Self-consistent quasiparticle-phonon coupling for superfluid nuclei*

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The utility of the anomalous Green's function formalism is physically motivated as a description of superfluid finite Fermi systems. The quasiparticle self-energy is then derived from a generalized six-point response function with the assumption that three-body correlations are negligible. An approximate calculation of the self-energy includes the usual core-polarization diagram, a backward-exchange diagram with ground state correlations, and higher-order diagrams that correct for exclusion-principle violations in the propagation of the quasiparticle and the microscopic phonon. Intermediate lines in the self-energy are described self-consistently, thereby including multiple-phonon core excitations. It is shown that the equation for the self-energy may be solved by matrix diagonalization with coupling of the even- and odd-nuclear eigenmodes. Renormalized phonons are calculated by taking into account the distribution of quasiparticle strength in the solution of the Bethe-Salpeter equation. The self-energy and phonon are calculated self-consistently.

[NUCLEAR STRUCTURE Green's function method with pairing. Self-consistent RPA, collective effects in odd-A nuclei.]

I. INTRODUCTION

In a previous article,[1] we presented a self-consistent treatment of quasiparticle-phonon coupling (SCQPC) for shape vibrations in nonsuperfluid nuclei. It was shown that writing the mass operator in terms of the six-point function permits a natural extension of Fermi liquid theory to phonon states in odd-mass nuclei. A further result of applying Green's function techniques was an unambiguous prescription for treating higher-order corrections to the usual quasiparticle-phonon coupling (QPC) approach. In particular, the diagrams in Fig. 1 (see caption for explanation) can be calculated according to the equations derived in Ref. 1. The idea of extending QPC to superfluid nuclei is not new.[2-5] In the present article, we extend the many-body SCQPC of Ref. 1 to such systems. In following articles[6] the theory will be employed for spherical and deformed nuclei.

In this paper we use the Migdal-Gorkov[7] formalism for treating superfluidity. In Sec. II we review this formalism and discuss why it is necessary for superfluid nuclei. In Sec. III we define one-, two-, and three-particle Green's functions. These are in fact supermatrices containing the normal Green's functions, as well as the anomalous Green's function.[7] The dynamical and analytical properties of these matrices are then given. In the case of the three-particle Green's function, the equations for a superfluid are derived for the first time.

In Sec. IV, an approximation for the six-point function is developed which leads to a matrix diagonalization problem for the energies and spectroscopic factors of odd-mass nuclear states. The equations for the even nucleus, which must be solved self-consistently with the odd nucleus, are also derived. The reader who wishes to skip over the algebra in Secs. III and IV will find the final matrix equations at the end of Secs. IV A and IV B, Eqs. (52)–(53) and (80)–(83). A number of lengthy derivations have been relegated to an unpublished appendix,[8] and a less rigorous development of our equations is contained in Ref. 9. We shall omit in this paper the Landau renormalization which transcribes the particle Green's function into a quasiparticle Green's function, the irreducible particle-hole (ph) and particle-particle (pp) blocks

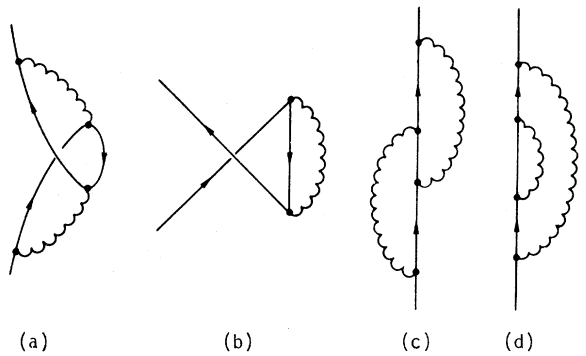

FIG. 1. Higher-order corrections to QPC: (a) corrects for exclusion-principle violations (EPV's); (b) is the so-called "backward" diagram which results from ground state correlations; (c) and (d) are two-phonon contributions to the self-energy; (d) can be obtained by "dressing" the intermediate line in the lowest-order QPC diagram of Fig. 2(b).

I and J into F and F', and the bare interaction V into T. The philosophy and details of this transformation are described in Ref. 1, including the splitting of the quasiparticle by QPC. We intend the term quasiparticle in the Landau sense: A "dressed" elementary excitation. To refer to those particular elementary excitations in superfluid nuclei which are both particle and hole, we introduce the word "quasibogolon."

In Sec. V we summarize the previous sections and explain the physics contained in the complicated equations describing SCQPC. The renormalization of pairing by backward QPC and the physical basis for the small backward amplitude approximation are also discussed. Finally, methods developed by other authors for nonrotational structure in superfluid nuclei are compared with this work.

II. MIGDAL-GORKOV PAIRING FORMALISM

Ideally one would like an identical treatment of pairing and shape correlations in superfluid nuclei. In this paper we consider an even-even nucleus to be superfluid when its ground state energy is related to the ground state energies of neighboring even-even nuclei by

$$E_{N+2}-E_N \approx E_N - E_{N-2}, \tag{1}$$

where N is the number of protons or neutrons. Superfluid and normal nuclei are different only because Eq. (1) facilitates a different approximation. The shell-model configuration interaction approach to pairing vibrations and shape vibrations in normal nuclei is well known. It is also true, however, that the low energy 0+ states of superfluid nuclei can be described by a matrix diagonalization of the pairing and quadrupole interactions among appropriate configurations.[10] The model space must be severely truncated in practice, but the shell model can in principle describe superfluid as well as normal nuclei. Therefore, in finite systems, the BCS transformation[11] is not a mathematical necessity.

In recent years most of the theoretical developments in structure physics have been based on many-body field theoretic techniques, because of the power and clarity of such formalisms. In the Green's function approach one should include pairing correlations in normal nuclei by calculating a (pp) correlated contribution to the single-particle mass operator as well as the usual (ph) term. These are shown in Fig. 2. In the latter case, it is well known that in the $N+1$ nucleus, the extra particle should be coupled at least to the lowest excited state of the N nucleus. A similar technique couples a hole to the ground state of the $N+2$ nucleus. Figure 2(a) is just the usual pairing vibrational approach used to describe states in odd-mass nuclei.[12]

In principle one may apply this approach to superfluid nuclei using the following prescription:

First, solve the (pp) random phase approximation (RPA) equations with dressed particles, i.e., with single-particle strength fractionated among the $N+1$ and $N-1$ nuclear states. The resultant two-particle Green's function with poles in the $N+2$ and $N-2$ nuclei has the approximate form

$$L(\omega)\approx \frac{S_+}{\omega-(E_{N+2}-E_N)+i\delta}-\frac{S_-}{\omega+(E_{N-2}-E_N)-i\delta}, \tag{2}$$

where the pairing correlations (S_+, S_-) are concentrated in the $N\pm2$ ground states.

Second, calculate the $N\pm1$ states by solving Dyson's equation for the one-nucleon Green's function

$$G = G^{(0)} + G^{(0)}MG \tag{3}$$

with

$$M = VLGV = \sum_{N\pm1} \frac{(VS_\mp)^2 z_{N\pm1}}{\omega \mp E_{N\pm1} \mp E_{N\mp2}}. \tag{4}$$

Steps one and two above should be done self-consistently. Further, step two requires as an input the distribution of single-particle strengths z into the $N\pm1$ nuclei, even though this is the object of the same step—a further self-consistency problem. This requirement follows because the one-particle propagator in the approximation for the mass operator must be dressed. The theory would be incorrect if the zeroth-order propagator were employed.

In fact, the above scheme was followed in a series of model calculations by Zawischa.[13] The unhappy result is that when the intermediate G is dressed (theoretically correct), the single-parti-

FIG. 2. Lowest-order quasiparticle-phonon-coupling diagrams. (a) shows the coupling of a pairing or (pp) vibration to a hole; (b) shows the coupling of a shape or (ph) vibration to a particle.

cle strength is fractionated not only into the $N-1$ nucleus, as well as the $N+1$ [the v^2 and u^2 of (d,p) reaction analysis], but also into many states within each nucleus. This is experimentally incorrect, since one generally populates only one state of a given spin and parity in each of the two sister nuclei. The only way to remedy this deficiency is to explicitly include some very complicated energy-dependent contributions to the (pp) or pairing interaction, which are required to fulfill the Pauli principle. These corrections would not be so crucial if the pairing vibration energy in the energy denominators of perturbation theory were not approximately zero, that is

$$E_{N+2}-E_N \approx E_N - E_{N-2} \approx 2\lambda \tag{1'}$$

or, when λ is taken as the zero for single-particle energies,

$$E_{N+2} \approx E_N \approx E_{N-2}, \tag{1''}$$

thus permitting a large number of such excitations to contribute and interfere with one another. Unfortunately, the cumbersome equations of Ref. 13 make it impossible to calculate a normal nucleus with both (pp) and (ph) correlations and a large single-particle basis set.

The approach of Gorkov[7] and later Migdal[7] is to take advantage of Eq. (1). Although this approach does not preclude going back and forth between the pairing and shape correlations to achieve self-consistency, it is based on a separate and unique treatment of the pairing. The assumption implicit in Eq. (1) is that by removing two nucleons from the N nucleus to obtain the $N-2$, the percentage of particles removed from each orbital is the same as the percentage added in going from N to $N+2$ (or removed in returning from $N+2$ to N). It follows that it is redundant to simultaneously construct an $N+2$ pairing vibration and an $N-2$ pairing vibration. The result is just to return to the ground state of the N nucleus. This is shown schematically in Fig. 3. Obviously, this argument is not applicable to a nucleus like ^{208}Pb where "$(N+2)+(N-2)$" leads, not back to the ground state, but to the state at 4.87 MeV.

