Separable expansions for local potentials with Coulomb interactions*

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If two particles are interacting via a short range potential and a repulsive Coulomb potential the t matrix can be written as a sum of the Coulomb and the "nuclear" t matrices. In order to solve the three-nucleon problem with Coulomb interactions usually we need a separable representation of this nuclear t matrix. A recently proposed method for finding a separable expansion for local potentials is here extended to find a rapidly convergent separable expansion, with analytic form factors, for the nuclear part of the t matrix of a local potential, in the presence of Coulomb interactions. The method is illustrated for a two-term Malfliet-Tjon potential. In each rank the nuclear phase shift is close to the corresponding phase shift when the Coulomb interaction is switched off.

[NUCLEAR REACTIONS Finite rank approximations to the t matrix of local potential considered in presence of Coulomb interactions, t matrices, phase shifts calculated at different energies.]

I. INTRODUCTION

Separable potentials and the corresponding t matrices are very interesting in practice because of the great simplicity they bring to the three-body problem. There are quite a few separable expansions already available for the short-range nuclear potentials[1-3] and some of them have been successfully used in three-body calculations.

Because of the long-range nature of Coulomb interactions it is not easy to find a separable representation for the pure Coulomb interactions.[4] However, in the presence of Coulomb interactions the t matrix for a local short-range potential can be written as a sum of the two terms—the "nuclear" and Coulomb t matrices. The nuclear t matrix t_{Cn} is defined by

$$t_{Cn} = (1 + V_C G_C)(V_n + V_n G V_n)(1 + G_C V_C),$$

where V_C and V_n are the Coulomb and nuclear potentials. Here G_C and G are the Coulomb and full Green's functions to be defined by Eq. (4). This nuclear t matrix is well behaved and we can find a rapidly convergent separable expansion for the nuclear t matrix. But it is this nuclear t matrix that is called for in the approximate formulation of the three-body problem in the presence of Coulomb interactions such as given in Ref. 5.

There has been some work in this direction and people considered (mostly Yamaguchi type) rank one potentials in the presence of Coulomb interactions.[6-8] But rank one potentials cannot accurately represent the actual t matrix for the problem. Here in this paper we develop a scheme for finding finite rank approximations to the nuclear t matrix.

In this paper we work with a screened Coulomb potential. The main difficulty in working with a screened Coulomb potential is that the phase shift does not have a definite limit [see Eq. (24)] as the screening radius goes to infinity. But in this approach the screened Coulomb phase shift gets cancelled and in the end, when we go to the limit of a pure Coulomb potential, the only limit we have to take is that of the screened Coulomb wave function, which converges to the pure Coulomb wave function.

A recently proposed method[1,2] for finding a separable expansion for the t matrix is here extended to the case of the nuclear t matrix. Hence as in the separable expansion of Refs. 1 and 2, the present separable expansion for the nuclear t matrix does not require the explicit solution of an eigenvalue problem or an integral equation. The present separable expansion of any rank N also satisfies exact two-particle unitarity and time reversal symmetry, because it is derived from a real Hermitian potential.

The present separable expansion has analytic form factors in momentum space for a special choice of expansion functions and for a linear combination of Yukawa potentials. The use of analytic form factors is particularly convenient if the expansion is to be used in applications such as the three-body problem, since it allows the contour rotation method to be applied very easily. In fact the only numerical operation required to find the rank N nuclear t matrix is the evaluation of N^2 one-dimensional integrals and the inversion of an $N \times N$ matrix.

In Sec. II we give the general formulation of the problem in the presence of Coulomb interactions

and the method for finding the separable expansion for the nuclear t matrix. In Sec. III we give the explicit partial wave forms for all the quantities concerned. In Sec. IV we report numerical results for the Malfliet-Tjon potential and finally in Sec. V we give a brief discussion of the method.

II. GENERAL FORMULATION

Let us consider two particles interacting via a short-range potential V_n and a Coulomb potential V_C, so that the total potential is given by[8]

$$V = V_n + V_C. \tag{1}$$

The Lippmann-Schwinger equation can be written as

$$t = V + VG_0t, \ = V + tG_0V, \tag{2}$$

where

$$G_0 = (s - H_0)^{-1} \tag{3}$$

is the two-particle free Green's function, with s the complex energy parameter and H_0 the free Hamiltonian.

