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Effect of vacuum polarization on the solar p-p reaction rate*

G. E. Bohannon† and L. Heller

Theoretical Division, Los Alamos Scientific Laboratory, University of California, Los Alamos, New Mexico 87545

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We have put very tight upper and lower bounds on the effect of vacuum polarization on the proton-proton scattering wave function, in the region where the nuclear potential is known, but have not succeeded in doing the same at very small distances. Simply making a smooth extrapolation of the bounds leads to the expectation that vacuum polarization reduces the proton-proton reaction rate by between 0.8% and 1.2% assuming the ${}^1S_0$ potential is known beyond 2 fm. An independent argument following the work of Picker and Haftel leads to the conclusion that $1.7 < \Lambda^2 < 7.2$ with vacuum polarization included, assuming the ${}^1S_0$ and ${}^3S_1$-${}^3D_1$ potentials are known beyond 3 fm.

[NUCLEAR REACTIONS Effect of vacuum polarization on scattering length, wave function. Bounds on $pp \to de^+\nu$ reaction rate.]

I. INTRODUCTION

Some time ago one of us made a crude estimate[1] of the effect of vacuum polarization (v.p.) on the rate of the solar reaction $p+p \to d+e^+ + \nu$. Shortly thereafter—in connection with a study of the effect of v.p. on elastic p-p scattering[2]—some of the numerical work was performed which is needed for an improved estimate, but the calculation was not completed. Since interest has been raised in this reaction in connection with the solar neutrino puzzle, we decided to restudy the matter.

It was first pointed out by Foldy and Eriksen[3,4] that the modification of the electrostatic interaction between two protons due to v.p. might be experimentally observable, and Durand[5] showed that this is mainly due to the fact that the long range of the v.p. potential produces scattering in many angular momentum states, even at low energies. The electrostatic interaction between two protons is taken to be the sum of the Coulomb potential and the Uehling potential[6]

$$V_E(r) = V_C(r) + V_{\text{v.p.}}(r) = (e^2/r)[1+\lambda I(r)], \tag{1}$$

where $\lambda = 2\alpha/3\pi = 1.549 \times 10^{-3}$ and the function $I(r)$ can be written

$$I(r) = \int_1^\infty dx\, e^{-2\kappa r x}\left(\frac{1}{x^2}+\frac{1}{2x^4}\right)(x^2-1)^{1/2} \tag{2}$$

with $(2\kappa)^{-1} = \hbar/2mc = 193.1$ fm and m is the electron mass. $I(r)$ is plotted in Ref. 2 and Fig. 1, and convenient numerical approximations are given in Ref. 7. The above expressions are correct for point particles. The modification to $I(r)$ for the finite extent of the proton charge distribution will be discussed in Sec. III.

The matrix element which is needed for the p-p reaction is the overlap of the p-p scattering wave function u in the energy region around $E_{\text{lab}} = 12$ keV with the deuteron wave function u^D. At these energies, only the s-wave parts of the wave functions are needed:

$$M = \int_0^\infty dr\, u^D(r) u(r). \tag{3}$$

Since the v.p. potential is weak, one expects that its effect upon M is small. The fact that the nuclear phase shift is even smaller than the v.p. phase shift at solar energies does not alter this expectation for the following reason. Even though the nuclear potential produces a very small effect on the wave function at large distances, it has a large effect at small distances. The modification of the wave function due to the v.p. potential is *everywhere small*, even though it is bigger than the nuclear effect at large distances. These ideas will be confirmed quantitatively below.

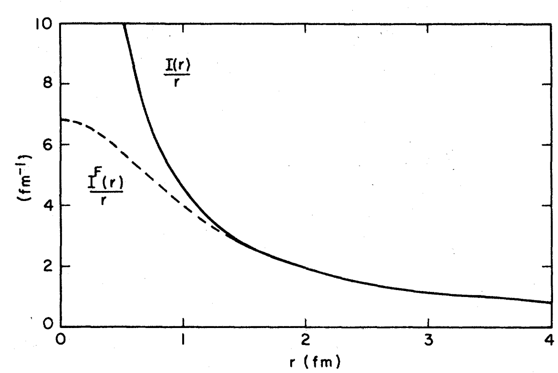

FIG. 1. The vacuum polarization integral divided by r. The solid curve is the point interaction form; the dashed curve includes the exponential charge distribution of the protons with an r.m.s. charge radius of 0.8 fm.