The far-reaching result of Eq. (1) and Fig. 3, however, is that all graphs which superpose pairing vibrations are eliminated for superfluid nuclei. The "modified" Migdal prescription[7] is inconsistent because although $E_{N+2}-E_N \neq E_N - E_{N-2}$ (a clear violation of Fig. 3), higher-order graphs [e.g., Fig. 4(a)] are neglected due to Fig. 3. As an example, even the simple contribution to the self-energy shown in Fig. 4(a) is disallowed because it is equal (by the logic of Fig. 3) to Fig. 4(b) which is already calculated by an iteration of the lowest-order graph. The mass operator is then simply

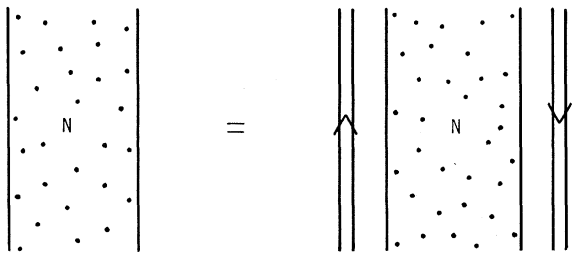

FIG. 3. In the Migdal-Gorkov approximation, the superposition of the collective 0+ (pp) and (hh) vibrations upon the N-nucleon 0+ ground state gives the same superfluid 0+ ground state.

$$M \simeq VLG^0V, \tag{5}$$

where the intermediate G^0 is undressed. Also our logic implies that S_+ and S_- are equal, as well as E_{N+2} and E_{N-2}. The solution of Dyson's equation is then just the BCS system of equations[11]:

$$G = \frac{u^2}{\omega - E + i\delta} + \frac{v^2}{\omega + E - i\delta},$$

$$\begin{pmatrix} \epsilon - \lambda & \Delta \\ \Delta & \lambda - \epsilon \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = E \begin{pmatrix} u \\ v \end{pmatrix}, \tag{6}$$

$$\Delta = S_+ V,$$

$$S_+ = \sum uv.$$

Of course, having obtained G in this way, one can still solve the usual Bethe-Salpeter equation to obtain the (ph) excitations, thereby including the

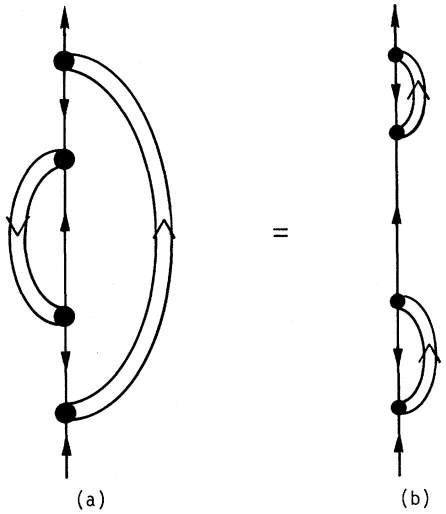

FIG. 4. An example of the Migdal-Gorkov approximation showing that the (pp) and (hh) 0+ vibrations cancel whenever they coincide in time.

dressed RPA bubbles such as those shown in Fig. 5. We proceed from the homogeneous integral equation

$$\rho_{13} = A_{13} I_{1234} \rho_{42},$$

where

$$A_{13}(\omega) = \frac{-i}{2\pi} \int G_1(\omega_1) G_3(\omega_1 - \omega) d\omega_1$$

$$= \frac{u_1^2 v_3^2}{\omega - E_1 - E_3} - \frac{u_3^2 v_1^2}{\omega + E_1 + E_3}. \quad (7)$$

If we define

$$X_{13} = \frac{u_1 v_3}{\omega - E_1 - E_3} \frac{\rho_{13}}{A_{13}}, \quad Y_{13} = \frac{-v_1 u_3}{\omega + E_1 + E_3} \frac{\rho_{13}}{A_{13}},$$

we obtain

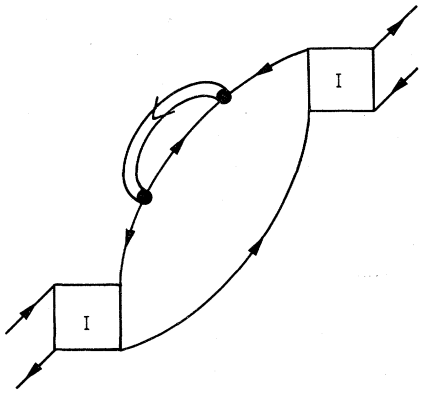

FIG. 5. An RPA bubble which is effectively calculated by using normal Green's functions which include (pp) correlations in the self-energy.

$$\omega(X_{13} \mp Y_{13}) = (u_1 v_3 \pm v_1 u_3)[I_{1234}(u_4 v_2 X_{42} + u_2 v_4 Y_{42}) - I_{1432}(u_2 v_4 X_{42} + u_4 v_2 Y_{42})] + (E_1 + E_3)(X_{13} \pm Y_{13}). \quad (8)$$

If we compare this to the well-known superfluid RPA with a (ph) interaction

$$\omega(X_{13} \mp Y_{13}) = (u_1 v_3 \pm v_1 u_3)(u_4 v_2 \pm v_4 u_2)(I_{1234} \mp I_{1432})(X_{42} \pm Y_{42}) + (E_1 + E_3)(X_{13} \pm Y_{13}) \quad (9)$$

it is clear that half of the terms are missing. The reason is shown in Fig. 6. If (pp) correlations are important, then there are some pairing vibration contributions to the (ph) interaction which should be explicitly taken into account.

In the approach of Migdal[7] this is accomplished as shown in Fig. 7. The pairing vibration exchange diagram is replaced by a diagram in which the pair-addition and pair-subtraction modes are initiated by different lines—this is legal because of Fig. 3. The anomalous Green's function F is defined to be the one-nucleon graph that begins as a particle on the N nucleus and ends as a hole on the $N+2$ nucleus. With the same assumptions that give the BCS [Eq. (6)],

$$F = \frac{uv}{\omega - E + i\delta} + \frac{uv}{\omega + E - i\delta}. \quad (10)$$

Note that F does not contain any energy dependence of the pairing vibration because of Eq. (1). Figure 7 thus constitutes an anomalous RPA bubble that must be added to Eq. (7) giving

$$\rho_{13} = A_{13} I_{1234} \rho_{42} + A'_{13} I_{1432} \rho_{42},$$

where

$$A'_{13} = \frac{-i}{2\pi} \int F_1(\omega_1) F_3(\omega_1 - \omega) d\omega_1. \quad (11)$$

The second term turns out, after a good deal of algebra,[14] to be responsible for the other half of the terms in Eq. (9).

It is possible to show in a similar way how the

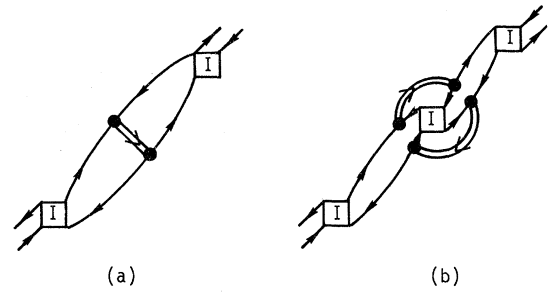

FIG. 6. RPA bubbles which are effectively calculated by including anomalous Green's functions.

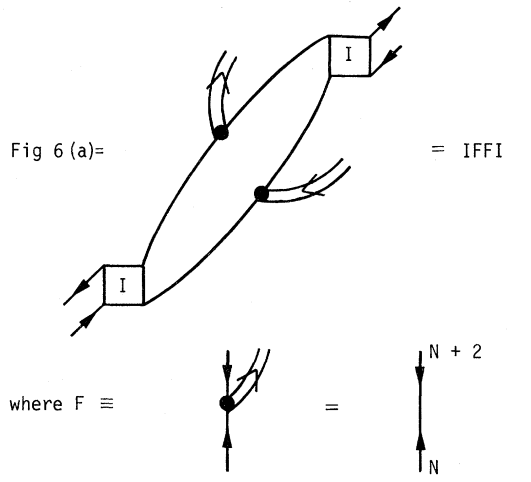

FIG. 7. Illustrating how anomalous Green's functions represent more complicated diagrams such as those of Fig. 6.

(pp) correlations are added to the (ph) interaction. But we have already demonstrated the essential points: In the case that Eq. (1) holds approximately, it is not feasible to treat both (pp) and (ph) correlations on the same footing. In a Green's function formalism it is necessary to make the Migdal-Gorkov approximation shown in Fig. 3. This approximation is implemented by defining the anomalous Green's function and treating it on an equal footing with the normal Green's function. We conclude with the comment that this approach implies number nonconservation since we are assuming that it is impossible to distinguish the N and $N+2$ nuclei.

III. EQUATIONS FOR ONE-, TWO-, AND THREE-PARTICLE GREEN'S FUNCTIONS

As Secs. III and IV develop in considerable detail the theory and approximations of SCQPC, we begin by outlining the presentations.

Nuclear structure information is contained in Green's functions as follows: The n-point function describes the propagation or correlation in time of $n/2$ particles or holes created upon some core. They may scatter or excite core excitations. The Fourier transform (from time to energy variable) exhibits singularities (poles) whenever the energy variable approaches eigenvalues of that nucleus which is $n/2$ particles or holes away from the core. The residues at such poles are products of spectroscopic amplitudes. Typically then, the two-point function describes the energies and particle amplitudes of the odd-A nucleus, while the four-point function gives the energies and amplitudes of particle-hole (or particle-particle) excitations built on the core. The Green's function is the solution of an integral or Bethe-Salpeter equation which incorporates the effective interaction between particles or between a particle and the core.

In Sec. III A the Green's or n-point function is defined in a general way so that particles and holes are effectively mixed. By partially integrating the equation for the two-point function, an effective one-nucleon potential M is defined. This quantity is usually called the self-energy or mass operator. It depends, however, on the four-point or two-nucleon Green's function, which in turn, depends on both higher and lower orders. The scheme is formally closed as follows: A Bethe-Salpeter equation for a four-point (response) function is defined, in which all dependence on higher orders is collected into an effective interaction I or (as it turns out graphically) an irreducible block, whose behavior we approximate later. A similar technique yields a Bethe-Salpeter equation for the six-point function. However, if real three-body forces are neglected, the effective interaction for three nucleons K is a sum of two-nucleon interactions I. The mass operator turns out to be the sum of a Hartree-Fock-like potential and a correlated term involving the product of two-nucleon interactions and the six-point function. Thus, we finally obtain a set of general formulas connecting the two-, four-, and six-point functions, coupled by an effective interaction which we discuss in Sec. III B. This interaction may be calculated in perturbation theory or parametrized, as in Fermi liquid theory. We follow the latter approach.

The approximations and simplifications for evaluating the above quantities are the subject of Sec. IV. In particular, an RPA-like equation emerges for the four-point function, for which collective excitations of the core (or phonons) are particular solutions. The six-point function is approximately factorized into a product of the two-point function (particle) and the four-point function (phonon).

A. Bethe-Salpeter and analytic representations

Guided by the considerations of the previous section, we want to extend the ideas of Refs. 1 and 15 to superfluid nuclei by introducing anomalous Green's functions. It is convenient to combine normal and anomalous quantities into a generalized matrix notation.[16,17] The advantages are that most of the equations can be derived using the same matrix multiplication techniques employed for normal systems and that normal and anomalous quantities are treated to the same order. The n-point Green's function is defined by

$$G^{AB\cdots MA'B'\cdots M'}_{ab\cdots ma'b'\cdots m'} \equiv (-i)^n \langle | \, T(\psi^A_a \psi^B_b \cdots \psi^M_m \psi^{\dagger M'}_{m'} \cdots \psi^{\dagger B'}_{b'} \psi^{\dagger A'}_{a'}) \, | \rangle \,, \tag{12}$$

where n is the number of particle indices $a,b,\ldots,m$. The even-even ground state is represented by $|\,\rangle$, and we shall neglect differences between neighboring nuclei (see the arguments in Sec. II):

$$|\,\rangle = |N\rangle = |N\pm 2\rangle = |N\pm 4\rangle, \text{ etc.} \tag{13}$$

T is the time ordering operator.