If we define two more Green's functions by

$$G_C(s) = (s - H_0 - V_C)^{-1}$$

and (4)

$$G(s) = (s - H_0 - V_C - V_n)^{-1},$$

we have

$$1 + GV = (1 + GV_n)(1 + G_CV_C)$$

and (5)

$$1 + VG = (1 + V_CG_C)(1 + V_nG).$$

Then it is easy to see, with the help of Eq. (5), that the total t matrix defined by

$$t = V + VGV$$

can be written as

$$t = t_C + t_{Cn}, \tag{6}$$

where

$$t_C = V_C + V_CG_0t_C = V_C + t_CG_0V_C \tag{7}$$

is the Coulomb t matrix and t_{Cn} is the nuclear t matrix defined by

$$t_{Cn} = (1 + V_CG_C)t_n(1 + G_CV_C), \tag{8}$$

where t_n satisfies

$$t_n = V_n + V_nG_Ct_n \ = V_n + t_nG_CV_n. \tag{9}$$

It is to be noted that neither t_n nor t_{Cn} is the pure nuclear t matrix but t_C is the actual Coulomb t matrix.

In Eq. (9) V_n is the nuclear potential and if we have a finite rank representation for V_n, t_n can be found in a finite rank form. This has been done in Refs. 1 and 2 when G_C in Eq. (9) is replaced by G_0. The derivations of Refs. 1 and 2 can be easily extended to this case and we give a brief account of the derivations here.

We construct a separable expansion $V_n^{(N)}$ of rank N for the short-range local potential V_n, according to the prescription[9]

$$V_n^{(N)} = \sum_{\tau} \sum_{\tau'} V_n|f_\tau\rangle \Delta_{\tau\tau'}\langle f_{\tau'}|V_n, \tag{10}$$

where

$$(\Delta^{-1})_{\tau'\tau} = \langle f_{\tau'}|V_n|f_\tau\rangle. \tag{11}$$

The τ represents a complete set of labels for the real functions f_τ. As $V_n^{(N)}$ is real and Hermitian, the resulting t matrix will satisfy the correct two-body unitarity and time reversal symmetry conditions in any rank. The approximate t matrix $t_n^{(N)}$ of any rank N is then obtained by replacing V_n in Eq. (9) by $V_n^{(N)}$ and solving the equation for t_n. The solution is

$$t_n^{(N)} = \sum_{\tau} \sum_{\tau'} V_n|f\rangle D_{\tau\tau'}\langle f_{\tau'}|V_n \tag{12}$$

where

$$(D^{-1})_{\tau'\tau}^{(N)} = \langle f_{\tau'}|(V_n - V_nG_CV_n)|f_\tau\rangle. \tag{13}$$

The only difference with the pure nuclear case is that in the pure nuclear case V_C is zero and G_C in Eqs. (9), (12), and (13) is replaced by G_0. Equations (12) and (13) define the required separable expansion.

An alternative derivation of Eqs. (12) and (13) can be given starting from a modified form of the Schwinger variational principle[2] for the t matrix t_n in the Coulomb state representation

$$\langle \vec{p}_C'^{(-)}|t_n|\vec{p}_C^{(+)}\rangle = \langle \vec{p}_C'^{(-)}|V_n|\psi_{\vec{p}C}^{(+)}\rangle + \langle \psi_{\vec{p}'C}^{(-)}|V_n|\vec{p}_C^{(+)}\rangle - \langle \psi_{\vec{p}'C}^{(-)}|(V_n - V_nG_CV_n)|\psi_{\vec{p}'C}^{(+)}\rangle, \tag{14}$$

where

$$|\vec{p}_C^{(+)}\rangle = [1 + G_C(u)V_C]|\vec{p}\rangle = [1 - G_0(u)V_C]|\vec{p}\rangle$$

and (15)

$$\langle \vec{p}_C'^{(-)}| = \langle \vec{p}'|[1 + V_CG_C(u)] = \langle \vec{p}'|[1 - V_CG_0(u)]$$

are the outgoing and ingoing Coulomb states, where u [the complex energy parameter in Eq. (15)] is $(p^2/2\mu) + i\epsilon$, and the Coulomb potential is screened at a suitable very large distance. The expression (14) is stationary under variations of $\psi_{\vec{p}C}^{(+)}$ and $\psi_{\vec{p}'C}^{(-)}$,

the off-shell functions defined by

$$|\psi^{(+)}_{\vec{p}\,C}\rangle = |\vec{p}^{\,(+)}_C\rangle + G_C V_n |\psi^{(+)}_{\vec{p}\,C}\rangle,$$
$$\langle\psi^{(-)}_{\vec{p}\,'C}| = \langle\vec{p}^{\,(-)}_C| + \langle\psi^{(-)}_{\vec{p}\,'C}| V_n G_C. \tag{16}$$