In attempting to obtain the p-p wave function, it is important to remember that most of the contribution to M comes from beyond the range of the nuclear force.[8,9] In this region, and at solar energies, the wave function is completely determined by just one piece of nuclear information: the scattering length. But some care is required in the definition of this quantity when v.p. is present. If one solves the problem of a nuclear plus Coulomb plus v.p. potentials, $V_N + V_E$, for the wave function $u^E(r)$, it has the asymptotic behavior

$$u^E(r) \underset{\mu r \gg 1}{\sim} \cos\delta^E S(r) + \sin\delta^E T(r), \tag{4}$$

where S and T are the regular and irregular solutions[2] in the potential V_E, and μ is the π-meson mass. We have used the symbol $\underset{x \gg 1}{\sim}$ to mean *equality* in the limit $x \to \infty$. The effective range function appropriate to δ^E is very well given at solar energies by its zero energy limit $-1/a^E$. According to recent analyses[10-12] of the elastic scattering data in the few MeV range with the inclusion of v.p., $a^E = -7.815 \pm 0.005$ fm.

If one determines a particular nuclear potential V_N such that the phase shifts δ^E agree with the data, then it is a straightforward problem to arbitrarily turn off $V_{\text{v.p.}}$ and solve the $V_N + V_C$ problem for the wave function $u^C(r)$. Using this procedure, Gari and Huffman[13] estimated the effect of vacuum polarization on the p-p reaction rate to be at most 1%. For orientation purposes the wave functions u^C and u^D computed with the Reid soft core potentials[14] are shown in Fig. 2, where it is seen that the product falls off very slowly with increasing distance.

In Sec. II we have made a more general examination of $u^E - u^C$ to find out how much it can vary with the choice of the nuclear potential, and the following points are demonstrated. (i) One already gets a rough value for the ratio u^E/u^C just from the penetrability of the v.p. potential; this is the quantity $1+\chi \equiv (S/F)_{r=0}$, where F is the regular Coulomb function. At solar energies $\chi \simeq -4.3 \times 10^{-3}$. (ii) For separations r beyond the range of the nuclear potential V_N, $u^E - u^C$ is largely independent of the behavior of V_N at short distances. The small dependence which remains is contained in just one number, $a^E - a^C$, the difference of the scattering lengths. a^C is defined by the conventional effective range expansion involving the phase shift δ^C, where

$$u^C(r) \underset{\mu r \gg 1}{\sim} \cos\delta^C F(r) + \sin\delta^C G(r) \tag{5}$$

and G is the irregular Coulomb function.

In Sec. III we have shown that $a^E - a^C$ is itself rather well constrained by the experimental values of the scattering length and effective range, together with the fact that V_N is well determined in to a separation of about 2 fm. Combining this with the work in Sec. II leads to a very small uncertainty in $u^E - u^C$ beyond the range of V_N, and in fact all the way down to 2 fm.

We have not succeeded in bounding $u^E - u^C$ in the region $r < 2$ fm where the nuclear potential is not known. In Sec. IV, however, we have put upper and lower bounds directly on the matrix element M, Eq. (3), by simply repeating, with vacuum polarization, what Picker and Haftel[20] did without it.

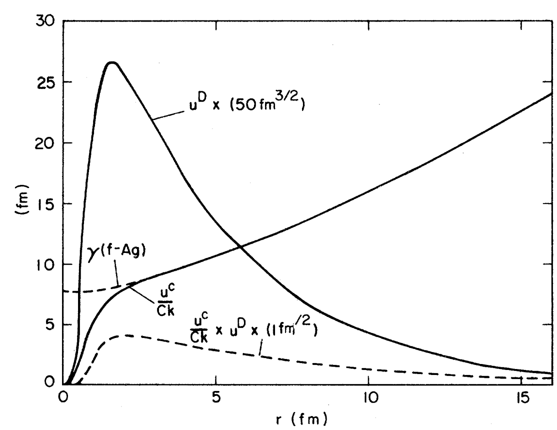

FIG. 2. The ${}^1S_0$ p-p wave function u^C/Ck and the deuteron wave function u^D, both generated with the Reid soft core potential. Also shown is the asymptotic form $\gamma(f - a^C g)$ and the product $(u^C/Ck)u^D$. The difference between u^C/Ck at zero energy and at 12 keV is less than 1% over the range of distances shown.

