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Soft photon nucleon-nucleon bremsstrahlung in the potential model*

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The soft photon theorem for bremsstrahlung has been derived nonrelativistically for two particles interacting via a nonlocal spin and isospin dependent potential. The result is in a simple form easily applicable to the special cases in which one or both particles have zero charge, spin, or isospin.

[NUCLEAR REACTIONS $N(N, N\gamma)$; derived nucleon-nucleon bremsstrahlung amplitude in soft photon limit, potential model, including nonlocality spin and isospin.]

I. INTRODUCTION

The soft photon theorem for bremsstrahlung was originally proved by Low for two spin-zero particles and for one spin-zero, one spin-one-half particle in the framework of quantum field theory.[1] According to the theorem, the two leading terms of the expansion of the bremsstrahlung amplitude in powers of the photon momentum depend only on the on-shell elastic scattering amplitude. The derivation was extended to the two-nucleon case by Nyman.[2] Feshbach and Yennie proved the theorem in the potential model for two spin-zero particles. Their approach avoided expansion of the T-matrix with respect to the energy, making the result better suited to the energy regions in which resonances occur in the elastic scattering. Their result, however, contained derivatives of the T matrix with respect to the scattering angle.[3] It was shown by Nyman that the angle derivatives can also be avoided if the result is expressed in terms of overlap integrals outside the range of the potential, which depend only on elastic scattering phase shifts.[4] Heller derived the theorem for local potentials in the case of two spin-zero as well as two spin-one-half particles.[5] Liou extended Heller's result to the case of nonlocal potentials,[6] as did Woloshyn.[7] It was shown by Liou and Sobel that the theorem also holds for isospin-dependent potentials, and they obtained an expression for the spinless case.[8] An excellent review of the nucleon-nucleon bremsstrahlung problem, including soft photon theorems, has been given by Nyman.[9]

The purpose of the present work is to include nonlocality, spin, and isospin simultaneously, while maintaining the simple appearance of the original results obtained by Low. We achieve this by (a) avoiding expansion of the T matrix with respect to either energy or angle, and (b) by avoiding the parametrization of the T matrix in terms of different scalar invariants in the final result. In addition, we demonstrate the complete equivalence of obtaining the bremsstrahlung amplitude either directly from the full electromagnetic interaction Hamiltonian or using gauge invariance to calculate the "internal" part of the amplitude from the singular "external" part.

II. DERIVATION OF BREMSSTRAHLUNG AMPLITUDE

Using the two-potential formula[10] with the electromagnetic interaction treated linearly, one gets for the bremsstrahlung amplitude in the center of mass frame

$$\langle \vec{p}_1'', \vec{p}_2'', \vec{k}, \hat{\epsilon} \,|\, \mathfrak{M} \,|\, \vec{p}_1', \vec{p}_2' \rangle = \delta(\vec{p}_1'' + \vec{p}_2'' + \vec{k}) \langle \vec{p}_1'', \vec{p}_2'' \,|\, \vec{M}\cdot\hat{\epsilon} \,|\, \vec{p}_1', \vec{p}_2' \rangle, \quad (1)$$

with

$$\vec{M}\cdot\hat{\epsilon} = (1 + t''g'')V_{em}(1 + g't'), \quad (2)$$

where the g's and t's are the usual propagators and elastic t matrices corresponding to the initial and final relative energies of the nucleons:

$$g' \equiv \frac{1}{E' - H_0 + i\epsilon}; \quad g'' \equiv \frac{1}{E'' - H_0 + i\epsilon}, \quad (3)$$

$$t' = V(1 + g't') = (1 + t'g')V;$$
$$t'' = V(1 + g''t'') = (1 + t''g'')V, \quad (4)$$

H_0 is the relative kinetic energy operator for the nucleons, and V their interaction potential. V_{em} is the interaction Hamiltonian with the electromagnetic field $\vec{A}(\vec{x})$, where

$$\vec{A}(\vec{x}) = \hat{\epsilon} e^{-i\vec{k}\cdot\vec{x}}, \quad \hat{\epsilon}\cdot\vec{k} = 0. \quad (5)$$

