Wave function approach to reaction matrix theory. II*

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The wave function approach to reaction matrix theory has been adapted to the K matrix calculation for potentials which behave like r^{-1} near the origin. The results are used to obtain the Hulthén K matrix in terms of tabulated transcendental functions.

[NUCLEAR REACTIONS Scattering theory, K matrix for singular potentials, Hulthén K matrix obtained in closed form.]

I. INTRODUCTION

In the preceding paper[1] (cited as paper I hereafter) two of us have presented a wave function method for computing the two-particle K matrix. The matrix elements for the K operator have been expressed as a single quadrature over the potential sandwiched between a plane wave and an appropriate off-shell wave function. Similar treatment can also be made for other important regular potentials like the Morse and Woods-Saxon potentials. In a forthcoming publication we shall present the results for these potentials. Our object in this paper is, however, somewhat different. We shall show, in particular, that with some modifications the wave function approach can be used to obtain the fully-off-shell K matrix for potentials which have $1/r$ singularity. This modification will amount to deriving an expression for the K matrix which does not involve the potential explicitly. The procedure will naturally avoid certain integrals which are difficult to perform because of the above noted singularity. Such an approach to potential scattering has been used earlier for T matrix calculations.[2]

In Sec. II we present the method for computing the K matrix for singular interactions. Using these results we obtain in Sec. III a closed form expression for the s-wave part of the Hulthén K matrix. We conclude by noting that the wave function approach to reaction matrix theory may in fact form a general framework for K matrix calculations.

II. K MATRIX FOR A SINGULAR POTENTIAL

Following the notation of paper I we introduce the wave operator

$$\Omega(E) = 1 + G_0^S(E)K(E). \tag{1}$$

Therefore

$$G_0^S K(E) = \Omega(E) - 1. \tag{2}$$

Taking Eqs. (1) and (2) in the mixed representation we get

$$\langle \vec{r} \,|\, \Omega(E) \,|\, qlm\rangle = (2/\pi)^{1/2} j_l(qr) y_{lm}(\hat{r}) + \int \langle \vec{r} \,|\, G_0^S(E) \,|\, \vec{r}'\rangle d\vec{r}' \langle \vec{r}' \,|\, K(E) \,|\, qlm\rangle \tag{3}$$

and

$$(k^2 - p^2)^{-1} \langle plm \,|\, K(E) \,|\, qlm\rangle = \frac{2}{\pi}\int_0^\infty r^2 dr\, j_l(pr)[\Omega_l(k,q,r) - j_l(qr)]. \tag{4}$$

In writing Eqs. (3) and (4) we have assumed the potential to be central. The object $\Omega_l(k,q,r)$ is related to the off-shell wave function $\phi_l(k,q,r)$ of paper I by

$$\Omega_l(k,q,r) = \frac{\phi_l(k,q,r)}{qr}. \tag{5}$$

The function $\Omega_l(k,q,r)$ satisfies the differential equation

$$\left[k^2 + \frac{1}{r^2}\frac{d}{dr} r^2 \frac{d}{dr} - \frac{l(l+1)}{r^2} - V(r)\right] \Omega_l(k,q,r) = (k^2 - q^2) j_l(qr). \tag{6}$$

In terms of Jost solutions, the appropriate off-shell wave function is given by

$$\phi_l(k,q,r) = -\tfrac{1}{4}\pi q \langle klm \,|\, K(E) \,|\, qlm\rangle [e^{-il\pi/2} f_l(k,r) + e^{il\pi/2} f_l(-k,r)] + (1/2i)[e^{-il\pi/2} f_l(k,q,r) - e^{il\pi/2} f_l(k,-q,r)], \tag{7}$$

where the half off-shell K matrix is

$$\langle klm|K(E)|qlm\rangle = \frac{2}{i\pi q}\left(\frac{k}{q}\right)^l \frac{f_l(k,q)-f_l(k,-q)}{f_l(k)+f_l(-k)}. \tag{7a}$$

Outside the range of interaction the solution of Eq. (6) is given by

$$\Omega_l(k,q,r) = j_l(qr) + A_l(k,q)\eta_l(kr). \tag{8}$$

In writing Eq. (8) we have included the spherical Neumann function η_l in order to incorporate the standing wave boundary condition involved in the K matrix determination.

To facilitate the calculation we now rewrite Eq. (4) in the form

$$\langle plm|K(E)|qlm\rangle = (2/\pi)(k^2-p^2)\int_0^\infty r^2 dr\, j_l(pr)A_l(k,q)\eta_l(k,r) + (2/\pi)(k^2-p^2)\int_0^\infty r^2 dr\, j_l(pr)[\Omega_l(k,q,r) - j_l(qr) - A_l(k,q)\eta_l(k,r)]. \tag{9}$$

Using the integral

$$\int_0^\infty r^2 dr\, j_l(pr)\eta_l(kr) = -k^{-1}(k^2-p^2)^{-1}(p/k)^l \tag{10}$$

in the first term on the right-hand side of Eq. (9), we get

$$\langle plm|K(E)|qlm\rangle = -(2/\pi)k^{-1}\left(\frac{p}{k}\right)^l A_l(q,k) + (2/\pi)(k^2-p^2)\int_0^\infty r^2 dr\, j_l(pr)[\Omega_l(k,q,r) - j_l(qr) - A_l(q,k)\eta_l(k,r)]. \tag{11}$$