The lower case indices refer to the single-particle states occupied by a particle, as well as the time of creation or annihilation. The upper case indices have the following meaning:

$$\psi^A_a = \begin{cases} \psi_a \,, & A=1 \\ \psi^\dagger_{-a} \,, & A=2 \,, \end{cases} \qquad \psi^{\dagger A}_a = \begin{cases} \psi^\dagger_a \,, & A=1 \\ \psi_{-a} \,, & A=2 \,, \end{cases} \tag{14}$$

$$\bar{A} \equiv 3-A, \quad \bar{a} = -a \,.$$

G^{11} is thus the usual form of the Green's function defined for normal systems, while G^{12} is the anomalous function defining amplitudes of a particle upon the N nucleus and a hole upon the $N+2$.

A systematic treatment of the Green's functions is based on their equations of motion[18] and analytical behavior.[7,16,17] The dynamical equations relating superfluid Green's functions of different order may be derived[8] as in normal systems:

$$G^{AA'}_{aa'} = (G^0)^{AA'}_{aa'} - i(G^0)^{AK'}_{ak'}\, V^{K'P'KP}_{k'p'kp}\, G^{KPA'P'}_{kpa'p'}\,, \tag{15a}$$

$$G^{AB\,A'B'}_{ab\,a'b'} = (G^0)^{AA'}_{aa'}\, G^{BB'}_{bb'} - (G^0)^{AB'}_{ab'}\, G^{BA'}_{ba'} + (G^0)^{A\bar{B}}_{a\bar{b}}\, G^{\bar{B}'A'}_{\bar{b}'a'} + i(G^0)^{AK'}_{ak'}\, V^{K'P'KP}_{k'p'kp}\, G^{BKPA'B'P'}_{bkpa'b'p'}\,, \tag{15b}$$

where

$$V^{K'P'KP}_{k'p'kp} = \delta_{KK'}\delta_{PP'}\delta_{KP} \times (\delta_{K1}\, V_{k'p'kp} + \delta_{K2}\, V_{-k'-p'-k-p})\,.$$

The above equations suggest (on account of their complicated appearance) a change of notation, namely to replace the subscript and superscript by a single subscript, e.g.,

$$G^{AA'}_{aa'} \to G_{AA'}, \quad \text{etc.}$$

We shall follow this notation frequently through remaining parts of the paper.

If we make the definition

$$-iV_{K'P'KP}G_{KPA'P'} \equiv M_{K'K}\, G_{KA'} \tag{16}$$

we obtain a Dyson equation for the two-point function

$$G_{AA'} = G^0_{AA'} + G^0_{AK'}\, M_{K'K}\, G_{KA'}\,. \tag{17}$$

Equation (17) turns out to be useful in practical applications only when the analytical properties of $G_{AA'}$ are detailed:

$$G_{AA'} = \sum_i \left(\frac{\langle |\, \psi^A_a |\, i\rangle \langle i |\, \psi^{\dagger A'}_a |\rangle}{\omega - E^{(i)} + i\delta} + \frac{\langle |\, \psi^{\dagger A'}_a |\, i\rangle \langle i |\, \psi^A_a |\rangle}{\omega + E^{(i)} - i\delta} \right). \tag{18}$$

This can be derived by inserting a complete set of odd-mass nuclear states between the Heisenberg operators and Fourier transforming. The $E^{(i)}$ are the energies of the odd nucleus measured relative to the even nucleus. This statement of the analytic behavior of G embodies the Migdal-Gorkov approximation, made in the present theory; Eqs. (15)–(17) do not. Since neighboring even nuclei are assumed identical, neighboring odd nuclei have the same eigenvalues and spectroscopic amplitudes. From this point on, we must expect that spurious states and the other unfortunate consequences of number nonconservation are an inherent part of the development based on Eq. (18).

Using the technique of functional differentiation,[18,19] a more practical equation for the four-point function may be derived. With the definitions

$$L_{ABA'B'} \equiv G_{ABA'B'} - G_{AA'}\, G_{BB'}\,, \tag{19a}$$

$$L^{\text{free}}_{ABA'B'} \equiv -G_{AB'}\, G_{BA'} + G_{A\bar{B}}\, G_{\bar{B}'A'}\,, \tag{19b}$$

$$-iI_{K'P'KP} \equiv \frac{\delta M_{K'K}}{\delta G_{PP'}}\,, \tag{19c}$$

Weigel[16] has obtained

$$L_{ABA'B'} = L^{\text{free}}_{ABA'B'} - iG_{AK'}\, G_{KA'}\, I_{K'P'KP}\, L_{PBP'B'} \tag{20}$$

which bears an obvious resemblance to the Bethe-Salpeter equation for the particle-hole response function in normal systems. I is the generalized irreducible block for scattering of excitations in the superfluid system. Its structure is discussed later in this section.

The above dynamical description of L is supplemented by its analytical representation

$$L_{ABA'B'} = -i \sum_{|S\rangle \neq |\rangle} \left(\frac{\chi^{(S)}_{AA'}\chi^{(S)\dagger}_{B'B}}{\omega - E^{(S)} + i\delta} - \frac{\chi^{(S)}_{BB'}\chi^{(S)\dagger}_{A'A}}{\omega + E^{(S)} - i\delta} \right), \tag{21}$$

where the sum is over states of the even-even (or odd-odd) nucleus with excitation energy $E^{(S)}$. The energy variable ω equals $\omega_A - \omega_{A'}$ and the spectroscopic amplitudes or generalized density matrix for the excited states may be obtained from $\chi^{(S)}_{AA'}$ by

$$\frac{-i}{2\pi} \int \chi^{(S)}_{AA'}\, d\omega_A = \langle |\, \psi^{\dagger A'}_{a'} \psi^A_a |\, S\rangle \equiv \rho^{(S)}_{AA'}\,. \tag{22}$$

The dynamical descriptions of Eqs. (17), (19c), and (20) for both the two- and four-point functions are dependent on the mass operator M. The approach to determining M for superfluid nuclei is analogous to that of Refs. 1, 15, and 20 for normal systems, namely to write M in terms of a Hartree-Fock-Bogoliubov (HFB) contribution and the six-point response function. The latter is, in turn, determined by the assumption that only certain correlations involving two particles are essential to the dynamical treatment of three particles.

With Eq. (17) and the definition

$$L_{CABA'B'C'} \equiv G_{CABA'B'C'} - G_{ABC'U}\, G_{UV}{}^{-1}\, G_{VCA'B'} \tag{23}$$

for the six-point response function, G^0 may be removed from Eq. (15b) to obtain

$$G_{ABA'B'} = G^{\text{free}}_{ABA'B'} + iG_{AK'}\, V_{K'P'KP}\, L_{BKPA'B'P'}\,, \tag{24}$$

where

$$G^{\text{free}}_{ABA'B'} = G_{AA'}\, G_{BB'} - G_{AB'}\, G_{BA'} + G_{A\bar{B}}\, G_{\bar{B}'A'}\,.$$

From Eqs. (16) and (24) the mass operator may be written symmetrically[8,20]

$$M_{D'D} = M^{(\mathrm{HFB})}_{D'D} + V_{D'C'AB} L_{CABA'B'C'} V_{A'B'DC} , \tag{25}$$

where

$$M^{(\mathrm{HFB})}_{D'D} G_{DA'} \equiv -iV_{D'P'DP} G^{\mathrm{free}}_{DPA'P'}$$

or

$$M^{(\mathrm{HFB})}_{D'D} = -2iV_{D'P'DP} G_{PP'} - iV_{D'\bar{D}P\bar{P}'} G_{P\bar{P}'} . \tag{26}$$

The six-point response function L contains all graphs connecting excitations in orbits A, B, and C' to excitations in A', B', and C, with the exception of those graphs that can be separated by cutting a single line, i.e., a 2p-1h excitation cannot evolve into a 1p excitation. Graphical considerations and the analogy to low orders suggest that a Bethe-Salpeter equation for the six-point function be written down. Generalizing the development in Refs. 1, 15, and 20:

$$\begin{aligned} L_{CABA'B'C'} = {} & L^{\mathrm{free}}_{CABA'B'C'} \\ & + G_{AP} G_{BQ} G_{R'C'} \\ & \times K_{RPQP'Q'R'} L_{CP'Q'A'B'R} , \end{aligned} \tag{27}$$

where

$$\begin{aligned} L^{\mathrm{free}}_{CA'BA'B'C'} = {} & G_{AA'} G_{BB'} G_{CC'} - G_{AB'} G_{BA'} G_{CC'} \\ & - G_{A\bar{C}} G_{BB'} G_{\bar{A}'C'} + G_{A\bar{C}} G_{BA'} G_{\bar{B}'C'} \\ & - G_{AA'} G_{B\bar{C}} G_{\bar{B}'C'} + G_{AB'} G_{B\bar{C}} G_{\bar{A}'C'} \end{aligned} \tag{28}$$

and

$$\begin{aligned} iK_{RPQP'Q'R'} = {} & I_{PRR'P'} G_{QQ'}{}^{-1} + I_{QRR'Q'} G_{PP'}{}^{-1} \\ & - I_{P\bar{Q}'\bar{Q}P'} G_{RR'}{}^{-1} . \end{aligned} \tag{29}$$

The analytical properties of L follow from Eqs. (23) and (25) and the *ansatz* that the mass operator (minus the HFB part) has the form

$$M_{D'D} = \sum_{\gamma} \frac{\sigma^{(\gamma)}_{D'} (\sigma^{(\gamma)}_{D})^{\dagger}}{\omega - \omega_{\gamma}} . \tag{30}$$

It can then be shown[15] that

$$L_{CABA'B'C'} = \sum_{\gamma} \frac{\rho^{(\gamma)}_{ABC'} (\rho^{(\gamma)}_{A'B'C})^{\dagger}}{\omega - \omega_{\gamma}} \tag{31}$$

with

$$\begin{aligned} \rho^{(\gamma)}_{ABC'} = -\sigma^{(\gamma)}_{D'} \sum_{i} \Bigg\{ & \frac{\langle | \psi_A \psi_B \psi^{\dagger}_{C'} | i \rangle \langle i | \psi^{\dagger}_{D'} | \rangle}{E^{(i)} - \omega_{\gamma}} \\ & - \frac{\langle | \psi^{\dagger}_{D'} | i \rangle \langle i | \psi_A \psi_B \psi^{\dagger}_{C'} | \rangle}{E^{(i)} + \omega_{\gamma}} \Bigg\} \end{aligned} \tag{32}$$

and

$$\sigma^{(\gamma)}_{D'} = V_{D'C'AB} \rho^{(\gamma)}_{ABC'} . \tag{33}$$

B. Irreducible block

The irreducible block, or effective interaction between quasiparticles, is formally defined in the derivation of the Bethe-Salpeter equation for the four-point response function:

$$-iI_{K'P'KP} = \frac{\delta M_{K'K}}{\delta G_{PP'}} . \tag{19c}$$

Actually, the choice of this matrix is basic to Fermi liquid theory. In principle it can be calculated, and there are numerous efforts in this direction.[21,22] More commonly the (ph) block and (pp) block for normal systems (I and J) are determined phenomenologically,[23] and the results are extrapolated to superfluid nuclei. Justification for the latter procedure has not been thoroughly investigated.