II. EFFECT OF VACUUM POLARIZATION ON THE p-p WAVE FUNCTION

Since u^E and u^C differ by the effect of the weak potential $V_{\text{v.p.}}(r)$, Eq. (1), we can write the first order perturbation expression

$$u^E(r) - u^C(r) = -\frac{1}{k}\left[u(r)\int_r^\infty dr' U(r') v(r') u(r') + v(r)\int_0^r dr' U(r') u^2(r')\right], \tag{6}$$

where $U(r) \equiv M V_{\text{v.p.}}(r)$ and M is the proton mass. On the right-hand side of Eq. (6) and in the following equations we have not distinguished between type C and E functions where the difference contributes in second order (λ^2). $v^C(r)$ is an irregular solution to the $V_N + V_C$ problem with the asymptotic behavior

$$v^C(r) \underset{\mu r \gg 1}{\sim} \cos\delta^C G(r) - \sin\delta^C F(r). \tag{7}$$

Using the asymptotic behavior given in Eqs. (5) and (7), one obtains

$$u^E(r) \underset{2\kappa r \gg 1}{\sim} \cos(\delta^C + \Delta) F(r) + \sin(\delta^C + \Delta) G(r), \tag{8}$$

where

$$\Delta \equiv -\frac{1}{k}\int_0^\infty dr\, U(r) u^2(r)\,. \tag{9}$$

It follows[2] from the definition of the v.p. phase shift τ that

$$\delta^C + \Delta = \delta^E + \tau\,, \tag{10}$$

where τ is given by

$$\tau = -\frac{1}{k}\int_0^\infty dr\, U(r) F^2(r)\,. \tag{11}$$

From its analytic behavior at $k=0$, one has[15]

$$\tan\delta^C = -C^2 k\gamma A^C\,, \tag{12}$$

where $\gamma = (1 + hA^C/R)^{-1}$, $C^2 = 2\pi\eta[\exp(2\pi\eta) - 1]^{-1}$, $\eta = (2kR)^{-1}$, $h = \mathrm{Re}[\Gamma'(i\eta)/\Gamma(i\eta)] - \ln\eta$, $R = \hbar^2/Me^2 = 28.82$ fm, and the effective range function is conventionally expanded as

$$-\frac{1}{A^C} = -\frac{1}{a^C} + \tfrac{1}{2} r^C k^2 + \cdots\,. \tag{13}$$

The corresponding expression for δ^E is[2]

$$\tan\delta^E = -\frac{C^2 k (1+2\chi) A^E}{1 + (h+l)A^E/R}\,, \tag{14}$$

where

$$\chi = -\frac{1}{k}\int_0^\infty dr\, U(r) F(r) G(r)\,, \tag{15}$$

$$l = -R\int_0^\infty dr\, U(r)\Big([CG(r)]^2 - [CG(r)]^2_{k=0}\Big)\,, \tag{16}$$

and A^E is given by Eq. (13) with the superscripts C replaced by E.

It is useful to record the approximate numerical values of these quantities at 12 keV. $k^2 = 1.45 \times 10^{-4}$ fm^{-2}, $\eta = 1.44$, $h = 0.0420$, and $C^2 = 1.05 \times 10^{-3}$. The v.p. quantities below 20 keV are shown in Fig. 3. At 12 keV they have the values $\tau = -3.1 \times 10^{-4}$, $\chi = -4.3 \times 10^{-3}$, and $l = -1.6 \times 10^{-4}$. With the approximate values of the effective range parameters $a^C = -7.8$ fm and $r^C = 2.8$ fm, the phase shift has the value

$$\delta^C(12\ \text{keV}) = 1.0 \times 10^{-4}\,.$$

In the sequel, therefore, we have for convenience neglected the difference between δ and $\tan\delta$.