The half-off-shell version of Eq. (11) is obtained by substituting $p=k$. We thus have

$$A_l(q,k) = -(\tfrac{1}{2}\pi)k\langle klm|K(E)|qlm\rangle. \tag{12}$$

Inserting Eq. (12) in Eq. (11) we get

$$\langle plm|K(E)|qlm\rangle = (p/k)^l\langle klm|K(E)|qlm\rangle + \left(\frac{2}{\pi}\right)\frac{(k^2-p^2)}{pq}\int_0^\infty dr\, prj_l(pr)[\phi_l(k,q,r) - qrj_l(qr) + \tfrac{1}{2}\pi q\langle klm|K(E)|qlm\rangle kr\eta_l(kr)]. \tag{13}$$

To deduce Eq. (13) we have also employed the relation (5). Equation (13) represents the basic formula for computing the off-shell K matrix for potentials singular at the origin. This equation does not involve the potential explicitly. The term inside the squared bracket is singular at the origin. In fact, as $r\to 0$ this term goes as

$$\tfrac{1}{2}\pi q\langle klm|K(E)|qlm\rangle(2l-1)!!(kr)^{-l}.$$

Since

$$prj_l(pr) \underset{r\to 0}{\sim} (pr)^{l+1},$$

the integrand on the right-hand side of Eq. (13) is regular despite this singularity.

III. S-WAVE HULTHÉN K MATRIX

The s-wave Jost solution $f(k,q,r)$ for the Hulthén potential satisfies the inhomogeneous differential equation

$$\left[k^2 + \frac{d^2}{dr^2} - \frac{V_0 e^{-r/a}}{1-e^{-r/a}}\right]f(k,q,r) = (k^2-q^2)e^{iqr}. \tag{14}$$

Equation (14) can be solved by introducing the transformation

$$Z = e^{-r/a} \tag{14'}$$

and using the standard techniques given in Babister to yield the off-shell Jost solution in the form[3]

$$f(k,q,r) = V_0 a^2 e^{ikr} f_{\sigma+1}(A,B,C,e^{-r/a}) + e^{iqr}, \tag{15}$$

where

$$A = -ika + ia(V_0+a^2)^{1/2},$$
$$B = -ika - ia(V_0+a^2)^{1/2}, \tag{16}$$
$$C = 1-2ika,$$

and

$\sigma = i(ka - qa)$.

The quantity f_σ is related to the generalized hypergeometric function[4]

$$ {}_mF_n\left(\begin{matrix}\alpha_1 & \alpha_2 & \cdots & \alpha_m \\ \beta_1 & \beta_2 & \cdots & \beta_n\end{matrix}\middle| Z\right) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k(\alpha_2)_k\cdots(\alpha_m)_k}{(\beta_1)_k(\beta_2)_k\cdots(\beta_n)_k}\frac{Z^k}{k!} \tag{17}$$

by

$$ f_\sigma(A,B,C;Z) = \frac{Z^\sigma}{\sigma(\sigma+C-1)}\, {}_3F_2\left(\begin{matrix}1 & \sigma+A, & \sigma+B \\ & \sigma+1, & \sigma+C\end{matrix}\middle| Z\right) . \tag{18}$$

The series in Eq. (18) converges when $|Z|<1$; it converges when $|Z|=1$ provided that $\mathrm{Re}(c-A-B)>0$, which from Eq. (16) is true in our case.

Using Eq. (15) and its on-shell version, i.e., when $q\to k$ in the s-wave part of Eq. (7), we get

$$\begin{aligned}\phi(k,q,r) - \sin qr + \tfrac{1}{2}\pi q\langle k|K(E)|q\rangle\cos kr = \frac{1}{2i}\{ & V_0a^2e^{ikr}[f_{\sigma+1}(A,B,C,e^{-r/a}) - f_{\sigma'+1}(A,B,C,e^{-r/a})] \\ & - \tfrac{1}{4}\pi q V_0a^2\langle k|K|q\rangle[f_1(A,B,C,e^{-r/a})e^{ikr} + f_1(A,B,C,e^{-\alpha/a}e^{-ikr}]\} ,\end{aligned} \tag{19}$$

where

$\sigma' = i(ka+qa)$.