In general, we expect the supermatrix I to have 16 elements since each superscript can assume the value of 1 and 2. But the following approximation has been universally employed

$$\begin{aligned} & I^{1\,1\,11}_{k'\,p'\,kp} = I^{2\,2\,22}_{\bar{k}\,\bar{p}\,\bar{k}\bar{p}} = I_{k'p'kp} , \\ & I^{1\,2\,21}_{k'\,p'\,kp} = I^{2\,1\,12}_{\bar{k}'\,\bar{p}'\,\bar{k}\bar{p}} = J_{k'\bar{k}p\bar{p}'} , \end{aligned} \tag{34}$$

where the I and J without superscripts are the interactions in normal systems, and all other superscript combinations are zero. There are several reasons for Eqs. (34). One is that if they are combined with the BCS approximation for the two-point function, Eqs. (6) and (10), the Bethe-Salpeter equation (20) yields the usual superfluid RPA equations with (pp) and (ph) interactions. Then, in the limit that all u's and v's are 0 or 1, these give the (pp) RPA, with interaction J, and the (ph) RPA, with interaction I. Moreover, the close correspondence of Eqs. (34) with lowest-order perturbation theory should be noted:

$$I_{K'P'KP} \approx i \frac{\delta M^{(\mathrm{HFB})}_{K'K}}{\delta G_{PP'}}$$

$$= \begin{cases} 2V_{k'p'kp} & \text{for } K'K = PP' \text{ and } K' = K = 1 \\ 2V_{\bar{k}'\bar{p}'\bar{k}\bar{p}} & \text{for } K'K = PP' \text{ and } K' = K = 2 \\ V_{k'\bar{k}p\bar{p}'} & \text{for } K'K = PP' \text{ and } K' = \bar{K} = 1 \\ V_{\bar{k}'k\bar{p}p'} & \text{for } K'K = PP' \text{ and } K' = \bar{K} = 2 \end{cases} \tag{35}$$

Namely, all but 4 of the 16 supermatrix elements are zero. This is advantageous for obvious calculational reasons, as well as the physical ones that the matrix elements which begin with higher orders and mixed superscripts are difficult to interpret and could not be determined via phenomenology.

Nevertheless, Eq. (34) is subject to several

criticisms. One is that

$$I^{1111} \neq I(\text{normal}); \tag{36a}$$

the reasons are simply that while the Green's functions are the same

$$G^{11} = G(\text{normal}), \tag{36b}$$

the mass operators are not [recall Eq. (19c)]

$$M^{11} \neq M(\text{normal}). \tag{36c}$$

For example, Fig. 2(a), the coupling of a hole and a pairing vibration, contributes to the mass operator for normal systems, although for superfluid systems the same graph is calculated by iterating the anomalous mass operator, i.e.,

$$\text{Fig. 1(a)} = M^{12} G^{22} M^{21}.$$

Similarly, I (normal) includes the pairing vibration exchange contribution, Fig. 6, but I^{1111} does not. The answer to this technical point is that Fig. 6 is not important for normal systems, although for superfluid systems it is so dominant that its influence must be handled by anomalous Green's functions in order for I to remain approximately energy independent. A less easily discounted argument is that the contribution of Fig. 8, viz., the exchange of a (ph) excitation in the other (ph) channel, influences I differently in the normal and superfluid regions. This is because the 2+ or 3− states can be at greatly different energies. But this same argument would seem to apply to every different nucleus in the Periodic Table, so we shall not comment further on its importance in the present paper on superfluids. We shall adopt the above approximation for I, as have other authors.

It is also of some interest to examine the relation of the generalized I block to the Hartree-Fock (HF) and BCS (or HFB) approximations. The functional definition (19c) may be integrated to give

$$\begin{aligned} M^{11}_{k'k} &= -iI_{k'p'kp} G^{11}_{pp'}, \\ M^{12}_{k'k} &= -iJ_{k'\bar{k}p\bar{p}'} G^{12}_{pp'}. \end{aligned} \tag{37}$$

Formally, these are the HF and BCS equations

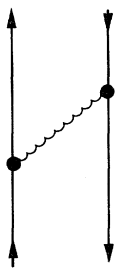

FIG. 8. The phonon-exchange contribution to the (ph) interaction.

but with Migdal's irreducible blocks, instead of the bare interaction. In the case that I and J are energy-independent, G^{11} and G^{12} will have only one pole. On the other hand, it should be realized that Eq. (37) is not a rigorous (or even an improved) form of HF or HFB. In principle, this approach causes the single-particle energy (or M^{11}) to include Fig. 2(b), core polarization. Similar comments hold for the pairing gap (or M^{12}). Thus the interpretation of single-particle strength is different from that of Baranger,[24] for example.

The advantage of doing HFB this way is that if HFB and RPA are done with the same interaction or, more generally, when the mass operator and the response function are calculated with the same block, as in Eqs. (19c) and (20), the approximation is conserving[18]; consequently the spurious states (1−, center of mass, or 0+, number nonconservation) are isolated.[25]

This points up a major shortcoming of the theory which we propose in this paper, namely that M is improved (i.e., energy-dependent or time-delayed) while I is a phenomenological and unretarded (ph) block. Spurious states will have to be eliminated explicitly, since the system of equations is no longer conserving. In principle, we could eliminate this difficulty by determining I by functional differentiation; in practice, this is not simple.

IV. SCQPC EQUATIONS

In this part of the paper, the Bethe-Salpeter equations for the two-, four-, and six-point functions are simplified by a set of self-consistent approximations, which are suitable for odd-A nuclei in which the collective excitations of the even-A nuclei persist. A capsule summary of the section follows:

(a) Phonons of the even-even nucleus:

1. One-nucleon Green's function. We first define a generalized set of particle and hole amplitudes u and v and relate these to forward (ψ) and backward (ϕ) amplitudes, the latter being small except in the case of ground state correlations.
2. Bethe-Salpeter equation. Utilizing the u's and v's of the previous paragraph, as well as the multiplicity of odd-A states with particle residue, an RPA-like equation for superfluid nuclei is derived. Its most important feature is that it includes renormalization due to QPC-induced fractionation of single-particle strength.

(b) Odd-mass problem:

1. Approximate treatment of the six-point function. A partial summation of terms for the six-point function is performed, so that, in lowest-order perturbation theory, it is a product of the four-point function and the two-point function. The

former contains information about the collective states of the even-even system.

2. Approximate diagonalization for the poles of the mass operator. The generalized (2p-1h) or three-nucleon amplitudes of the six-point function are factorized into phonon and particle contributions. This development of the analytic representation of the six-point function, combined with the approximation for its dynamic behavior mentioned above, admits a matrix solution for the poles of the mass operator. The effective interaction between phonon-plus-quasibogolon configurations is actually equivalent to the higher-order graphs which account for the exclusion principle violations and two-phonon intermediates discussed in the introduction.

3. A second matrix-diagonalization problem connects the poles of the mass operator to the energies and spectroscopic amplitudes of the odd-mass nuclei.

4. The final step is to combine the two matrix problems into one.

It is intended that consistency between the even- and odd-mass nuclei is achieved by iterating back and forth between the two problems, hence, the title self-consistent quasiparticle-phonon coupling.

A. Phonons of the even-even nucleus

1. One-nucleon Green's function

The exact analytic representation of the two-point function is given in Eq. (18). As commented upon previously, the $N\pm1$ nuclei have the same energies. This enforces a special symmetry in the equations. We shall also assume that the Green's function is diagonal in the single-particle representation. Making use of an obvious notation,

$$\langle |\psi_a^A| i\rangle = \begin{cases} u_a^{(i)}, & A=1 \\ -s_a v_a^{(i)}, & A=2, \end{cases} \tag{38}$$

where s_a is the usual time-reversal phase factor, the Green's function may be written

$$G_{AK'} = G_{ak'}^{AK'} = \delta_{ak'} \sum_i \frac{B_{AK'}^{(i)}}{\omega - E_a^{(i)} + i\delta} + \frac{C_{AK'}^{(i)}}{\omega + E_a^{(i)} - i\delta}, \tag{39}$$

where

$$B_{AK'}^{(i)} = \begin{pmatrix} u_a^{(i)}u_a^{(i)} & -s_a v_a^{(i)}u_a^{(i)} \\ -s_a v_a^{(i)}u_a^{(i)} & v_a^{(i)}v_a^{(i)} \end{pmatrix}$$

and

$$C_{AK'}^{(i)} = \begin{pmatrix} v_a^{(i)}v_a^{(i)} & s_a v_a^{(i)}u_a^{(i)} \\ s_a v_a^{(i)}u_a^{(i)} & u_a^{(i)}u_a^{(i)} \end{pmatrix}.$$

It turns out from the QPC development in Sec. IV B that the occupation amplitudes obey the following equation:

$$\begin{pmatrix} u_a^{(i)} \\ -s_a v_a^{(i)} \end{pmatrix} = \begin{pmatrix} u_a & v_a \\ -s_a v_a & s_a u_a \end{pmatrix} \begin{pmatrix} \psi_a^{(i)} \\ \phi_a^{(i)} \end{pmatrix}. \tag{40}$$

That is, the QPC amplitudes (of which there are $2n$, n being the number of poles i) are related to another set of amplitudes by the u's and v's of HFB. The amplitudes ψ and ϕ have the properties for physical states $|\phi|^2 \ll 1$ and also $\psi \simeq 1$ for states with little or no phonon admixture, i.e., quasibogolons. We shall henceforth refer to ψ and ϕ as the "forward" and "backward" amplitudes. Although a more sophisticated development (eventually requiring a much larger computational program) is possible, we shall henceforth make the "small backward amplitudes" approximation, viz.,

$$\begin{aligned} u_a^{(i)} &\approx u_a \psi^{(i)} \equiv u_a (z_a^{(i)})^{1/2}, \\ v_a^{(i)} &\approx v_a \psi^{(i)} \equiv v_a (z_a^{(i)})^{1/2}, \\ G_{AK'} &= \delta_{ak'} z_a^{(i)} \left(\frac{B_{AK'}}{\omega - E_a^{(i)} + i\delta} + \frac{C_{AK'}}{\omega + E_a^{(i)} - i\delta} \right), \end{aligned} \tag{41}$$

with

$$B_{AK'} = \begin{pmatrix} u_a^2 & -s_a v_a u_a \\ -s_a v_a u_a & v_a^2 \end{pmatrix}$$

and

$$C_{AK'} = \begin{pmatrix} v_a^2 & s_a v_a u_a \\ s_a v_a u_a & u_a^2 \end{pmatrix}. \tag{42}$$

Thus, the effect of backward coupling on energies ($E^{(i)}$ pushed up when $\phi \neq 0$) is being retained, while the redistribution of single-particle strength across the Fermi surface is being neglected. For single-particle states nearest the Fermi surface, this is a good approximation since pairing effects dominate backward coupling in determining the relative amounts of particle (u^2) and hole (v^2) strength in a given odd-mass nuclear state. We discuss the small backward amplitude approximation further in Sec. V.