It is also useful to isolate the main energy dependence of the Coulomb functions[15] with the following definitions:

$$f(r) = F(r)/Ck\,,$$
$$g(r) = CG(r) - (h/R) f(r)\,; \tag{17}$$

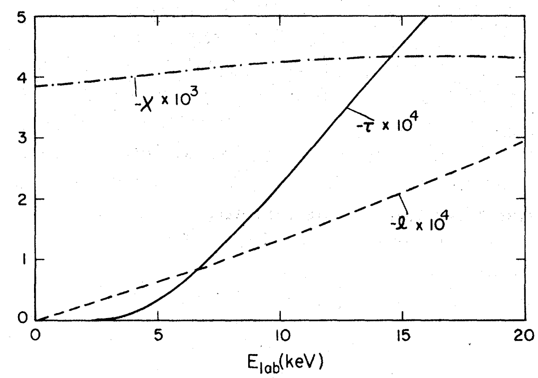

FIG. 3. The vacuum polarization quantities τ, χ, and l versus the laboratory kinetic energy.

f and g vary by less than 3% from zero energy up to solar energies for $r < 20$ fm. Also define

$$\begin{aligned}
M(r) &= \int_0^r dr' U(r') f^2(r')\,,\\
N(r) &= \int_0^r dr' U(r') f(r') g(r')\,,\\
P(r) &= \int_r^\infty dr' U(r') g^2(r')\Big|_{k=0}\,,\\
Q(r) &= \int_0^r dr' U(r') [g^2(r') - g^2(r')|_{k=0}]\,,
\end{aligned} \tag{18}$$

and

$$J(r) = \int_0^r dr' U(r') \left[\frac{u(r')}{Ck}\right]^2,$$

with the following connections to previously introduced quantities

$$\begin{aligned}
-\tau &= C^2 k M(\infty)\,,\\
-\chi &= N(\infty) - \frac{h}{R}\frac{\tau}{C^2 k}\,,\\
-\frac{l}{R} &= Q(\infty) - \frac{2h}{R}\chi + \left(\frac{h}{R}\right)^2 \frac{\tau}{C^2 k}\,.
\end{aligned} \tag{19}$$

Since $V_{\text{v.p.}}$ has a range which is much greater than that of V_N it is sensible to take Eq. (9) for Δ and break up the region of integration into two intervals[4] $(0, r)$ and (r, ∞) with $\mu r \gg 1$. In the outer region one has

$$\frac{u^C}{Ck} \underset{\mu r \gg 1}{\sim} \gamma (f - A^C g) \tag{20}$$

as a consequence of Eqs. (5), (12), and (17). Therefore

$$-\frac{\Delta}{C^2 k} \underset{\mu r \gg 1}{\sim} \int_0^r dr' U(r') \left[\frac{u(r')}{Ck}\right]^2 + \gamma^2 \int_r^\infty dr' U(r') [f^2(r') - 2Af(r')g(r') + A^2 g^2(r')]\,. \tag{21}$$

Adding τ/C^2k to both sides, and using Eqs. (10), (18), and (19) gives

$$\frac{\delta^C-\delta^E}{C^2k} \underset{\mu r\gg 1}{\sim} J(r)+\gamma^2\{-M(r)+2AN(r)+A^2[P(r)-Q(r)-l/R]\}+2\gamma A\chi . \tag{22}$$

Using Eqs. (12) and (14) for δ^C and δ^E, Eq. (22) becomes

$$A^E-A^C \underset{\mu r\gg 1}{\sim} \gamma^2 J(r)-M(r)+2AN(r)+A^2[P(r)-Q(r)] . \tag{23}$$

M and N have practically the same values (to better than 3%) at zero energy and 12 keV for $r<20$ fm. The term involving Q in Eq. (23) is negligible. In this same region $\gamma(f-Ag)$ varies by less than 1% from zero to 12 keV, and since $V_N \gg E$ the same is true of u/Ck inside the range of the potential. Therefore J varies less than 2% and the zero energy limit of Eq. (23) is useful to record:

$$a^E-a^C \underset{\mu r\gg 1}{\sim} [J(r)-M(r)+2aN(r)+a^2P(r)]_{k=0} . \tag{24}$$

It is worth pointing out that Eq. (24) is quite different from the change in the scattering length due to a weak *short range* potential (which would just be the zero energy limit of $-\Delta/C^2k$) because a^E and a^C are obtained from differently defined phase shifts.