The trial wave functions in the variational expression (14) are taken to be of the form

$$|\psi^{(+)}_{\vec{p}\,C}\rangle = \sum_\tau a_\tau(\vec{p})|f_\tau\rangle$$

and (17)

$$\langle\psi^{(1)}_{\vec{p}\,'C}| = \sum_\tau \langle f_\tau|b_\tau(\vec{p}').$$

To find a variational expression for t_n we substitute Eq. (17) in Eq. (14) and demand that the right hand side be stationary with respect to variations of $a_\tau(\vec{p})$ and $b_\tau(\vec{p}')$. In this way we can easily solve for $a_\tau(\vec{p})$ and $b_\tau(\vec{p}')$ and hence recover Eqs. (12) and (13).

III. EXPLICIT PARTIAL WAVE FORM

In this section we develop explicit partial wave expressions for the separable expansion, assuming that the potential for each partial wave is local. We follow the conventions and normalizations of Ref. 2. We also assume that the Coulomb interaction is repulsive so that it does not have any bound state.

We work in the outgoing and ingoing Coulomb state representation $|\vec{p}^{\,(\pm)}_C\rangle$, because G_C will be diagonal in this representation and the partial wave projection of Eq. (9) will be identical in appearance to an ordinary partial wave Lippmann-Schwinger equation.

The partial wave representation of Eq. (9) in the Coulomb representation is[6,10]

$$(p|T^{Cn}_L|p') = (p|V^{Cn(+)}_L|p') + \frac{2}{\pi}\int_0^\infty q^2 dq \frac{(p|V^{Cn(-)}_L|q)(q|T^{Cn}_L|p')}{(2\mu S/\hbar^2) - q^2}, \tag{18}$$

where μ is the reduced mass and where the partial wave t-matrix element $(p|T^{Cn}_L|p')$ is defined by

$$\langle\vec{p}^{\,(-)}_C|t_n|\vec{p}^{\,\prime(+)}_C\rangle = \frac{2}{\pi}\frac{\hbar^2}{2\mu}\sum_{L,M} Y_{LM}(\hat{p})Y^*_{LM}(\hat{p}')(p|T^{(n)}_L|p') \tag{19}$$

and the partial wave elements $(p|V^{Cn(\pm)}_L|p')$ are defined by

$$\langle\vec{p}^{\,(-)}_C|V_n|\vec{p}^{\,\prime(\pm)}_C\rangle = \frac{2}{\pi}\frac{\hbar^2}{2\mu}\sum_{L,M} Y_{LM}(\hat{p})Y^*_{LM}(\hat{p}') \times (p|V^{Cn(\pm)}_L|p'). \tag{20}$$

It is to be noted that Eq. (18) is identical in appearance to an ordinary Lippmann-Schwinger equation.

With this normalization the on-shell Coulomb and the total t matrices are given by[6,10]

$$(k|T^C_L|k) = -\frac{1}{k}e^{i\sigma^S_L(k)}\sin\sigma^S_L(k) \tag{21}$$

and

$$(k|T^{Cn}_L|k) + (k|T^C_L|k) = -\frac{1}{k}e^{i\delta_L(k)}\sin\delta_L(k), \tag{22}$$

where the energy is $\hbar^2k^2/2\mu + i\epsilon$ and where $(k|T^C_L|k)$ is the partial wave component of t_C defined by

$$\langle\vec{p}|t_C|\vec{p}'\rangle = \frac{2}{\mu}\frac{\hbar^2}{2\mu}\sum_{L,M} Y_{LM}(\hat{p})Y^*_{LM}(\hat{p}')(p|T^C_L|p') \tag{23}$$

and where $\delta_L(k)$ is the total phase shift. Here $\sigma^S_L(k)$ is the phase shift for the screened Coulomb potential and is related to the Coulomb phase shift $\arg\Gamma(L+1+i\eta)$ by

$$\sigma^S_L(k) = \arg\Gamma(L+1+i\eta) - \eta\ln 2kR, \tag{24}$$

where $\eta = \mu e^2/\hbar^2 k$ and e is the charge on each particle and R is the cutoff radius of the screened Coulomb potential.