2. Bethe-Salpeter equation

The derivation of the RPA with appropriate QPC modifications depends primarily on a transformation of the normal and anomalous amplitudes $\rho_{AA'}^{(S)}$ defined in Eq. (22). We shall outline the steps as simply as possible, since this development of RPA has never been published. More details on the algebra are contained in Ref. 8.

Substituting Eq. (21), the analytic behavior of the four-point functions, into Eq. (20) and setting $\omega = E^{(S)}$ one obtains the homogeneous Bethe-Salpeter equation

$$\chi^{(S)}_{AA'} = -iG_{AK'}G_{KA'}I_{KP'KP}\chi^{(S)}_{PP'}. \tag{43}$$

If I is taken to be energy-independent, Eq. (43) may be integrated to obtain

$$\rho^{(S)}_{AA'} = A_{AKA'K'}I_{K'P'KP}\rho^{(S)}_{PP'},$$

where

$$A_{AKA'K'} \equiv \frac{-i}{2\pi}\int G^{AK'}_{ak'}(\omega_a)G^{KA'}_{ka'}(\omega-\omega_a)d\omega_a$$

$$= \delta_{ak'}\delta_{a'k}\sum_{i,j} z^{(i)}_a z^{(j)}_{a'} \times\left(\frac{B_{AK'}C_{KA'}}{\omega - E^{(i)}_a - E^{(j)}_{a'}} - \frac{C_{AK'}B_{KA'}}{\omega + E^{(i)}_a + E^{(j)}_{a'}}\right). \tag{44}$$

The algebra involving superscripts is cumbersome in the four-dimensional $(2\times2\times2\times2)$ notation. We transform all quadruply superscripted quantities to a two-dimensional (4×4) matrix notation, for example,

$$A^{AKA'K'} \to A^{AA',K'K}, \tag{45}$$

where AA' or $K'K = 11$, 22, 12, and 21 correspond to row or column 1, 2, 3, and 4 of the 4×4 matrix. It is then not difficult to show that

$$B^{AK'}_{aa}C^{KA'}_{a'a'} = MM^{\dagger}, \tag{46a}$$

where

$$M = \begin{pmatrix} u_a s_{a'} v_{a'} & 0 & 0 & 0 \\ -s_a v_a u_{a'} & 0 & 0 & 0 \\ u_a u_{a'} & 0 & 0 & 0 \\ -s_a s_{a'} v_a v_{a'} & 0 & 0 & 0 \end{pmatrix}$$

and

$$C^{AK'}_{aa}B^{KA'}_{a'a'} = NN^{\dagger}, \tag{46b}$$

where

$$N = \begin{pmatrix} 0 & s_a v_a u_{a'} & 0 & 0 \\ 0 & -u_a s_{a'} v_{a'} & 0 & 0 \\ 0 & -s_a s_{a'} v_a v_{a'} & 0 & 0 \\ 0 & u_a u_{a'} & 0 & 0 \end{pmatrix}$$

Note also that the I matrix is 4×4 and diagonal

$$I^{K'P'KP}_{k'p'kp}\delta_{ak'}\delta_{a'k} = \begin{pmatrix} I_{ap'a'p} & 0 & 0 & 0 \\ 0 & s_a s_{a'} s_p s_{p'} I_{ap'a'p} & 0 & 0 \\ 0 & 0 & J_{a-a'p-p'} & 0 \\ 0 & 0 & 0 & s_a s_{a'} s_p s_{p'} J_{a-a'p-p'} \end{pmatrix}, \tag{47}$$

where we have used the time-reversal properties of I and J in rows 2 and 4.

Finally, we introduce the unitary matrix which transforms the superscripted amplitudes into the more conventional amplitudes (X and Y) of superfluid RPA (cf. Baranger[25]):

$$U_{aa'} \equiv \begin{pmatrix} u_a s_{a'} v_{a'} & -s_a v_a u_{a'} & u_a u_{a'} & -s_a s_{a'} v_a v_{a'} \\ s_{a'} v_a u_{a'} & -u_a s_a v_{a'} & -v_a v_{a'} & u_a u_{a'} s_a s_{a'} \\ u_a u_{a'} & s_a s_{a'} v_a v_{a'} & -u_a s_{a'} v_{a'} & -s_a v_a u_{a'} \\ s_a s_{a'} v_a v_{a'} & u_a u_{a'} & s_a v_a u_{a'} & u_a s_{a'} v_{a'} \end{pmatrix}. \tag{48}$$

Multiplying Eq. (44) from the left by $U_{aa'}$, inserting $(U_{pp'})^{\dagger}U_{pp'}$, and defining the column matrix (four elements) $X = U\rho$, we obtain the equation

$$X_{aa'} \equiv U_{aa'}\rho_{aa'}$$

$$= A_{aa'}(\omega)U_{aa'}M_{aa'}(M_{aa'})^{\dagger}I_{ap'a'p}(U_{pp'})^{\dagger}X_{pp'} + A_{aa'}(-\omega)U_{aa'}N_{aa'}(N_{aa'})^{\dagger}I_{ap'a'p}(U_{pp'})^{\dagger}X_{pp'}, \tag{49}$$

where for convenience we have deleted the superscripts from the column matrix X and from the 4×4 matrices U, M, N, and I. The function $A_{aa'}(\omega)$ is not a matrix, rather

$$A_{aa'}(\omega) = \sum_{ii'}\frac{z^{(i)}_a z^{(i')}_{a'}}{\omega - E^{(i)}_a - E^{(i')}_{a'}}. \tag{50}$$

Because of the simplicity of the matrices U, M,

N, and I defined above, the matrix multiplications are actually rather easy. It turns out that

$$UMM^{\dagger}IU^{\dagger}=\begin{bmatrix} P & Q & P' & Q' \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$UNN^{\dagger}IU^{\dagger}=\begin{bmatrix} 0 & 0 & 0 & 0 \\ Q & P & Q' & P' \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{51}$$

Rows 3 and 4 being identically zero, the corresponding amplitudes X are also zero and the values of the matrix elements P' and Q' do not enter the calculation. If we write the first and second elements of the column matrix X, as X and Y, we arrive at the matrix problem

$$\begin{pmatrix} X_{aa'} \\ Y_{aa'} \end{pmatrix} = \begin{pmatrix} A_{aa'}(\omega) & 0 \\ 0 & A_{aa'}(-\omega) \end{pmatrix} \times \begin{pmatrix} P_{ap'a'p} & Q_{ap'a'p} \\ Q_{ap'a'p} & P_{ap'a'p} \end{pmatrix} \begin{pmatrix} X_{pp'} \\ Y_{pp'} \end{pmatrix}, \tag{52}$$

where

$$(P \pm Q) = \xi^{\pm}_{aa'}\xi^{\pm}_{pp'}J_{a-a'p-p'} + \eta^{\pm}_{aa'}\eta^{\pm}_{pp'}s_{a'}s_{p'}I_{ap'a'p},$$

$$\xi^{\pm}_{aa'} \equiv (u_a u_{a'} \mp v_a v_{a'}),$$

$$\eta^{\pm}_{aa'} \equiv (u_a v_{a'} \pm v_a u_{a'}).$$

Equation (52) represents the self-consistent version of the RPA. In the HFB limit where

$$A_{aa'}(\omega) = (\omega - E_a - E_{a'})^{-1}$$

we obtain the usual superfluid RPA equations, including different interactions in the (pp) and (ph) channels. The importance of the modification to $A(\omega)$ is that forward QPC, which fragments the single-particle strength in the odd nucleus, lowers the energy of the phonon by admixing certain anharmonicities. On the other hand, backward QPC, which primarily pushes odd-nuclear eigenenergies upwards, similarly affects the energy of the even-nuclear state by reducing the amount of ground state correlation.

We shall not present the details of the normalization equation here. It follows from the inhomogeneous term of Eq. (20). The algebra is similar to that in the Appendix of Ref. 1 with two modifications: The matrix algebra involving the u's and v's must be faithfully carried out[8] and the inhomogeneous term is antisymmetric and contains two terms rather than one. A factor of 2 is thereby introduced into the normalization formula:

$$1 = \frac{1}{2}\sum_{aa'} X_{aa'}{}^2 W_{aa'}(E^{(S)}) - Y_{aa'}{}^2 W_{aa'}(-E^{(S)}), \tag{53}$$

where

$$W_{aa'}(E^{(S)}) = \frac{dA^{-1}_{aa'}}{d\omega}\bigg|_{\omega=E^{(S)}}.$$

B. Odd-mass problem

1. Partial summation of bubble diagrams in the six-point function

The three-nucleon interaction given in Eq. (27) may be substituted graphically into the Bethe-Salpeter equation for L. One obtains the following:

$$L_{CABA'B'C'} = L^{(21)}_{CABA'B'C'} - L^{(21)}_{CBAA'B'C'} - L^{(21)}_{CABB'A'C'} + L^{(21)}_{CBAB'A'C'} + L^{(21)}_{\bar{A}'\bar{C}'B\bar{C}B'\bar{A}} - L^{(21)}_{\bar{A}'AB\bar{C}B'C'} + L^{(21)}_{\bar{A}'BA\bar{C}B'C'} - L^{(21)}_{C\bar{C}'BA'B'\bar{A}} + L^{(21)}_{C\bar{C}'BB'A'\bar{A}} - 2L^{\text{free}}_{CABA'B'C'}. \tag{54}$$

The function $L^{(21)}$ obeys an integral equation appropriate for a correlated pair and weakly interacting particle:

$$L^{(21)}_{CABA'B'C'} = -L_{BCC'B'}G_{AA'} - iL_{BR'C'Q}G_{AP}H^{(21)}_{RPQP'Q'R'}L^{(21)}_{CP'Q'A'B'R}, \tag{55}$$

where

$$H^{(21)}_{RPQP'Q'R'} = I_{QS'R'P'}G_{SS'}I_{PRSQ'} + iI_{PS'TP'}G_{SS'}G_{TT'}I_{QT'R'U}G_{UU'}I_{U'RSQ'} + \cdots. \tag{56}$$