In Eq. (19) we have omitted the subscripts l and m. Specializing Eq. (13) for the s waves and using Eq. (19), we obtain the off-shell Hulthén K matrix in the form

$$\begin{aligned}\langle p|K(E)|q\rangle = \langle k|K(E)|q\rangle & \\ + \frac{(k^2-p^2)V_0a^2}{2\pi pq}\bigg\{ & \int_0^\infty dr\, e^{-\rho r/a}[f_{\sigma+1}(A,B,C,e^{-r/a}) - f_{\sigma'+1}(A,B,C,e^{-r/a})] \\ & - \int_0^\infty dr\, e^{-\rho' r/a}[f_{\sigma+1}(A,B,C,e^{-r/a}) - f_{\sigma'+1}(A,B,C;e^{-r/a})] \\ & - \frac{\pi q}{2i}\langle k|K(E)|q\rangle\bigg[\int_0^\infty dr\, e^{-\rho' r/a}f_1(A,B,C;e^{-r/a}) - \int_0^\infty dr\, e^{-\rho'' r/a}f_1(A,B,C;e^{-r/a}) \\ & + \int_0^\infty dr\, e^{-\rho''' r/a}f_1(A,B,C;e^{-r/a}) - \int_0^\infty dr\, e^{-\rho r/a}f_1(A,B,C;e^{-r/a})\bigg]\bigg\} ,\end{aligned} \tag{20}$$

where

$$\left.\begin{aligned}\rho &= i(p-k)a , \\ \rho' &= -i(p+k)a , \\ \rho'' &= -\rho' , \\ \rho''' &= -\rho .\end{aligned}\right\} \tag{21}$$

If we now make the change in variable given by Eq. (14′) in the integrals arising in Eq. (20), all the integrations can be performed by using the result

$$\int_0^1 dZ\, Z^{\alpha-1}f_\beta(a,b,c;z) = [\beta(\beta+c-1)(\alpha+\beta)]^{-1}\, {}_4F_3\left(\begin{matrix}1, & \beta+a, & \beta+b, & \alpha+\beta \\ & \beta+1, & \beta+c, & \alpha+\beta+1\end{matrix}\middle| 1\right) . \tag{22}$$

We thus obtain

$$\langle P|K(E)|q\rangle = \frac{(p^2-k^2)a}{2\pi pq}\left[\frac{f(k,q)-f(k,-q)}{f(k)+f(-k)}\{x(p,k)-x(-p,K)+x(p,-k)-x(-p,-k)\}\right.$$
$$\left. - V_0a^2\{Y(p,q,-k)-y(p,-q,-k)-y(-p,q-k)+y(-p,-q,-k)\}\right], \quad (23)$$

where we have used the value of the half-off-shell K matrix

$$\langle k|K(E)|q\rangle = \frac{2}{i\pi q}\frac{f(k,q)-f(k,-q)}{f(k)+f(-k)}\,. \quad (24)$$

The functions $x(p,k)$ and $y(p,q,k)$ are given by

$$x(p,k) = \rho^{-1}{}_3F_2(A,B,\rho;c,\rho+1;1) \quad (25a)$$

and

$$y(p,q,k) = (\sigma+1)^{-1}(\sigma+c)^{-1}(\rho+\sigma+1)^{-1}{}_4F_3\left(\begin{matrix}1, & \sigma+A+1, & \sigma+B+1, & \rho+\sigma+1\\ & \sigma+2, & \sigma+c+1, & \rho+c+2\end{matrix}\middle|\,1\right). \quad (25b)$$

We have performed several checks on this fairly complicated result for the off-shell K matrix. We have seen that Eq. (23) yields the correct on-shell limit. We have tied Eq. (23) with the real part of the T matrix given by Bahethi and Funda[2] to get the relation given by Kauri and Levin.[5]

IV. CONCLUSION

The wave function approach to K matrix theory presented in papers I and here appears to represent an effective way to calculate the off-shell K matrices for a number of realistic N-N potentials. The results for the Morse potential, which we propose to communicate in a forthcoming publication, will be of particular importance for studying N-N scattering reactions. The Morse function represents a static soft core potential and can be used to account for the behavior of $^1s_0$ phase shifts at high energies. The calculation of the K matrix for the Morse function will be facilitated by using the results of our work on the Morse T matrix.[6]

Recently we have derived the off-shell T matrix for the Woods-Saxon potential in terms of elementary transcendental functions. The Woods-Saxon potential is one of the most important phenomenological nucleon-nucleus potentials. The off-shell K matrix for this potential can be obtained in close analogy with our work on the T matrix.[7]

The next logical step of our work will be to examine how one could introduce the effect of tensor forces in the wave function approach. More ambitiously, one might then like to use the results of such calculations in nuclear reaction studies.

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[1]B. Talukdar, N. Mallick, and S. Mukhopdhyaya, Phys. Rev. C 15, 1252 (1977), preceding paper, hereafter referred to as I.
[2]H. S. Picker, E. F. Redish, and G. J. Stephenson, Jr., Phys. Rev. C 4, 287 (1971); O. P. Bahethi and M. G. Fuda, J. Math. Phys. 12, 2076 (1971).
[3]M. G. Fuda and J. S. Whiting, Phys. Rev. C 8, 1255 (1973).
[4]A. W. Babister, *Transcendental Functions Satisfying Nonhomogeneous Linear Differential Equations* (MacMillan, New York, 1967).
[5]D. J. Kouri and F. S. Levin, Phys. Lett. 48B, 203 (1974).
[6]B. Talukdar, M. N. Sinha Roy, N. Mallick, and D. K. Nayek, Phys. Rev. C 12, 370 (1975).
[7]M. N. Sinha Roy, B. Talukdar, and D. Chattarji, J. Math. Phys. 17, 1763 (1976).