[26]

The result obtained here is based on the Thomas-Fermi approximation for the kinetic energy density. For the static case, the Thomas-Fermi approximation is a good approximation.[27] In the dynamical case, one expects that it remains a good approximation when the time scale for the collective motion is long compared to the microscopic relaxation time. What we then obtain is a hydrodynamical-type equation of sound propagation presented here. How these sound waves are related to the various sound waves in the Landau-Fermi liquid theory is still not completely resolved, although it is known that in the long wavelength limit, the TDHF equation for the Wigner function corresponds to the equation of motion for the Landau-Fermi liquid theory with no collision terms.[28]

On the other hand, the comparison[3] of the generalized hydrodynamical equations obtained from TDHF and from the exact many-body Schrödinger equation indicates that Eq. (7.3) and Eq. (7.4) could have been obtained from the exact many-body Schrödinger equation where only the two-body correlation function has been approximated by the use of a single Slater determinant while the thermal stress tensor and the quantum stress tensor can take on other forms of approximation for the many-body problem. Thus, the use of the Thomas-Fermi approximation can be justified on a different context, namely, as approximations to the exact many-body problem, which need not coincide with the results for these quantities in a TDHF calculation.

Finally, we may mention that in cases where the spin and isospin degrees of freedom are not important, one can initiate many interesting investigations on the dynamics of the nuclear fluid with the simplified TDHF Eq. (6.9).

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[1]C. Y. Wong, J. A. Maruhn, and T. A. Welton, Nucl. Phys. A253, 469 (1975), hereafter referred to as I.

[2]S. E. Koonin and A. K. Kerman (unpublished); S. E. Koonin, Ph.D. thesis, Massachusetts Institute of Technology, June, 1975 (unpublished).

[3]C. Y. Wong and J. A. McDonald (unpublished).

[4]M. Goldhaber and E. Teller, Phys. Rev. 74, 1046 (1948).

[5]W. Wild, Bayr. Akad. Wiss. Mat.-Nat. Klasse 18, 371

(1956).
[6]A. E. Glassgold, W. Heckrotte, and K. W. Watson, Ann. Phys. (N.Y.) 6, 1 (1959).
[7]R. Raphael, H. Uberall, and C. Werntz, Phys. Rev. 152, 899 (1966); H. Uberall, *Electron Scattering from Complex Nuclei* (Academic, New York, 1971).
[8]A. Bohr and B. Mottelson, *Nuclear Structure* (Benjamin, New York, 1975), Vol. 2, and references cited therein.
[9]N. N. Bogoliubov, *Lectures on Quantum Statistics* (Gordon and Breach, New York, 1976), Vol. 1.
[10]M. Born and H. S. Green, *A General Kinetic Theory of Liquids* (Cambridge U.P., Cambridge, 1949), p. 51.
[11]D. L. Hill and J. A. Wheeler, Phys. Rev. 89, 1102 (1953).
[12]J. J. Griffin and K. K. Kan, in *Proceedings of the Third International Symposium on the Chemistry and Physics of Fission, Rochester, New York, 1973*, (IAEA, Vienna, 1974), SM174/58; K. K. Kan, Ph.D. thesis, University of Maryland, June, 1975 (unpublished).
[13]C. Y. Wong, J. Math. Phys. 17, 1008 (1975).
[14]D. Vautherin and D. M. Brink, Phys. Rev. C 5, 626 (1972).
[15]J. W. Negele and D. Vautherin, Phys. Rev. C 5, 1472 (1972).
[16]M. Golin and L. Zamick, Nucl. Phys. A251, 129 (1975).
[17]L. Zamick, Nucl. Phys. A260, 241 (1976).
[18]C. Y. Wong (unpublished).
[19]C. Y. Wong (unpublished).
[20]G. F. Bertsch, Ann. Phys. (N.Y.) 86, 138 (1974); G. F. Bertsch and K. Stricker, Phys. Rev. C 13, 1312 (1976).
[21]K. T. R. Davies and G. R. Satchler, Nucl. Phys. A222, 13 (1974).
[22]G. F. Bertsch and S. F. Tsai, Phys. Rep. 18C, 125 (1975).
[23]B. D. Chang, Phys. Lett. 56B, 205 (1975); S. O. Bäckman, A. D. Jackson, and J. Speth, *ibid.* 56B, 209 (1975).
[24]P. Bonche, S. E. Koonin, and J. W. Negele, Phys. Rev. C 13, 1226 (1976).
[25]H. Steinwedel and J. H. Jensen, Z. Naturforsch. 5A, 413 (1950).
[26]B. Gatty, D. Guerreau, M. Lefort, J. Pouthes, X. Tarrago, J. Galin, B. Gauvin, J. Girard, and H. Nifenecker, Z. Phys. 273, 65 (1975).
[27]C. Y. Wong, Phys. Lett. 63B, 395 (1976).
[28]J. Goldstone and K. Gottfried, Nuovo Cimento 13, 849 (1959).