The graphical description of the interaction is given in Fig. 9. Note that $L^{(21)}$ is not antisymmetric with respect to interchange of the particle indices, C' and A or A and B. We have taken advantage of the fact that $L^{(2)}$, and therefore $L^{(21)}$, are antisymmetric with respect to interchange of B and C. The free part (GGG) is seen to correct the lowest-order contribution to $L^{(3)}$ for double counting, namely

$$L^{\text{free}} = 6 \times GGG,$$

$$9 \times L^{(21)} = 9 \times 2 \times GGG + \cdots.$$

Equation (54) may be substituted into Eq. (25) to obtain

$$M_{D'D} = V_{D'C'AB}\ (9 \text{ terms like } L^{(21)} - 12 \text{ terms like } GGG) V_{A'B'DC}. \tag{57}$$

If we neglect the free part and define an interaction W, which has the same symmetry properties as I,

$$W_{D'C'AB} \equiv 2V_{D'C'AB} + V_{D'\bar{A}B\bar{C}'}, \tag{58}$$

the mass operator involves only a single $L^{(21)}$ function

$$M_{D'D} \approx W_{D'C'AB} L^{(21)}_{CABA'B'C'} W_{A'B'DC}. \tag{59}$$

2. *Approximate diagonalization for the poles of the mass operator*

In order to develop a matrix equation for the six-point function (and therefore the mass operator), we would like to follow the usual procedure of substituting an analytical representation into a homogeneous Bethe-Salpeter equation. We have already shown that M is related to $L^{(21)}$ and further that $L^{(21)}$ obeys an integral equation. We shall assume, however, that the analytic behavior of $L^{(21)}$ may be taken as that of $L^{(3)}$ itself, viz., Eq. (31). We point out that the justification for this step of the procedure is more schematic than rigorous. Further comment is postponed until after Eq. (66). At poles of the mass operator ω_γ this substitution yields the equation:

$$\rho^{(\gamma)}_{ABC'} = A_{R'ABPQC'}(\omega_\gamma) H^{(21)}_{RPQP'Q'R'} \rho^{(\gamma)}_{P'Q'R}, \tag{60}$$

where

$$A_{R'ABPQC'}(\omega) = \frac{-i}{2\pi} \int L_{BR'C'Q} G_{AP} = \sum_{S,i} z^{(i)} \left(\frac{\rho^{(S)}_{BC'} B_{AP} \rho^{(S)}_{QR'}}{\omega - E^{(S)} - E^{(i)}} + \frac{\rho^{(S)}_{C'B} C_{AP} \rho^{(S)}_{R'Q}}{\omega + E^{(S)} + E^{(i)}} \right).$$

Note that in writing down an expression for ρ, we assume that the unnatural energy variables have been integrated upon, i.e., ρ is obtained by applying to the more general quantity the operator

$$\int d\omega_a d\omega_b d\omega_{c'} (\omega - \omega_a - \omega_b + \omega_{c'}).$$

Equation (60) also presumes that $H^{(21)}$ is independent of the unnatural energy variables, which will be the case when the I block is energy-independent.

It should also be pointed out that because of the symmetries of the Migdal-Gorkov approximation, the poles of the mass operator will occur in pairs. We shall assume that the amplitudes $\rho^{(\gamma)}_{ABC}$ and $\rho^{(\bar{\gamma})}_{ABC}$ are uncoupled. This reduces the dimensions of our problem by a factor of 2. A further reduction (as in the phonon case) to a manageable and physically intuitive form hinges upon an approximate decomposition and unitary transformation of the amplitudes.

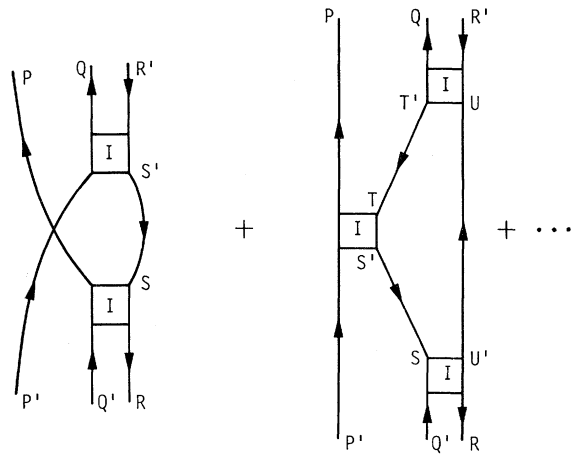

FIG. 9. The "interaction" between the four-point response function and an extra particle. The ingoing and outgoing "stubs" are not part of the interaction. Intermediate lines labeled S, T, U, etc. are exact or "dressed" propagators. I is the irreducible interaction. All possible time orderings of the I blocks are implied in this diagram.

A decomposition of the amplitude $\rho^{(\gamma)}_{ABC'}$ which leads naturally to a linear algebra is the following:

$$-\langle | \psi_A \psi_B \psi^\dagger_{C'} | i \rangle \approx \rho^{(S)}_{BC'} \langle S | \psi_A | i \rangle \tag{61a}$$

and

$$-\langle i | \psi_A \psi_B \psi^\dagger_{C'} | \rangle \approx \rho^{(\bar{S})}_{C'B} \langle i | \psi_A | \bar{S} \rangle = \rho^{(S)}_{BC'} \langle i | \psi_A | \bar{S} \rangle, \tag{61b}$$

where we have inserted a complete set of states S in Eq. (61a) and truncated to the collective phonons. In Eq. (61b) we have inserted the even-mass reflection states $\bar{S}$ appropriate in the case of coupling to achieve the odd-mass reflection states with energy $-E^{(i)}$. The algebraic difficulties associated with commuting the ψ's and deleting the ground state from the summation are discussed in Refs. 1 and 15. These decompositions permit the definition of a quasibogolon-phonon amplitude

$$U^A_a U^{BC}_{bc} \rho^{ABC(\gamma)}_{abc} \equiv X^{(S)}_{bc} \rho^{(\gamma)}_{aS}, \tag{62}$$

where the unitary transformations U diagonalize a one-quasibogolon excitation, Eq. (40), or a two-quasibogolon excitation, Eq. (48).

Then with definition

$$H_{aSa'S'} \equiv X^{(S)}_{bc'} U^{BC'}_{bc'} U^A_a H^{CABA'B'C'}_{cabA'b'c'} U^{A'}_{a'} U^{B'C}_{b'c} X^{(S')}_{b'c}, \tag{63}$$

Eq. (60) becomes

$$\rho_{aS}^{(\gamma)}/A_{aS}(\omega_\gamma)=H_{aSa'S'}\rho_{a'S'}^{(\gamma)}, \tag{64}$$

with

$$A_{aS}(\omega_\gamma)=\sum_k \frac{Z^{(k)}}{\omega_\gamma - E_a^{(k)} - E^{(S)}}$$

and for the reflection states

$$\rho_{aS}^{(\bar\gamma)}/A_{aS}(-\omega_\gamma)=H_{aSa'S'}\rho_{a'S'}^{(\bar\gamma)}. \tag{65}$$

As expected, the poles $\bar\gamma$ are a reflection of the positive and we can choose any convenient phase factor. We take

$$\rho_{aS}^{(\gamma)}=\rho_{aS}^{(\bar\gamma)}.$$

The foregoing development shows why we chose to replace $L^{(21)}$ by the analytical expression for $L^{(3)}$ and subsequently develop a phonon-like factorization of the $L^{(3)}$ amplitudes. Namely

$$L^{(21)}=\frac{X_{bc'}^{(S)}\rho_{aS}^{(\gamma)}\rho_{a'S'}^{(\gamma)}X_{b'c}^{(S')}}{\omega-\omega_\gamma}+\cdots \tag{66a}$$

is similar to the analytic form for the free part of $L^{(21)}$:

$$-L^{(2)}G=\frac{X_{bc'}^{(S)}Z^{(k)}X_{b'c}^{(S)}}{\omega-E_a^{(k)}-E^{(S)}}+\cdots. \tag{66b}$$

Nevertheless, it was implicitly assumed that the quasiparticle and phonon are distinguishable: Even though $L^{(21)}$ is antisymmetric and $L^{(21)}$ includes exclusion principle violation (EPV) corrections via $H^{(21)}$, there will be double counting and consequently normalization difficulties when more than one phonon is considered. A simple example is that the configuration $a=a, S=(bc)+\cdots$ has nonzero overlap with $a'=b, S'=(ac)+\cdots$. In the present development, it is assumed that this does not give rise to large errors.

Equation (64) prescribes that an amplitude for a particle-plus-phonon configuration is connected to other amplitudes of the same type by a higher-order sort of effective interaction. Following the development of Ref. 1 this coupling equation can be transcribed to a matrix-diagonalization problem. If we define

$$\rho_{kS}^{(\gamma)}\equiv\frac{(z_a^{(k)})^{1/2}\rho_{aS}^{(\gamma)}}{(\omega_\gamma-E_a^{(k)}-E^{(S)})A_{aS}} \tag{67}$$

Eq. (64) may be rewritten

$$(\omega_\gamma-E_a^{(k)}-E^{(S)})\rho_{kS}^{(\gamma)}=(z_a^{(k)}z_{a'}^{(k')})^{1/2}H_{aSa'S'}\rho_{k'S'}^{(\gamma)}. \tag{68}$$

A transcription of Eq. (65) is similarly accomplished. If we take H to be independent of the natural energy variable (a rather drastic approximation since it is at least second order in the I block and therefore retarded) and make use of the fact that it is symmetric,

$$H_{aSa'S'}=H_{a'S'aS}, \tag{69}$$

it is clear that Eq. (68) is a Hermitian matrix problem and that the following unitary transformation holds

$$\tilde R H\tilde R^{-1}=\delta_{\gamma\gamma'}\omega_\gamma,$$

where

$$\tilde R^{-1}=\begin{pmatrix}\rho_{kS}^{(\gamma)} & \rho_{kS}^{(\gamma')}\cdots\\ \rho_{k'S'}^{(\gamma)} & \rho_{k'S'}^{(\gamma')}\cdots\end{pmatrix} \tag{70}$$

and

$$\langle kS|H|k'S'\rangle\equiv\delta_{kk'}\delta_{SS'}(E_a^{(k)}+E^{(S)}) + (z_a^{(k)}z_a^{(k')})^{1/2}H_{aSa'S'}. \tag{71}$$

The normalization of Eq. (67) is

$$1=\sum_k|\rho_{kS}^{(\gamma)}|^2 \tag{72}$$

subject to the reservations discussed in connection with Eq. (66).