In Sec. III we have estimated how much a^E-a^C depends on the choice of V_N. To proceed further here we have made use of the fact that beyond the range of V_N, all the nuclear information is incorporated in the functions A^E and A^C which at 12 keV differ from the numbers a^E and a^C by only 0.2%. Using Eqs. (7), (12), and (17),

$$Cv^C(r) \underset{\mu r\gg 1}{\sim} g(r)+\frac{h}{R}f(r) , \tag{25}$$

where the term $C^4k^2\gamma A^C f$ has been neglected since it is of the same size as δ^2. Putting Eqs. (20) and (25) into Eq. (6) gives

$$\frac{u^E(r)-u^C(r)}{Ck} \underset{\mu r\gg 1}{\sim} -\left[g(r)+\frac{h}{R}f(r)\right]\int_0^r dr'U(r')\left[\frac{u(r')}{Ck}\right]^2 -\gamma^2[f(r)-Ag(r)]\int_r^\infty dr'U(r')\left[g(r')+\frac{h}{R}f(r')\right][f(r')-Ag(r')] , \tag{26}$$

and making use of Eqs. (18) and (19), this becomes

$$\frac{u^E(r)-u^C(r)}{u^C(r)} \underset{\mu r\gg 1}{\sim} \chi-\gamma\frac{A}{R}l+\frac{1}{f(r)-Ag(r)}\Big\{Af(r)P(r)+[f(r)+Ag(r)]N(r)-g(r)M(r)-Af(r)Q(r) -\gamma\left[g(r)+\frac{h}{R}f(r)\right](A^E-A^C)\Big\} . \tag{27}$$

The main difference between the values of the right-hand side of Eq. (27) at zero energy and at 12 keV comes from χ, which varies from -3.85×10^{-3} to -4.30×10^{-3}. The quantity $(u^E-u^C)/u^C$ at 12 keV obtained from Eq. (27) is plotted in Fig. 4. Also shown in Fig. 4 is the same quantity computed with the Reid soft core potential.[14] The last term in Eq. (27) is the only one which depends on the choice of the nuclear potential, and it is seen in Fig. 4 that this term accounts for only a small part of the answer. In Fig. 4 we have used the Reid soft core value of a^E-a^C, 0.01 fm, to evaluate this term. Constraints on a^E-a^C are discussed in the following section.

III. BOUNDS ON $a^E - a^C$

All dependence of a^E-a^C on the short distance behavior of the wave function is contained in $J(r)$. The values of the other functions which appear in Eq. (24) are given in Table I, evaluated beyond the range of the strong interaction, at $r=4$ fm. We have determined bounds on J, and thus on a^E-a^C, allowing for the most extreme variations in the short range wave function consistent with certain experimental constraints.

It is possible to enhance the wave function at small distance where $U(r)$ diverges and obtain a large value for $a^E - a^C$. This divergent behavior of $U(r)$ is unphysical and arises from neglecting the finite extent of the proton charge distribution. A static exponential charge distribution for each proton gives the following modified form of $V_{\text{v.p.}}$;

$$V^F_{\text{v.p.}}(r) = \frac{\lambda e^2}{r} I^F(r),$$

$$I^F(r) = \int_1^\infty dx \{ D^4(x) e^{-2\kappa r x} + D(x) e^{-\rho} [3\rho + 3\rho^2 + \rho^3 - 6D(x)(\rho + \rho^2) + 24 D^2(x)\rho - 48 D^3(x)]/48 \} \left(\frac{1}{x^2} + \frac{1}{2x^4} \right) (x^2 - 1)^{1/2},$$

$$\rho = (12)^{1/2} r / r_p, \quad D(x) = [(2\kappa r_p x)^2/12 - 1]^{-1}, \tag{28}$$

where r_p is the proton r.m.s. charge radius for which we used 0.8 fm. Although it is not obvious, the integrand in Eq. (28) is finite at $\frac{1}{12}(2\kappa r_p x)^2 = 1$. The derivation of this expression follows a similar treatment of the Coulomb potential.[16] $I^F(r)/r$ and the point form $I(r)/r$ are shown for small r in Fig. 1.