E' and E'' are the relative energies of the nucleons in the initial and final states, respectively, so that in the center of mass frame

$$E'=\frac{p'^2}{2m}=\frac{p_1'^2}{2m_1}+\frac{p_2'^2}{2m_2},$$
$$E''=\frac{p''^2}{2m}=\frac{p_1''^2}{2m_1}+\frac{p_2''^2}{2m_2}-\frac{k^2}{2(m_1+m_2)}, \tag{6}$$

$$\vec{p}'\equiv\frac{m_2\vec{p}_1'-m_1\vec{p}_2'}{m_1+m_2}=\vec{p}_1'=-\vec{p}_2',$$
$$\vec{p}''\equiv\frac{m_2\vec{p}_1''-m_1\vec{p}_2''}{m_1+m_2}=\vec{p}_1''+\frac{m}{m_2}\vec{k}=-\vec{p}_2''-\frac{m}{m_1}\vec{k}, \tag{7}$$
$$\vec{p}_1'+\vec{p}_2'=\vec{p}_1''+\vec{p}_2''+\vec{k}=0,\quad m=m_1m_2/(m_1+m_2).$$

The part of $\vec{M}\cdot\hat{\epsilon}$ which is singular at $k=0$ and corresponds to photon emission before and after scattering is

$$\vec{M}_{\text{ext}}\cdot\hat{\epsilon}=V_{\text{em}}^{(1)}g't'+t''g''V_{\text{em}}^{(1)}, \tag{8}$$

with

$$V_{\text{em}}^{(1)}=-ie\sum_{\alpha=1}^{2}\vec{A}(\vec{r}_\alpha)\cdot[H_0,Z_\alpha\vec{r}_\alpha]-\sum_{\alpha=1}^{2}\mu_\alpha\vec{\sigma}_\alpha\cdot\vec{\nabla}_\alpha\times\vec{A}(\vec{r}_\alpha)$$
$$=-\tfrac{1}{2}\hat{\epsilon}\cdot\sum_{\alpha=1}^{2}\left\{\frac{eZ_\alpha\vec{p}_\alpha}{m_\alpha}+i\mu_\alpha\vec{k}\times\vec{\sigma}_\alpha,\ e^{-i\vec{k}\cdot\vec{r}_\alpha}\right\}. \tag{9}$$

Here Z_α and μ_α are the charge and magnetic moment operators for nucleon α:

$$Z_\alpha=\tfrac{1}{2}(1+\tau_{\alpha 3});\quad \mu_\alpha=[\mu_n+(\mu_p-\mu_n)Z_\alpha], \tag{10}$$
$$\alpha=1,2,\quad \mu_p=2.793e/2m_p;\quad \mu_n=-1.913e/2m_p.$$

Note that since we are working with relative coordinates, the operators $\vec{r}_1,\vec{r}_2$ or $\vec{p}_1,\vec{p}_2$ are not independent:

$$\vec{p}_\alpha=(-1)^{\alpha+1}\vec{p},\quad \vec{r}_\alpha=(-1)^{\alpha+1}\frac{m}{m_\alpha}\vec{r},\quad \alpha=1,2, \tag{11}$$

where $\vec{r}$ and $\vec{p}$ are the relative position and momentum operators, respectively.

We may obtain the remaining part of $\vec{M}\cdot\hat{\epsilon}$, denoted by $\vec{M}_{\text{int}}\cdot\hat{\epsilon}$, in two equivalent ways. In the first method, we invoke charge conservation to calculate $\vec{M}_{\text{int}}\cdot\vec{k}$ and, relying on the analyticity of $\vec{M}_{\text{int}}$ as $\vec{k}\to 0$, read off the k^0 term of $\vec{M}_{\text{int}}$. The second method involves the straightforward calculation of $\vec{M}_{\text{int}}$ using an expression for $V_{\text{em}}^{(2)}$ valid to the same accuracy.