3. *Diagonalization for the solutions of Dyson's equation*

At a pole of the two-point function, $\omega\approx E_d^{(i)}$, Dyson's equation (17) reduces to the homogeneous equation

$$\langle|\psi_d^k|i\rangle=(G_d^0)^{KD'}M_d^{D'D}\langle|\psi_d^D|i\rangle, \tag{73}$$

where we have already incorporated the assumption that G and M are diagonal in the single-particle representation. Multiplying both sides by the unitary matrix U_d, and inserting U^+U at appropriate points, we obtain

$$\begin{pmatrix}\psi_d\\ \phi_d\end{pmatrix}=U\begin{pmatrix}\langle|\psi_d|i\rangle\\ \langle|\psi_d^\dagger|i\rangle\end{pmatrix} = \begin{pmatrix}(\omega-E_d+i\delta)^{-1} & 0\\ 0 & (\omega+E_d-i\delta)^{-1}\end{pmatrix} \times(UMU^\dagger)\begin{pmatrix}\psi_d\\ \phi_d\end{pmatrix} \tag{74}$$

which may be written

$$(\omega-E_d)\psi_d=(UMU^\dagger)^{11}\psi_d+(UMU^\dagger)^{12}\phi_d, \tag{74a}$$

$$(\omega+E_d)\phi_d=(UMU^\dagger)^{21}\psi_d+(UMU^\dagger)^{22}\phi_d. \tag{74b}$$

As M is energy-dependent this is not a convenient formulation of the problem. Recall the analytic behavior of the mass operator, Eq. (20), and also that the poles ω_γ occur in pairs $(\omega_\gamma, -\omega_\gamma)$. Then

$$(UMU^{\dagger})^{D'D} = \sum_{\omega_\gamma > 0} \frac{\tilde{\sigma}^{D'(\gamma)}\tilde{\sigma}^{D(\gamma)}}{\omega - \omega_\gamma} + \frac{\tilde{\sigma}^{D'(\bar{\gamma})}\tilde{\sigma}^{D(\bar{\gamma})}}{\omega + \omega_\gamma}$$

with

$$\tilde{\sigma}^{D'(\gamma)} \equiv U^{D'K'}\sigma^{K'(\gamma)}. \tag{75}$$

If we define (for $\omega = E^{(i)}$)

$$C_\gamma^{(i)}(\omega - \omega_\gamma) \equiv \tilde{\sigma}^{1(\gamma)}\psi_d + \tilde{\sigma}^{2(\gamma)}\phi_d,$$
$$C_{\bar{\gamma}}^{(i)}(\omega - \omega_\gamma) \equiv \tilde{\sigma}^{1(\bar{\gamma})}\psi_d + \tilde{\sigma}^{2(\bar{\gamma})}\phi_d,$$

and assume that we have chosen phase conventions such that all matrix elements are real, then Eq. (75) becomes the symmetric matrix problem

$$\begin{bmatrix} E_d & \tilde{\sigma}^{1(\gamma)} & 0 & \tilde{\sigma}^{1(\bar{\gamma})} \\ \tilde{\sigma}^{1(\gamma)} & \omega_\gamma & \tilde{\sigma}^{2(\gamma)} & 0 \\ 0 & \tilde{\sigma}^{2(\gamma)} & -E_d & \tilde{\sigma}^{2(\bar{\gamma})} \\ \tilde{\sigma}^{1(\bar{\gamma})} & 0 & \tilde{\sigma}^{2(\bar{\gamma})} & -\omega_\gamma \end{bmatrix} \begin{bmatrix} \psi_d^{(i)} \\ C_\gamma^{(i)} \\ \phi_d^{(i)} \\ C_{\bar{\gamma}}^{(i)} \end{bmatrix} = \omega \begin{bmatrix} \psi_d^{(i)} \\ C_\gamma^{(i)} \\ \phi_d^{(i)} \\ C_{\bar{\gamma}}^{(i)} \end{bmatrix}. \tag{76}$$

Since it can be shown that

$$\tilde{\sigma}^{2(\bar{\gamma})} = \tilde{\sigma}^{1(\gamma)}, \qquad \tilde{\sigma}^{2(\gamma)} = \tilde{\sigma}^{1(\bar{\gamma})} \tag{77}$$

the eigenvalues of Eq. (76) occur in pairs $(E^{(i)}, -E^{(i)})$ as expected. With the assumption that they are energy-independent, the matrix (called S) is diagonalized by a unitary matrix $\tilde{\tilde{R}}$, constructed from the eigenvectors:

$$\tilde{\tilde{R}}S\tilde{\tilde{R}}^{-1} = \omega. \tag{78}$$

The physical significance of the $\psi_d^{(i)}$, previously established in Eqs. (40) and (41), is valid only for the normalization

$$1 = \sum_{\gamma > 0} |C_\gamma^{(i)}|^2 + |C_{\bar{\gamma}}^{(i)}|^2 + |\psi_d^{(i)}|^2 + |\phi_d^{(i)}|^2. \tag{79}$$

4. *Final SCQPC matrix*

We have thus far written down two matrix problems. The first, diagonalized by $\tilde{R}$, combines quasiparticle-plus-phonon configurations to obtain "dressed" modes corresponding to poles of the mass operator. In the second problem, which is diagonalized by $\tilde{\tilde{R}}$, these modes are coupled to the quasiparticle degree of freedom. It is convenient to define an H_{QPC}, which directly couples the quasiparticles to the quasiparticle-plus-phonon configurations:

$$H_{\mathrm{QPC}} = \begin{bmatrix} E_d & \langle d|H|kS\rangle & \cdots & 0 & \langle d|H|\bar{k}\bar{S}\rangle & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \langle d|H|k'S\rangle & \langle k'S'|H|kS\rangle & \cdots & \langle d|H|\bar{k}\bar{S}\rangle & 0 & \cdots \\ 0 & \langle d|H|\bar{k}\bar{S}\rangle & \cdots & -E_d & -\langle d|H|kS\rangle & \cdots \\ \cdot & \cdot & & \cdot & \cdot & \\ \cdot & \cdot & & \cdot & \cdot & \\ \cdot & \cdot & & \cdot & \cdot & \\ \langle d|H|\bar{k}\bar{S}\rangle & 0 & \cdots & -\langle d|H|k'S'\rangle & -\langle k'S'|H|kS\rangle & \cdots \end{bmatrix} \tag{80}$$

with $\langle k'S'|H|kS\rangle$ as defined in Eq. (71),

$$\langle d|H|kS\rangle = U_d^{1D} W_{abdc}^{ABDC} (U_a^{A1})^{\dagger} (U_{bc}^{BC})^{\dagger} X_{bc} z_a^{(k)},$$

and

$$\langle d|H|\bar{k}\bar{S}\rangle = U_d^{1D} W_{abdc}^{ABDC} (U_a^{A2})^{\dagger} (U_{bc}^{BC})^{\dagger} X_{cb} z_a^{(k)}.$$

These three matrix elements represent the higher-order interaction between quasiparticle-plus-phonon configurations and the forward and backward QPC matrix elements, respectively. The matrix H_{QPC} may be diagonalized by applying successive transformations: first, rows 2 through n, where $n-1$ equals the number of configurations (kS), are diagonalized by $\tilde{R}$; the remaining problem is just that given by Eq. (78). Symbolically we may write

$$RH_{\mathrm{QPC}}R^{-1} = \pm E^{(i)}, \tag{81}$$

where

$$R = \tilde{\tilde{R}}\tilde{R}$$

and we denote the elements of R as

$$R^{-1} = \begin{bmatrix} \psi_d^{(i)} & \cdots & \phi_d^{(i)} & \cdots \\ \psi_{kS}^{(i)} & \cdots & \phi_{kS}^{(i)} & \cdots \\ \phi_d^{(i)} & \cdots & \psi_d^{(i)} & \cdots \\ \phi_{kS}^{(i)} & \cdots & \psi_{kS}^{(i)} & \cdots \end{bmatrix}.$$

Because of the analytical properties of $\rho_{aS}^{(\nu)}$, we may identify the elements of R with the quasiparticle-plus-phonon amplitudes discussed previously:

$$\sum_k (z_a^{(k)})^{1/2}\psi_{kS}^{(i)} = U^{1A}\langle S|\psi_a^A|i\rangle \equiv \psi_{aS}^{(i)}$$

and (82)

$$\sum_k (z_a^{(k)})^{1/2}\phi_{kS}^{(k)} = U^{1A}\langle i|\psi_a^A|\bar{S}\rangle \equiv \phi_{aS}^{(i)}.$$

The normalization of the amplitudes follows from the unitarity of $\tilde{R}$ and $\tilde{\tilde{R}}$; hence,

$$|\psi_d^{(i)}|^2 + |\phi_d^{(i)}|^2 + \sum_{kS}(|\psi_{kS}^{(i)}|^2 + |\phi_{kS}^{(i)}|^2) = 1. \quad (83)$$

Equations (80)–(83) constitute the odd-mass part of the SCQPC problem. The resultant residues and eigenvalues are inputs to the same problem and also the even-mass equations, which determine the phonon. This means that we have two iteration procedures: Solutions for different spin states of the odd-mass nuclei must be self-consistent, and an outer self-consistency must be achieved with the even-mass solutions.

V. DISCUSSION

The preceding sections have extended the self-consistent approach to quasiparticle-phonon coupling to superfluid nuclei. The theory for normal systems and a comparison to a number of other QPC developments have been presented in a previous work.[1]

In Sec. II we discussed why QPC and pairing correlations cannot be treated on the same footing. In a superfluid system, a number nonconserving assumption is a technological necessity. However, once this is accepted, equations of motion for a supermatrix of normal and anomalous Green's functions may be formulated, and the mass operator for a single nucleon may be related to the six-point (or pph) response function.

In Sec. IV the six-point function was approximated to yield a QPC description. It was also shown how a Bethe-Salpeter equation for the phonons is handled when the quasiparticle description is improved by the QPC mass operator. For odd-mass nuclei (and therefore the mass operator), the implications of our development are most easily seen by an inspection of Eq. (80), the final matrix diagonalization for odd-A energies and spectroscopic amplitudes. Note that it has the reflection property (similar to RPA) that positive diagonal energies are coupled to negative diagonal energies of the same magnitude.