Now if we define the nuclear phase shift δ^C_L by

$$\delta^C_L(k) = \delta_L(k) - \sigma^S_L(k), \tag{25}$$

then from Eqs. (21), (22), and (25) we see

$$(k|T^{Cn}_L|k) = -\frac{1}{k}\exp\{i[\delta^C_L(k) + 2\sigma^S_L(k)]\}\sin\delta^C_L(k). \tag{26}$$

Hence $(k|T^{Cn}_L|k)$ carries the nuclear phase shift.

The Coulomb scattering states defined by Eq. (15) have their partial wave projection

$$\langle\vec{r}|\vec{p}^{\,(\pm)}_C\rangle = \left(\frac{2}{\pi}\frac{\hbar^2}{2\mu}\right)^{1/2}\sum_{L,M}\langle\vec{r}|p^{(\pm)}_{CL}\rangle Y^*_{LM}(\hat{p})$$
$$= \left(\frac{2}{\pi}\right)^{1/2}\frac{1}{pr}\sum_{L,M} i^L e^{\pm i\sigma^S_L(p)}W_L(pr) \times Y_{LM}(\hat{r})Y^*_{LM}(\hat{p}), \tag{27}$$

where W_L is a solution of the Schrödinger equation with the screened Coulomb potential and in the limit $kR \gg L(L+1)+\eta^2(k)$ and $R \gg r$ approaches the Coulomb wave function $F_L(pr)$ defined by

$$F_L(pr) = (2i)^{-L-1} C_L(\eta) M_{i\eta, L+1/2}(2ipr), \tag{28}$$

where $C_L(\eta)$ is the Coulomb penetration factor defined by

$$C_L(\eta) = \frac{2^L e^{-\pi\eta/2}}{(2L+1)!} \left|\Gamma(L+1+i\eta)\right|, \tag{29}$$

where

$$\eta = \mu e^2/\hbar^2 p \tag{30}$$

and M is the Whittaker function.[11]

Now we take f_τ in Eq. (12) to be eigenfunctions of L^2 and L_z; thus $\tau \to m, L, M$, where m runs from 1 to N and we define the components of f_τ in momentum space by

$$\langle \vec{p}_C^{(\pm)} | f_{mLM} \rangle = \exp(\mp i\sigma_L^S) f_{mL}^C(p) Y_{LM}(\hat{p}) \tag{31}$$

and in coordinate space by

$$\langle \vec{r} | f_{mLM} \rangle = g_{mL}(r) i^L Y_{LM}(\hat{r}). \tag{32}$$

Then $f_{nL}^C(p)$ and $g_{nL}(r)$ are related by

$$\begin{aligned} g_{mL}(r) &= (2/\pi)^{1/2}(1/r) \int_0^\infty p\, dp\, W_L(pr) f_{mL}^C(p) \\ f_{mL}^C(p) &= (2/\pi)^{1/2}(1/p) \int_0^\infty r\, dr\, W_L(pr) g_{mL}(r). \end{aligned} \tag{33}$$

The partial wave form for the separable expansion (12) in the Coulomb state representation is given by

$$\begin{aligned} (p|T_L^{Cn}|p') = \sum_{m,m'=1}^{N} &\langle p_{CL}^{(-)} | V_n | f_{mL} \rangle (D_L)_{mm'} \\ &\times \langle f_{m'L} | V_n | p'^{(+)}_{CL} \rangle. \end{aligned} \tag{34}$$

We assume that the interaction in the Lth partial wave is a local potential $V_{nL}(r)$; then the form factor integral is explicitly given in coordinate space by

$$\begin{aligned} \langle p_{CL}^{(-)} | V_n | f_{mL} \rangle &= \langle f_{mL} | V_n | p_{CL}^{(+)} \rangle \\ &= (2\mu/\hbar^2)^{1/2}(1/p) e^{i\sigma_L^S(p)} \\ &\quad \times \int_0^\infty r\, dr\, W_L(pr) V_{nL}(r) g_{mL}(r). \end{aligned} \tag{35}$$