The first method assumes that the charge density operator for the system is known and is indeed localized at the position of the particles:

$$\rho(\vec{x})=\sum_{\alpha=1}^{n}eZ_\alpha\delta(\vec{x}-\vec{r}_\alpha), \tag{12}$$

where we have generalized the number of particles to n. Now, if in the formula

$$V_{\text{em}}=-\int\vec{j}(\vec{x})\cdot\vec{A}(\vec{x})d\vec{x} \tag{13}$$

we modify $\vec{A}(\vec{x})$ by changing $\hat{\epsilon}$ to $\vec{k}$, $\vec{A}$ becomes the gradient of $ie^{-i\vec{k}\cdot\vec{x}}$, and a partial integration on Eq. (13) followed by the use of charge-current continuity equation gives[11]

$$V_{\text{em}}(\hat{\epsilon}\to\vec{k})=-\int\frac{\partial\rho(\vec{x})}{\partial t}ie^{-i\vec{k}\cdot\vec{x}}d\vec{x}$$
$$=[H,\sum_{\alpha=1}^{n}eZ_\alpha e^{-i\vec{k}\cdot\vec{r}_\alpha}]$$
$$=[H,f], \tag{14}$$

where $H=H_0+V$ and

$$f\equiv\sum_{\alpha=1}^{n}eZ_\alpha e^{-i\vec{k}\cdot\vec{r}_\alpha}. \tag{15}$$

To find $\vec{M}\cdot\vec{k}$, we simply use Eq. (14) with $n=2$ in place of V_{em} in Eq. (2):

$$\vec{M}\cdot\vec{k}=(1+t''g'')[H,f](1+g't').$$

To get $\vec{M}_{\text{int}}\cdot\vec{k}$, we subtract from this $\vec{M}_{\text{ext}}\cdot\vec{k}$, which results from Eq. (8) and (9), namely,

$$\vec{M}_{\text{ext}}\cdot\vec{k}=[H_0,f]g't'+t''g''[H_0,f]. \tag{16}$$

The result is

$$\vec{M}_{\text{int}}\cdot\vec{k}=t''g''[H_0,f]g't'+(1+t''g'')[V,f](1+g't'), \tag{17}$$

where we have dropped the term $V_{\text{em}}^{(1)}$ from $\vec{M}\cdot\hat{\epsilon}$ because it does not contribute to radiation.

Before completing the calculation of $\vec{M}_{\text{int}}\cdot\hat{\epsilon}$, we demonstrate the equivalence of the two methods by writing the expression obtained for $\vec{M}_{\text{int}}\cdot\hat{\epsilon}$ using the second method. As shown by Ref. 12, the so-called minimal electromagnetic interaction Hamiltonian obtained through the gauge-invariant substitution

$$p_\alpha\to p_\alpha-eZ_\alpha\vec{A}(\vec{r}_\alpha),$$

together with the proper treatment of isospin dependence, gives

$$V_{\text{em}}^{(2)}\equiv V_{\text{em}}-V_{\text{em}}^{(1)}$$
$$=\left[V,-i\sum_{\alpha=1}^{2}eZ_\alpha\hat{\epsilon}\cdot\vec{r}_\alpha\right]+O(k)$$
$$=[V,h]+O(k), \tag{18}$$

where

$$h\equiv-i\sum_{\alpha=1}^{2}eZ_\alpha\hat{\epsilon}\cdot\vec{r}_\alpha. \tag{19}$$

We may also write the $\vec{k}\to 0$ limit of Eq. (9):

$$V_{\text{em}}^{(1)}=[H_0,h]+O(k). \tag{20}$$

Using Eq. (18) and (20), we find directly

$$\vec{M}_{\text{int}}\cdot\hat{\epsilon}=t''g''[H_0,h]g't'+(1+t''g'')[V,h](1+g't')+O(k), \tag{21}$$

which is identical to Eq. (17) except for the appearance of h instead of f within the commutators. As a matter of fact, when Eq. (15) for f is expanded in powers of k, the constant term gives zero in Eq. (17) by charge conservation, and the leading term comes from

$$f=-i\sum_{\alpha=1}^{2} eZ_\alpha\vec{k}\cdot\vec{r}_\alpha+O(k^2), \tag{22}$$

identical to Eq. (19) except for the substitution $\hat{\epsilon}\to\vec{k}$. Hence, the two methods, i.e., evaluation of Eq. (17) and Eq. (21), are indeed identical, and we only proceed with the former.