As expected from the Migdal-Gorkov assumption, there is a positive (physical) set of eigenvalues and an identical set of negative (unphysical) eigenvalues. The forward amplitude $\psi_d^{(i)}$ is a linear combination of particle and hole amplitudes. Although we have never explicitly introduced a canonical transformation from particles to BCS quasiparticles or "bogolons," $\psi_d^{(i)}$ corresponds to $\langle i|\alpha_d^\dagger|\rangle$ where $\alpha_d^\dagger$ is a bogolon:

$$\alpha_d^\dagger = u_d\psi_d - v_d s_d \psi_{-d}. \quad (84)$$

Similarly, the backward amplitude $\phi_d^{(i)}$ corresponds to $\langle i|\alpha_{-d}|\rangle$. The amplitudes $\psi_{kS}^{(i)}$ come from the coupling of a quasiparticle a, which has several poles k, to the phonon S. If we had neglected the self-consistency of the two-point function there would have been only one $\psi_{kS}^{(i)} = \psi_{aS}^{(i)} = \langle i|\alpha_a^\dagger|S\rangle$. Note that all matrix elements involving amplitudes are multiplied by the single-particle strength of the state k, i.e., $(z_a^{(k)})^{1/2}$. The interaction between $\psi_d^{(i)}$ and $\psi_{kS}^{(i)}$ is just the usual QPC [see Fig. 2(b)]. The interaction between $\psi_d^{(i)}$ and $\phi_{kS}^{(i)}$ or $\phi_d^{(i)}$ and $\psi_{kS}^{(i)}$ is the backward QPC [see Fig. 1(b)]. The interaction between the $\psi_{kS}^{(i)}$ comes from higher-order diagrams such as Figs. 1(a) and 1(c) which correct for EPV's and quasiparticle-phonon recoupling.

In order to carry out an SCQPC calculation, we need the following inputs: (i) the irreducible blocks I and J, as well as the "bare" interaction, which appears at the "one-three" vertices, and (ii) the unperturbed amplitudes u_a and v_a and the unperturbed bogolon energy E_a. The latter quantities might be determined from experiment or from a Migdal form of HFB, viz., Eq. (37). It should be noted that, to be consistent, the Green's functions or density matrices which appear in these equations are the exact ones. Hence, the calculation of the self-consistent field and the nuclear pairing should in principle be determined with a density matrix that has been corrected for QPC. Normally, this is not done; one assumes

$$M^{11} \gg M^{12} \text{ and } M^{12} > M_{\mathrm{QPC}}. \quad (85)$$

It is well known that for heavy nuclei the former assumption is quite reasonable. This means that HFB corresponds to HF plus BCS, because the nuclear pairing energy of the even-even nucleus is but a small fraction of the total binding energy, as computed from a variational calculation of the self-consistent field. For similar reasons, one might expect that

$$M^{11} \gg M^{\mathrm{QPC}}. \quad (86)$$

However, it is not clear that QPC does not renormalize pairing. With a constant pairing force and the usual Migdal-Gorkov assumption about the identity of N, $N\pm 2$ nuclei

$$\frac{\Delta}{G}=-\sum_b\sum_i\langle|\psi_b|i\rangle\langle i|\psi^\dagger_{-b}|\rangle s_b$$

$$=\sum_b\sum_i u_b^{(i)}v_b^{(i)}$$

from Eq. (40), we have

$$\frac{\Delta}{G}=\sum_b\sum_i\{u_b v_b([\psi^{(i)}]^2-[\phi^{(i)}]^2)+(v_b{}^2-u_b{}^2)\psi^{(i)}\phi^{(i)}\}$$

$$\approx\sum_b u_b v_b\sum_i([\psi^{(i)}]^2-[\phi^{(i)}]^2) \tag{87}$$

neglecting the term that changes sign at the Fermi surface.

This result illustrates a most important consequence of our theory, namely that forward QPC contributes nothing to the renormalization of nuclear pairing, because in this limit $\phi=0$ and the sum over physical states i of $|\psi^{(i)}|^2$ equals one. However, backward QPC ($\phi\neq 0$) has the effect of reducing the pairing field Δ. It is easy to understand this. Backward QPC is a consequence of ground state correlations, which are different from those induced by nuclear pairing, and the two interfere destructively.

One might in principle perform backward QPC calculations before calculating the pairing. The resultant energy levels of the N nuclei would be pushed apart (c.f., the matrix diagonalization in Ref. 1). This extra gap in the single-particle level scheme would then have the effect of reducing the calculated value for Δ. It turns out, however, that in the cases we have studied this effect is unimportant,[6,9] amounting to at most a 2% correction in Δ. Since this shows that pairing dominates backward QPC, we have further support for the small backward amplitude approximation (c.f., Sec. IV A), which neglects the redistribution of single-particle strength across the Fermi surface.

By way of comparison to the present work, not all of the QPC theories applicable to superfluid nuclei were discussed in Ref. 1. We also want to mention here various methods for the self-consistent treatment of even systems, which is by no means unimportant.

The most famous QPC equations for odd-mass superfluid nuclei are those of Kisslinger and Sorenson[2] and Soloviev.[2] One can obtain their problem from Eq. (80) by neglecting the backward coupling and the interaction between phonon-plus-quasiparticle configurations (and therefore EPV corrections), as well as using unperturbed intermediates in the self-energy (no self-consistency for the $\psi_{as}^{(i)}$ amplitudes). Their residual interaction is the quadrupole-quadrupole force, and this permits a considerable reduction of the remaining algebra. The even nucleus is not treated self-consistently, rather by the usual RPA.

There have also been several shell-model calculations,[5,26,27] in which states in odd-mass superfluid nuclei are described as linear combinations of one- and three-quasiparticle excitations. Of course, if the QPC is really valid, it should fall out of the shell-model diagonalization. Since configurations which violate the exclusion principle are excluded automatically,[26] diagrams like Fig. 1(a) are implicitly included. Also the higher-order contributions to the effective six-point interaction [Fig. 9(b)] are included. In theory, if these diagrams were not negligible, their energy dependence might break up the phonon structure.

Kuo and Baranger[27] have given a matrix problem which includes backward [extended Tamm-Dancoff approximation (ETDA) and extended RPA (ERPA)] diagrams. ETDA includes processes such as Fig. 1(b), while ERPA permits backward amplitudes into the phonon. Of course, like all shell-model calculations, the quasiparticles are not self-consistently dressed by the QPC mass operator. Recently, a quasispin formalism has been developed by Kuriyama, Marumori, and Matsuyanagi[28] to explain odd-mass nuclei. Arguing that recoupling vertices [e.g., middle part of Fig. 1(a) or Fig. 10(a)] are larger than "one-three" vertices [e.g., Fig. 2(b)] by a factor of $2uv/u^2-v^2$, coupling to one-quasiparticle states is neglected. Their method yields a matrix problem similar to Kuo's ERPA, with ETDA neglected. In a calculation of anomalous $(j-1)$ coupling states, the authors find that the "dressed three-quasiparticle modes" which describe these states are essentially phonon-plus-quasiparticle configurations with EPV corrections.

Goswami and co-workers,[4] as well as Klein with his collaboraters,[3] have authored on-going theoretical efforts to describe even- and odd-mass nuclear states self-consistently. The algebraic framework for both theories is the equations-of-motion method combined with spectral decomposition. While the "bootstrap" method of Goswami works in the bogolon or BCS representation, the generalized Hartree-Fock approximation of Klein works in the particle representation, i.e., the QPC and BCS parts of the Hamiltonian are diagonalized at the same time. We have argued earlier in this section that QPC does not interfere substantially with BCS in realistic cases. In both of the above-mentioned methods, the interaction between phonon-plus-quasiparticle configurations is equivalent to several third-order graphs of Fig. 9, while the second-order graphs are ignored. Also, the intermediate quasiparticles are not self-consistent.

However, as discussed in Ref. 1, the main difference between SCQPC and the spectral decomposition approach is in the calculation of the backward amplitudes, namely that ours have diagonal energies $-E_a-E_S$, in contrast to their value $-E_a+E_S$. Not only does their treatment result in a matrix diagonalization which does not have eigenvalues in pairs $(E^{(i)}, -E^{(i)})$ — hence violating the spirit of the Migdal-Gorkov approximation and making the separation of physical states ambiguous — but it seems to require an extensive orthogonalization procedure. In calculations in which the backwards quasiparticle amplitudes were properly orthonormal, their effect on energies was absent.[29]

We also want to mention that the method of bosons which has recently been applied to odd vibrational nuclei. Of course, the introduction of the ideal quasiparticle and the coupling to ideal bosons by a transcribed Hamiltonian represents an alternative picture of nuclear structure, when compared to the Green's function or spectral decomposition approaches. The most important result of the boson methods that we have noted is to include[30] an expression equivalent to the EPV diagram, Fig. 1(a). An earlier calculation by Simard and Banville[31] was shown to yield results equivalent to Kuo's ETDA.

The method of boson expansion has been widely applied to even nuclei.[32,33] In the limit of strong quadrupole correlations, the boson expansion can even yield a Hamiltonian which transcribes a strongly deformed potential.[33] A number of calculations have focused on the less ambitious goal of predicting the static quadrupole moment (due to anharmonic corrections to RPA) of the lowest $2+$ state. However, it should be pointed out that, in the calculation of $2+$ moments, it is unnecessary to explicitly include anharmonic admixtures into the wave function. An RPA-type wave function is sufficient if the Migdal theory of effective operators is used to deduce the anharmonic enhancement of the quadrupole operator.[34]

The second modification of RPA that is of wide interest is the effect of explicitly including ground state correlations [i.e., Fig. 1(b)] on the one-nucleon Green's function and the consequent raising of the phonon energy or prevention of it becoming imaginary. There are two causes for this phenomenon when viewed through the equations of Sec. IV A: first, the two-body matrix elements are reduced by a multiplicative factor of the order $(1-z)^2$ where z is the quasiparticle strength redistributed from the $N+1$ to the $N-1$ nucleus by ground state correlations; secondly, the quasiparticle diagonal energies are pushed up from their unperturbed values. In the approaches of Hara[35] and also Ikeda,[35] the reductions in residue strength are taken from the RPA Y amplitudes. The renormalization of quasiparticle energies is not considered. The latter effect is included by Neergard and Vogel[36] who estimate the energy change from perturbation theory. Due to the simplicity of the quadrupole-quadrupole interaction, they are able to include additionally the effects of the small components of the quasiparticle strength. While we have included these components for normal systems in Ref. 1, we have neglected them here in the small-backward-amplitude approximation. The open-shell RPA[37] and Hartree-Fock theory for nuclear bosons,[38] as well as the boson expansion methods,[32-33] give expressions for the even-even energy that include both the renormalization of residues and quasiparticle energies. The results for several of these approaches have been compared in Refs. 6 and 38 for a simple model. It was found that they yield very similar results. In the SCQPC[6] there is a slight improvement due to self-consistency of the intermediate Green's functions in the mass operator.

In summary, the foregoing sections have presented a consistent treatment of Green's functions for even- and odd-A nuclei. Not only does the SCQPC provide a more rigorous theoretical foundation for the time-honored QPC description of nuclear structure and a straightforward prescription for higher-order calculations, it also can be understood fairly simply from the Feynmann diagram equivalents of the Green's functions or by an inspection of the configuration interaction implicit in the SCQPC matrix diagonalization.

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