It is apparent from Eq. (26) and the fact that $2r_p \approx \mu^{-1}$ that the only significant effect of the charge distribution on $u^E - u^C$ outside the range of the nuclear potential is via the function $J(r)$. The same remark applies to Δ in Eq. (21). It is convenient, therefore, to continue to define the functions S and T; the quantities τ, χ, and l; and the functions M, N, P, and Q with the point form of the v.p. potential. Only the definitions of $J(r)$ in Eq. (18) and Δ in Eq. (9) will be modified for the finite size of the protons.

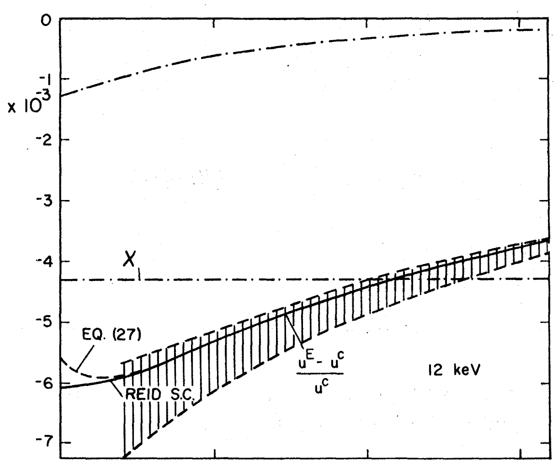

FIG. 4. $(u^E - u^C)/u^C$ computed with the Reid soft core potential (solid line) and from Eq. (27) (dashed line), which is valid outside the nuclear potential. Both curves were computed with the point v.p. potential. The shaded region indicates the maximum possible range of values due to changes in the nuclear potential for $r < 2$ fm. χ(12 keV) is shown by a dash-dot line, as is the small term in Eq. (27) proportional to $(A^E - A^C)$.

The value of $a^E - a^C$ is constrained by the following considerations. The $^1S_0$ potential appears to be unambiguously determined in to a separation of about 2 fm.[17] Furthermore, for the function $u^E(r)$ with asymptotic form $w^E(r) \equiv \cos\delta^E S(r) + \sin\delta^E T(r)$ the integral of $(u^E)^2$ is fixed by on-energy-shell quantities from the following relation[18]:

$$2\int_0^\infty dr \left\{ \left[\frac{u^E(r)}{Ck} \right]^2 - \left[\frac{w^E(r)}{Ck} \right]^2 \right\} = \frac{(1-\phi)^2 \sin^2\delta^E}{C^4 k^3} \frac{d}{dk}\left(\frac{1}{A^E} \right), \tag{29}$$

where to first order in λ one has $\phi = \chi$.

Therefore, we shall only consider wave functions for which the integral

$$\mathcal{I}(b) = \int_0^b dr \left[\frac{u^E(r)}{Ck} \right]^2 \tag{30}$$

has a specified value for each b. (Since $a^E - a^C$ is first order in λ, it is not necessary to include v.p. when computing $\mathcal{I}$. We have nevertheless included it so that these values of $\mathcal{I}$ may be used in Sec. IV.) The integral $\mathcal{I}$ was evaluated by first writing it as

$$\mathcal{I}(b) = \frac{(1-2\chi)\sin^2\delta^E}{2C^4 k^3} \frac{d}{dk}\left(\frac{1}{A^E} \right) + \int_0^b dr \left[\frac{w^E(r)}{Ck} \right]^2 - \int_b^\infty dr \left\{ \left[\frac{u^E(r)}{Ck} \right]^2 - \left[\frac{w^E(r)}{Ck} \right]^2 \right\}. \tag{31}$$

By using the Noyes values of $a^E = -7.814$ fm, $r^E = 2.7950$ fm to compute the first and second terms, and using the Reid soft core wave function to com-

TABLE I. Terms of Eq. (24) at $r = 4$ fm which do not depend on the short range wave function.