We use Eq. (3) to evaluate the first term and Eq. (4) to evaluate the second term in Eq. (17):

$$\begin{aligned}\vec{M}_{\rm int}\cdot\vec{k}&=-t''fg't'+t''g''E''fg't'+t''g''ft'\\&\quad-t''g''fE'g't'+t''f(1+g't')-(1+t''g'')ft'\\&=t''f-ft'+(E''-E')t''g''fg't'.\end{aligned} \tag{23}$$

Now, in the third term, $(E''-E')$ is already of order k, and we may use

$$f=\sum_{\alpha=1}^{2} eZ_\alpha+O(k), \tag{24}$$

which, being the total charge operator, can be commuted to the left. The first two terms may be written

$$t''f-ft'=[\tfrac{1}{2}(t'+t''),f]+\{\tfrac{1}{2}(t''-t'),f\}. \tag{25}$$

Here, in the commutator term, the leading term in f does not contribute, and we use Eq. (22) for f, whereas in the anticommutator term $t''-t'$ is already of order k, and we use Eq. (24). We get

$$t''f-ft'=\left[\frac{t'+t''}{2},\ -i\sum_{\alpha=1}^{2} eZ_\alpha\vec{k}\cdot\vec{r}_\alpha\right]+\left\{\frac{t''-t'}{2},\sum_{\alpha=1}^{2}eZ_\alpha\right\}+O(k). \tag{26}$$

Finally, we note that the third term in Eq. (23) is exactly equal to $(t'-t'')f$ by virtue of the Lippmann-Schwinger equations (4), and cancels the second term in the above equation as expected from the discussion preceding Eq. (22). Hence

$$\vec{M}_{\rm int}\cdot\vec{k}=\left[\frac{t'+t''}{2},\ -i\sum_{\alpha=1}^{2}eZ_\alpha\vec{k}\cdot\vec{r}_\alpha\right]+O(k^2) \tag{27}$$

or

$$\vec{M}_{\rm int}\cdot\hat{\epsilon}=-i\hat{\epsilon}\cdot\left[t',\ \sum_{\alpha=1}^{2}eZ_\alpha\vec{r}_\alpha\right]+O(k), \tag{28}$$

where we have again used the fact that the difference $t'-t''$ is of order k. We add Eq. (8) and (28) to get the total bremsstrahlung matrix:

$$\vec{M}\cdot\hat{\epsilon}=V^{(1)}_{\rm em}g't'+t''g''V^{(1)}_{\rm em}-i\hat{\epsilon}\cdot\left[t',\ \sum_{\alpha=1}^{2}eZ_\alpha\vec{r}_\alpha\right]+O(k), \tag{29}$$

with

$$V^{(1)}_{\rm em}=-\hat{\epsilon}\cdot\sum_{\alpha=1}^{2}\left(\frac{eZ_\alpha\vec{p}_\alpha}{m_\alpha}+i\mu_\alpha\vec{k}\times\vec{\sigma}_\alpha\right)e^{-i\vec{k}\cdot\vec{r}_\alpha}. \tag{30}$$

We recognize in the third term of Eq. (29) the commutator of the t matrix with the total electric dipole moment operator in the center of mass system. Making use of Eq. (11) and charge conservation, we may write this term as

$$\begin{aligned}&-i\hat{\epsilon}\cdot\left[t',e\sum_{\alpha=1}^{2}Z_\alpha\vec{r}_\alpha\right]\\&=\frac{me\hat{\epsilon}}{2}\cdot\sum_{\alpha=1}^{2}\left(\left[\{t',i\vec{r}\},\frac{(-1)^\alpha Z_\alpha}{m_\alpha}\right]+\left\{[t',i\vec{r}],\frac{(-1)^\alpha Z_\alpha}{m_\alpha}\right\}\right).\end{aligned} \tag{31}$$