It is clear from Eq. (13) that $(D^{-1})_{\tau'\tau}$ vanishes for a central potential unless $L' = L$ and $M' = M$ and is independent of M. There are two terms in $(D^{-1})_{\tau'\tau}$. It is easy to evaluate the first term in coordinate space in our case and the second term reduces to a one-dimensional integral in momentum space. So we give one part of D^{-1} in coordinate space and the other part in momentum space and we have[2]

$$\begin{aligned} (D_L^{-1})_{m'm} = &\int_0^\infty g_{m'L}(r) V_{nL}(r) g_{mL}(r) r^2 dr \\ &- \frac{2}{\pi} \int_0^\infty q^2 dq \frac{\langle f_{m'L} | V_n | q_{CL}^{(-)} \rangle \langle q_{CL}^{(-)} | V_n | f_{mL} \rangle}{(2\mu s/\hbar^2) - q^2}. \end{aligned} \tag{36}$$

We now define a t matrix $(p|T_L'^{Cn}|p')$ by

$$(p|T_L'^{Cn}|p') = e^{-i\sigma_L^S(p)} (p|T_L^{Cn}|p') e^{-i\sigma_L^S(p')}; \tag{37}$$

then we see by Eq. (26) that the on-shell quantity $(k|T_L'^{Cn}|k)$, given by

$$(k|T_L'^{Cn}|k) = -\frac{1}{k} e^{i\delta_L^C(k)} \sin\delta_L^C(k), \tag{38}$$

carries the nuclear phase shift.

Then from Eqs. (34), (35), (36), and (37) we have

$$(p|T_L'^{Cn}|p') = \sum_{m,m'=1}^{N} h_{mL}(p)(D_L)_{mm'} h_{m'L}(p'), \tag{39}$$

where

$$h_{mL}(p) = (2\mu/\hbar^2)^{1/2} p^{-1} \int_0^\infty r\, dr\, W_L(pr) V_{nL}(r) g_{mL}(r) \tag{40}$$

and

$$\begin{aligned} (D_L^{-1})_{m'm} = &\int_0^\infty g_{m'L}(r) V_{nL}(r) g_{mL}(r) r^2 dr \\ &- \frac{2}{\pi} \int_0^\infty q^2 dq \frac{h_{mL}(q) h_{m'L}(q)}{(2\mu s/\hbar^2) - q^2}. \end{aligned} \tag{41}$$

So far we have been working in terms of the screened Coulomb potential. To get the results for the Coulomb potential we have to take the limit when when the cutoff distance R goes to infinity. The main problem in taking this limit is that the screened Coulomb phase shift σ_L^S does not have a definite limit as R goes to infinity [see Eq. (24)]. But we do not face this problem here, because σ_L^S is cancelled and does not occur in our final formulas (39), (40), and (41). The only limit we have to take is that of W_L, the screened Coulomb wave function, which smoothly goes to the Coulomb wave function F_L, defined by Eq. (28), as R goes to infinity.

We consider potentials that are linear combinations of Yukawa potentials given by

$$V_{nL}(r) = \sum_{j=1}^{P} V_j r^{-1} e^{-\mu_j r}, \tag{42}$$

where usually $P = 1$, 2, or 3. The expansion functions used in coordinate space are

$$g_{mL}(r) = r^L m^{-1} e^{-\alpha m r}, \tag{43}$$

where $m = 1, 2, \ldots, N$ and α is a parameter, that can be varied to improve convergence.

In the limit when R goes to infinity W_L in Eq. (40) is replaced by the Coulomb wave function F_L and $h_{mL}(p)$ has the analytic form[12]

$$h_{mL}(p) = (2\mu/\hbar^2)^{1/2} C_L(\eta)(2L+1)!\, m^{-1} p^L \times \sum_j V_j (p^2 + A_j^2)^{-L-1} \exp[2\eta \arctan(p/A_j)], \tag{44}$$

where $A_j = \mu_j + m\alpha$. The first term in Eq. (41) has the analytic form[2]

$$(2L+1)!\,(mm')^{-1} \sum V_j (m'\alpha + m\alpha + \mu_j)^{-2L-2} \tag{45}$$

and the second term can be evaluated by breaking it into principal-value and imaginary parts. The principal-value part is evaluated by performing the integration by an even order Gaussian quadrature symmetrically located about the pole.