$-M$	-0.0017 fm
$2a^C N$	-0.0122 fm
$(a^C)^2 P$	0.0046 fm

pute the third term, we found the values for $\mathscr{G}$ given in Table II. The third term of Eq. (31) is 20%, 4%, and 1% of $\mathscr{G}$ at $b=2$, 3, and 4 fm, respectively. The functions S and T which are required to define w^E were obtained by solving the Schrödinger equation with the point interaction v.p. potential.

Bounds on $J(b)$ assuming the wave function to be unknown for $r<b$ were determined as follows. Since $V^F_{\text{v.p.}}(r)$ decreases monotonically, the maximum value of $J(b)$ was found by making $(u^E)^2$ for $r<b$ a δ function at $r=0$ with a coefficient adjusted to give the correct value of $\mathscr{G}(b)$. Similarly, the minimum value of $J(b)$ was found by making $(u^E)^2$ for $r<b$ a δ function at $r=b$. These extreme values of J are also given in Table II. (These values were computed at 12 keV but, as stated in Sec. II, the values of J at 12 keV differ very little from the ones at zero energy.) Adding $J(4\text{ fm})$ to the numerical values in Table I gives the bounds on a^E-a^C shown in the last column of Table II. These bounds are rigorous in that no restriction on the wave function at a distance less than 4 fm was used. Somewhat tighter but less rigorous bounds may be obtained by assuming the wave function to be known at distances less than 4 fm. By using the bounds on $J(b)$ for $b=2$ fm or 3 fm and using the Reid soft core wave function at larger distances one can find tighter limits on $J(4\text{ fm})$. These bounds on a^E-a^C which follow from these values of J are given in Table II. The limits on $(u^E-u^C)/u^C$ assuming the wave function to be known for $r>2$ fm are shown in Fig. 4. We have extended the bounds on $(u^E-u^C)/u^C$ in to 2 fm using Eq. (6) and the bounds on $J(2\text{ fm})$.

Noyes and Lipinski[12] have analyzed low energy ${}^1S_0$ p-p phase shifts using values of $\delta^E-\delta^C$ given by several potentials. They conjectured that the uncertainty in a^E-a^C due to the unknown short range wave function is ±0.0024 fm and chose as the central value that given by their repulsive core potentials. It is worth noting, however, that when the charge distribution is included the value of a^E-a^C given by the Reid potential is 0.009 fm. Because the repulsive core strongly suppresses the wave function near $r=0$, this value of a^E-a^C is not midway between the limits in Table II, but is closer to the lower limit.

TABLE II. Values of $\mathscr{G}(b)$ from Eq. (31) and bounds on $J(b)$ assuming the wave function to be unknown for $r<b$. Also shown are the corresponding bounds on a^E-a^C. $\mathscr{G}$ and J were evaluated at 12 keV.

b		2	3	4 fm
$\mathscr{G}(b)$		53.4	125.5	213.9 fm^3
$J(b)$	minimum	0.0055	0.0078	0.0092 fm
$J(b)$	maximum	0.0196	0.0462	0.0787 fm
a^E-a^C	minimum	0.006	0.003	0.000 fm
a^E-a^C	maximum	0.021	0.042	0.070 fm

IV. BOUNDS ON THE p-p REACTION RATE

Although we have been able to bound the effect of vacuum polarization on $u(r)$ only for $r>2$ fm, the overlap integral Eq. (3) can nevertheless be bounded. The p-p reaction rate is conventionally expressed in terms of

$$|\Lambda|^2=\tfrac{1}{2}\gamma^2\left|\int_0^\infty dr\frac{u^E(r)}{Ck}u^D(r)\right|^2,$$

where $\gamma^{-1}=4.317$ fm is the range of the deuteron wave function. Limits on the uncertainty in $|\Lambda|^2$, ignoring vacuum polarization, were considered by Picker and Haftel.[20] In this section we have followed their procedure, making the appropriate changes to include vacuum polarization.