Taking the matrix element of Eq. (29) with respect to the relative-momentum plane-wave final and initial states, we get

$$\begin{aligned}\langle\vec{p}_1'',\vec{p}_2''|\vec{M}|\vec{p}_1',\vec{p}_2'\rangle=\langle\vec{p}''|\vec{M}|\vec{p}'\rangle=\sum_{\alpha=1}^{2}\Big\{&\vec{J}_\alpha''\langle\vec{P}_\alpha''|t'|\vec{p}'\rangle+\langle\vec{p}''|t''|\vec{P}_\alpha'\rangle\vec{J}_\alpha'\\&+\frac{(-1)^\alpha me}{2m_\alpha}\Big([\langle\vec{p}''|\{t',i\vec{r}\}|\vec{p}'\rangle,Z_\alpha]+\{\langle\vec{p}''|[t',i\vec{r}]|\vec{p}'\rangle,Z_\alpha\}\Big)\Big\}+O(k),\end{aligned} \tag{32}$$

where

$$\vec{J}_\alpha''\equiv\frac{(-1)^\alpha(2m/m_\alpha)eZ_\alpha\vec{p}''-2im\mu_\alpha\vec{k}\times\vec{\sigma}_\alpha}{p'^2-P_\alpha''^2}, \tag{33}$$

$$\vec{J}_\alpha'\equiv\frac{(-1)^\alpha(2m/m_\alpha)eZ_\alpha\vec{p}'-2im\mu_\alpha\vec{k}\times\vec{\sigma}_\alpha}{p''^2-P_\alpha'^2}, \tag{34}$$

$$\vec{P}_\alpha''\equiv\vec{p}''+(-1)^{\alpha+1}\frac{m}{m_\alpha}\vec{k}=(-1)^{\alpha+1}(\vec{p}_\alpha''+\vec{k}), \tag{35}$$

$$\vec{P}_\alpha'\equiv\vec{p}'+(-1)^\alpha\frac{m}{m_\alpha}\vec{k}=(-1)^{\alpha+1}\left(\vec{p}_\alpha'-\vec{k}+\frac{m_\alpha}{m_1+m_2}\vec{k}\right). \tag{36}$$

We simplify the last two terms of Eq. (32) by writing

$$\begin{aligned}\langle\vec{p}''|\{t',i\vec{r}\}|\vec{p}'\rangle&=(\vec{\nabla}_{p'}-\vec{\nabla}_{p''})\langle\vec{p}''|t'|\vec{p}'\rangle\\&=-2\vec{\nabla}_q\langle\vec{p}''|t'|\vec{p}'\rangle,\end{aligned} \tag{37}$$

$$\langle \vec p''|[t',i\vec r]|\vec p'\rangle = (\vec\nabla_{p'}+\vec\nabla_{p''})\langle \vec p''|t'|\vec p'\rangle = \vec\nabla_Q\langle \vec p''|t'|\vec p'\rangle\,, \tag{38}$$

where $\vec q$ and $\vec Q$ are defined as follows:

$$\vec q \equiv \vec p'' - \vec p'\,,\quad \vec Q \equiv \tfrac{1}{2}(\vec p''+\vec p')\,, \tag{39}$$

and the gradient with respect to each is taken with the other kept constant.

The remaining task is to expand the half-off-shell t matrices occurring in the first two terms about suitably chosen on-shell "points" and to demonstrate that all the off-shell derivatives cancel against similar terms coming from the last two terms. Rather than expanding both t matrices about the same (average) value of the energy,[1,5,6,8] we shall be guided by the method of Ref. 3, which avoids an expansion with respect to the energy of the on-shell "leg" of the t matrices. This method makes the t matrices on shell by changing only the magnitude of the off-shell momentum. In this way, at least the singular part of the result remains valid when the energy lies in a resonance region whose width is comparable or smaller than the photon energy. In this method, the energy derivatives are replaced by angle derivatives in the final answer. We also avoid angle derivatives in the final answer by modifying *both* the on-shell and the off-shell relative momenta in such a way as to keep the momentum transfer vector unchanged. This can be done by adding a vector $\vec\Delta$ of order k to both the initial and final relative momenta of each off-shell t matrix. To leave the magnitude of the on-shell momentum unchanged (within order k^2), $\vec\Delta$ must be perpendicular to it, and to keep the plane of scattering intact, $\vec\Delta$ must be in the plane of scattering. These requirements uniquely determine $\vec\Delta$. We write