IV. NUMERICAL RESULTS

To see how the separable expansion works in practice we report S-wave numerical calculations with the Malfliet-Tjon potential defined by

$$V(r) = -V_A r^{-1} e^{-\mu_A r} + V_R r^{-1} e^{-\mu_R r}, \tag{46}$$

where $V_A = 181.5422\pi$ MeVfm, $V_R = 457.8828\pi$ MeVfm, $\mu_A = 1.55$ fm^{-1}, and $\mu_R = 3.11$ fm^{-1}, which has single bound state at an energy $E = -0.35$ MeV.

The parameter α in the expansion functions of Eq. (43) can be varied to improve the convergence rate. The value finally chosen as in Ref. 2 was $\alpha = 0.5$ fm^{-1}.

The potential considered has a bound state at an energy $E = -0.35$ MeV and the corresponding pure nuclear t matrix has a pole at this energy. As in Ref. 1 we found that low rank results were far from the converged results unless this pole is adequately included in the t matrix. This was done for the pure nuclear t matrix at zero energy as in Ref. 1, with a modification of the basis function, where the $m = 1$ basis function was replaced by a suitable linear

TABLE I. S-wave pure nuclear phase shift $\bar{\delta}_L$ and Coulomb corrected nuclear phase shift δ_L^C for different N. $N = 1, 2$ results have been calculated with a modification of the basis functions described in the text.

$E_{\text{c.m.}}$ (MeV)		Exact	N = 1	2	4	6	8
4	$\bar{\delta}_L$	1.4701	1.4614	1.4612	1.4665	1.4702	1.4701
	δ_L^C	1.4680	1.4603	1.4603	1.4636	1.4680	1.4680
8	$\bar{\delta}_L$	1.2420	1.2259	1.2252	1.2400	1.2420	1.2420
	δ_L^C	1.2513	1.2375	1.2371	1.2492	1.2513	1.2513
12	$\bar{\delta}_L$	1.0997	1.0732	1.0718	1.0982	1.0997	1.0997
	δ_L^C	1.1123	1.0885	1.0876	1.1108	1.1122	1.1123
24	$\bar{\delta}_L$	0.8370	0.7764	0.7721	0.8363	0.8370	0.8369
	δ_L^C	0.8513	0.7935	0.7899	0.8506	0.8513	0.8513
48	$\bar{\delta}_L$	0.5501	0.4358	0.4208	0.5457	0.5490	0.5498
	δ_L^C	0.5631	0.4503	0.4368	0.5593	0.5623	0.5630
72	$\bar{\delta}_L$	0.3730	0.2340	0.2005	0.3515	0.3722	0.3728
	δ_L^C	0.3850	0.2453	0.2142	0.3644	0.3841	0.3848
104	$\bar{\delta}_L$	0.2083	0.0790	0.0066	0.1515	0.2076	0.2074
	δ_L^C	0.2190	0.0868	0.0178	0.1633	0.2182	0.2181
152	$\bar{\delta}_L$	0.0358	0.0009	−0.1675	−0.0692	0.0312	0.0344
	δ_L^C		0.0018	−0.1591	−0.0592	0.0405	0.0437
176	$\bar{\delta}_L$	−0.0311	0.0058	−0.2237	−0.1507	−0.0374	−0.0323
	δ_L^C		0.0042	−0.2162	−0.1417	−0.0286	−0.0235

combination of the first few basis functions, chosen as to make the zero energy half-shell pure nuclear t matrix essentially exact. This simple technique ensures that the potential $V_n^{(1)}$ given by Eq. (10) is approximately the unitary pole approximation potential.[13] Four terms were taken to define the new $m=1$ basis, so that the results for $N \geq 4$ remain unchanged by this change of basis and the final convergence rate is unaffected.

In Table I we present nuclear elastic scattering S-wave phase shifts $\delta^C_{L=0}$ and the pure S-wave nuclear phase shift $\bar{\delta}_{L=0}$ in the absence of Coulomb interactions.

In Fig. 1 we show the real and imaginary parts of the half-shell S-wave t-matrix elements $(p|T_L'^{Cn(N)}|p')$, defined by Eq. (37) at $E_{c.m.}=72$ MeV and Fig. 2 shows the real and imaginary parts of some off-shell S-wave t-matrix elements $(p|T_L'^{Cn(N)}|p')$ at $E_{c.m.}=24$ MeV.