For r larger than the range of the nuclear force $u^E(r)$ and $u^D(r)$ are specified by the phase shift δ^E, the deuteron D-state fraction P_D, the deuteron binding energy, and the deuteron asymptotic normalization. The asymptotic normalization is fixed by P_D and an analytic continuation of the on-shell ${}^3S_1-{}^3D_1$ channel scattering amplitude to the deuteron pole. Examination of various potential models suggests that there is only a small uncertainty in the asymptotic normalization.[21] Just as in the previous section, this "external" region can be extended down to smaller separation distances, for which we have chosen $r\geq 2$ fm, where the ${}^3S_1-{}^3D_1$ potential is known approximately. In this external region we have used wave functions generated from the Reid soft core potential.[14]

The contribution from the internal region $r<b$ is bounded by the Schwartz inequality:

$$\left|\int_0^b dr\frac{u^E(r)}{Ck}u^D(r)\right| \leq\left\{\int_0^b dr\left[\frac{u^E(r)}{Ck}\right]^2\int_0^b dr[u^D(r)]^2\right\}^{1/2}. \quad (32)$$

The integral of $(u^D)^2$ is given by the normalization condition

$$\int_0^b [u^D(r)]^2=1-P_D-\int_b^\infty [u^D(r)]^2 \quad (33)$$

in terms of P_D and the external wave function. To obtain a conservative upper bound we have used $P_D=0$ on the right-hand side of Eq. (33) (in this respect we differ from Ref. 20). The integral of $(u^E)^2$ is the quantity $\mathscr{G}$ evaluated in the previous section. The values of these integrals required to bound $|\Lambda|^2$ are given in Table III. Since the bound on the internal contribution is smaller in magnitude than the external contribution one has both an upper and lower bound on $|\Lambda|^2$. The upper bound is not very sensitive to the radius outside of which one assumes the wave functions u^E and u^D to be known. The lower limit is much more sensitive to this radius.

TABLE III. Ingredients and results for the bounds on $|\Lambda|^2$ at 12 keV laboratory energy.

b	2	3	4 fm
$\int_b^\infty dr \frac{u^E(r)}{Ck} u^D(r)$	29.23	25.24	21.66 $\text{fm}^{3/2}$
$\int_0^b dr \left[\frac{u^E(r)}{Ck}\right]^2$	53.4	125.5	213.9 fm^3
$\int_0^b dr [u^D(r)]^2$	0.370	0.595	0.743 fm
Λ^2 lower bound	3.82	1.71	0.51
Λ^2 upper bound	7.05	7.14	7.30

V. SUMMARY

Vacuum polarization reduces the size of the proton-proton scattering wave function in the region where it overlaps the deuteron wave function, and the amount of this reduction at any distance depends upon the details of the nuclear potential at *all* distances—including $r<2$ fm where it is not well known. But the effect of this short distance behavior can be rigorously bounded for separations $r>2$ fm, and these bounds are shown in Fig. 4 for a laboratory energy of 12 keV. Right at $r=2$ fm, for example, v.p. reduces the wave function by a fractional amount which must fall between 0.0056 and 0.0073, and the amount of this reduction decreases smoothly with increasing distance. These bounds are probably not optimal, in the sense that there may not exist nuclear potentials which yield the extreme values.

Since most of the contribution to the proton-proton reaction matrix element comes from separations greater than 2 fm, comparison of Figs. 2 and 4 makes it very likely that the effect of v.p. on this matrix element is to reduce it by $(0.5\pm0.1)\%$. (Double these numbers for the reaction rate.) But we have not been able to make this argument ironclad by finding similar bounds right inside the unknown nuclear potential.

By applying the Schwartz inequality directly to the matrix element[20] with vacuum polarization included, we find that $1.7<\Lambda^2$ $(E=12 \text{ keV})<7.2$, assuming the nuclear potential is known for $r>3$ fm.

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†Present address: Center for Theoretical Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139.

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