$$\langle \vec P''_\alpha|t'|\vec p'\rangle = \langle \vec P''_\alpha+\vec\Delta''_\alpha|t'|\vec p'+\vec\Delta''_\alpha\rangle - \vec\Delta''_\alpha\cdot\vec\nabla_Q\langle \vec p''|t'|\vec p'\rangle + O(k^2)\,, \tag{40}$$

$$\langle \vec p''|t''|\vec P'_\alpha\rangle = \langle \vec p''+\vec\Delta'_\alpha|t''|\vec P'_\alpha+\vec\Delta'_\alpha\rangle - \vec\Delta'_\alpha\cdot\vec\nabla_Q\langle \vec p''|t'|\vec p'\rangle + O(k^2)\,, \tag{41}$$

where

$$\vec\Delta''_\alpha \equiv \frac{p'^2-P''^2_\alpha}{2(\vec p''\times\vec p')^2}\,\vec p'\times(\vec p''\times\vec p')\,,\qquad \vec\Delta'_\alpha \equiv \frac{P'^2_\alpha-p''^2}{2(\vec p''\times\vec p')^2}\,\vec p''\times(\vec p''\times\vec p')\,. \tag{42}$$

The first t matrices on the right-hand side of Eqs. (40) and (41) are now fully on shell, and the second ones can be replaced by on-shell t matrices *after* taking the gradients, as could those in Eqs. (37) and (38), without affecting the two leading terms of the final answer. Substituting Eqs. (37), (38), (40), and (41) into Eq. (32), we will have four gradient terms to simplify. These simplify as follows when Eqs. (39) and (42) are used:

$$\frac{-2Z_\alpha\vec p''}{p'^2-P''^2_\alpha}\,\vec\Delta''_\alpha\cdot\vec\nabla_Q\langle t\rangle + \vec\Delta'_\alpha\cdot\vec\nabla_Q\langle t\rangle\,\frac{-2Z_\alpha\vec p'}{p''^2-P'^2_\alpha} + \left\{\frac{Z_\alpha}{2},\vec\nabla_Q\langle t\rangle\right\} + [Z_\alpha,\vec\nabla_q\langle t\rangle]$$
$$= \left\{\frac{Z_\alpha}{2},\hat n(\hat n\cdot\vec\nabla_Q)\langle t\rangle\right\} + \left[\frac{Z_\alpha}{2},\left(2\vec\nabla_q - \frac{2\vec Q}{q^2}\,\vec q\cdot\vec\nabla_Q - \frac{\vec q}{2Q^2}\,\vec Q\cdot\vec\nabla_Q\right)\langle t\rangle\right]\,, \tag{43}$$

where

$$\langle t\rangle \equiv \langle \vec p''|t'|\vec p'\rangle\,; \tag{44}$$

$$\hat n \equiv \frac{\vec q\times\vec Q}{|\vec q\times\vec Q|} = \frac{\vec p''\times\vec p'}{|\vec p''\times\vec p'|}\,. \tag{45}$$

To simplify Eq. (43) further and to make it apparent that no off-shell derivatives survive, we resort to the rotation invariance of the t matrix to write, temporarily,

$$\langle \vec p''|t(E)|\vec p'\rangle = \sum_n A_n t_n\,, \tag{46}$$

where the t_n are functions of $Q^2+q^2/4$, q^2, $\vec q\cdot\vec Q$, E, $\vec\tau_1$, and $\vec\tau_2$ (the last three not exhibited):

$$t_n = t_n(Q^2+q^2/4,\,q^2,\,\vec q\cdot\vec Q)\,. \tag{47}$$

An on-shell t matrix is characterized by

$$Q^2+q^2/4 = 2mE\,,\quad \vec q\cdot\vec Q = 0\,. \tag{48}$$

The A_n have the general forms

$$1,\ \vec\sigma_\alpha\cdot\hat L_i\,,\ \text{ or }\ (\vec\sigma_1\cdot\hat L_i)(\vec\sigma_2\cdot\hat L_j)\,, \tag{49}$$

where $\hat L_i$ stands for any one of $\hat q$, $\hat Q$, or $\hat n$.[5] All the A_n satisfy the identities

$$(\hat n\cdot\vec\nabla_Q)A_n = \frac{1}{2iQ}\,[(\vec\sigma_1+\vec\sigma_2)\cdot\hat q, A_n]\,, \tag{50}$$

$$\left(\vec\nabla_q - \frac{\vec Q}{q^2}\,\vec q\cdot\vec\nabla_Q\right)A_n = \frac{1}{2iq}\,[(\vec\sigma_1+\vec\sigma_2)\times\hat q, A_n] \tag{51}$$

valid on shell, i.e., when $\vec q\cdot\vec Q=0$, which is true within order k.

The expression for the bremsstrahlung amplitude is arrived at by carefully evaluating Eq. (43) with the help of Eqs. (44) through (51) and adding the result to the nonderivative terms. It involves only the on-shell T matrix, which is a spin and isospin dependent function of energy, momentum transfer, and scattering plane, denoted by

$$T = T(E, \vec{q}, \hat{n}). \tag{52}$$

The final answer, valid in the center of mass frame, is

$$\begin{aligned}\langle \vec{p}_1'', \vec{p}_2'', \vec{k}, \hat{\epsilon} | \mathfrak{M} | \vec{p}_1', \vec{p}_2' \rangle &= \delta(\vec{p}_1'' + \vec{p}_2'' + \vec{k}) \hat{\epsilon} \circ \Bigg(\sum_{\alpha=1}^{2} [\vec{J}_\alpha'' \, T(E', \vec{q}_\alpha, \hat{n}'_\alpha) + T(E'', \vec{q}_\alpha, \hat{n}_\alpha'') \vec{J}'_\alpha] + \frac{em\hat{n}}{4Q} \left\{ \frac{Z_2}{m_2} - \frac{Z_1}{m_1}, [T, \; i(\vec{\sigma}_1 + \vec{\sigma}_2) \cdot \hat{q}] \right\} \\ &\quad + \frac{e}{2} \left[Z_2 - Z_1, \; \hat{q} \frac{\partial T}{\partial q} + \frac{1}{q} [T, i(\vec{\sigma}_1 + \vec{\sigma}_2) \times \hat{q}] \right] \Bigg) + O(k), \end{aligned} \tag{53}$$

where

$$\begin{aligned} \vec{J}_\alpha'' &= \frac{-2(m/m_\alpha) e Z_\alpha \vec{p}_\alpha'' - 2im\vec{k} \times \mu_\alpha \vec{\sigma}_\alpha}{p'^{\,2}_\alpha - (\vec{p}_\alpha'' + \vec{k})^2}, \\ \vec{J}'_\alpha &= \frac{-2(m/m_\alpha) e Z_\alpha \vec{p}'_\alpha - 2im\vec{k} \times \mu_\alpha \vec{\sigma}_\alpha}{\{\vec{p}_\alpha'' + [m_\alpha/(m_1+m_2)]\vec{k}\}^2 - \{\vec{p}'_\alpha - \vec{k} + [m_\alpha/(m_1+m_2)]\vec{k}\}^2}, \end{aligned} \tag{54}$$

$$\vec{q}_\alpha \equiv (-1)^\alpha (\vec{p}'_\alpha - \vec{p}_\alpha'' - \vec{k}), \quad \vec{q}_1 = \vec{p}_2' - \vec{p}_2'', \quad \vec{q}_2 = \vec{p}_1'' - \vec{p}_1', \tag{55}$$

$$\begin{aligned} \hat{n}'_\alpha &\equiv (-1)^\alpha \frac{\vec{p}'_\alpha \times (\vec{p}_\alpha'' + \vec{k})}{|\vec{p}'_\alpha \times (\vec{p}_\alpha'' + \vec{k})|}, \\ \hat{n}_\alpha'' &\equiv (-1)^\alpha \frac{\{\vec{p}'_\alpha - \vec{k} + [m_\alpha/(m_1+m_2)]\vec{k}\} \times \{\vec{p}_\alpha'' + [m_\alpha/(m_1+m_2)]\vec{k}\}}{|\{\vec{p}'_\alpha - \vec{k} + [m_\alpha/(m_1+m_2)]\vec{k}\} \times \{\vec{p}_\alpha'' + [m_\alpha/(m_1+m_2)]\vec{k}\}|} \end{aligned} \tag{56}$$

Here again, the terms under the summation are the pole terms as $k \to 0$, and the remaining terms go to a constant. That is why the arguments of the T matrices occurring in the latter are not specified. They may have the same arguments as any of the T matrices in the singular part or anything that differs by order k.

Note that the only surviving derivative of the T matrix occurs in the last term of (53), which term is nonzero only if the forces are isospin dependent. This derivative is with respect to the magnitude of momentum transfer, keeping the direction of momentum transfer, the energy, and the plane of scattering constant.

It should be pointed out that the only conditions imposed on the potential V to arrive at the result (53) are, translation, Gallilean, and rotation invariance, and charge conservation. The potential may have otherwise arbitrary nonlocality ($\vec{p}$ dependence) and spin and isospin dependence.[13]

Equation (53) can be evaluated from the elastic scattering data. For two nucleons, one uses the usual parametrization of the $T(E, \vec{q}, \hat{n})$ in terms of five operators:

$$1, \; (\vec{\sigma}_1 + \vec{\sigma}_2) \cdot \hat{n}, \; \vec{\sigma}_1 \cdot \vec{\sigma}_2, \vec{\sigma}_1 \cdot \hat{q} \vec{\sigma}_2 \cdot \hat{q}, \; \vec{\sigma}_1 \cdot \hat{n} \vec{\sigma}_2 \cdot \hat{n}, \tag{57}$$

the coefficients being functions of E, q^2, and $\vec{\tau}_1 \cdot \vec{\tau}_2$. Equation (53) needs to be sandwiched between the final and initial spin and isospin states, and then antisymmetrized. It sufficies to antisymmetrize only the final state with no $\sqrt{2}$ in the denominator. The masses must either be equal or isospin dependent for two nucleons.

For proton-proton bremsstrahlung $Z_2 - Z_1$ is zero, and the last two terms do not contribute. The singular terms also cancel to a large extent because the system has no electric dipole moment.

For isospin independent forces, the last term (commutator with $Z_2 - Z_1$) drops out, and for spin independent forces, the second term (anticommutator with $Z_2/m_2 - Z_1/m_1$) drops out.

Although Eq. (53) is derived for the nucleon-nucleon case, it applies to a large class of other cases as well. For example, if one or both particles have spin zero, the corresponding $\vec{\sigma}$ operator (s) is set to zero. We get for the simplest case, namely two spin-zero particles with only one having charge e,

$$\langle \vec{p}_1'', \vec{p}_2'', \vec{k}, \hat{\epsilon} | \mathfrak{M} | \vec{p}_1', \vec{p}_2' \rangle$$

$$= \frac{-2me}{m_1} \delta(\vec{p}_1'' + \vec{p}_2'' + \vec{k}) \hat{\epsilon} \cdot \left[\frac{\vec{p}_1'' T(E', (\vec{p}_2'' - \vec{p}_2')^2)}{p_1'^2 - (\vec{p}_1'' + \vec{k})^2} + \frac{\vec{p}_1' T(E'', (\vec{p}_2'' - \vec{p}_2')^2)}{[p_1'' + (m/m_2)\vec{k}]^2 - [\vec{p}_1' - \vec{k} + (m/m_2)\vec{k}]^2} \right] + O(k), \tag{58}$$

which further simplifies if the static approximation $k, p \ll m$ is made in the energy denominators. It then reduces to Low's equation (1.7 N.R.) if the T matrices are expanded about the average energy $\frac{1}{2}(E' + E'')$.

For nucleon-nucleus bremsstrahlung with an effective-potential interaction, we set Z_1 and Z_2 equal to the charges of the nucleon and the nucleus, respectively, which may be operators in isospin space. The ordinary spin dependence will carry through, however, only if the nucleus has a constant total spin of zero or one-half. The presence of resonances will not affect the accuracy of the singular term, but corrections of order k to the constant term can be large if the width of resonance is comparable or smaller than k.[3]

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