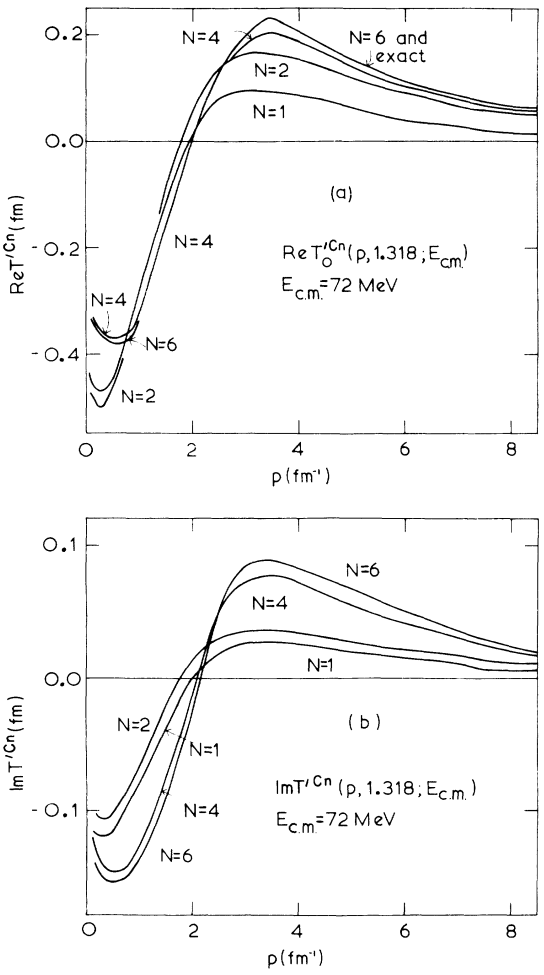

FIG. 1. The (a) real and (b) imaginary parts of the half-off-shell t-matrix elements $(p|T_L'^{cn(N)}|p')$ [see Eq. (37)] at $E_{c.m.}=72$ MeV. $N=1,2$ results have been calculated with a modification of basis function described in the text.

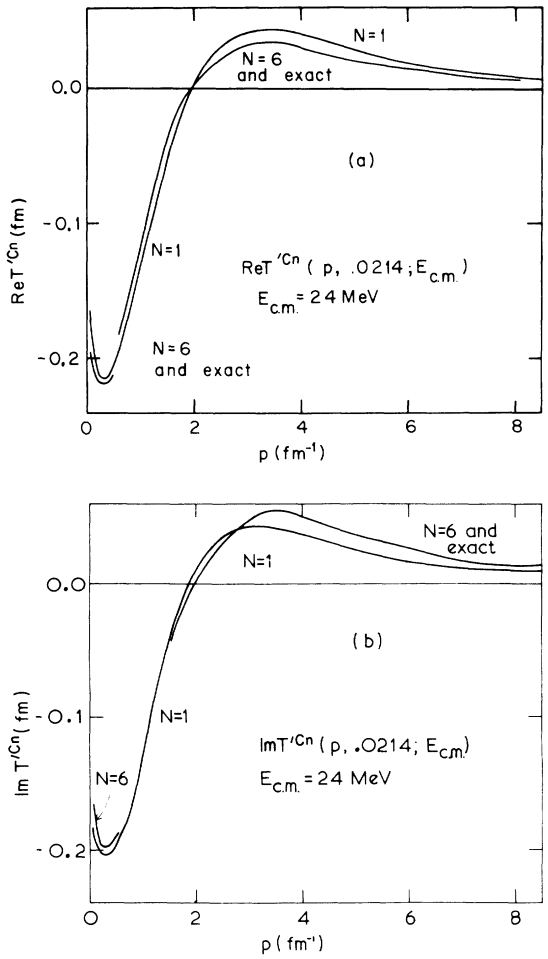

FIG. 2. The (a) real and (b) imaginary parts of the off-shell $(p|T_L'^{cn(N)}|p')$ at $E_{c.m.}=24$ MeV. The $N=1$ result has been calculated with a modification of basis functions described in the text.

V. DISCUSSION

The final results in Table I and Figs. 1 and 2 show that the approximate t matrix and the on-shell scattering phase shifts converge rapidly and uniformly in all cases. The convergence is comparable to the convergence in the absence of the Coulomb interactions.

This separable expansion and especially the rank one approximation will be very useful in more complicated three-body problems in nuclear physics in the presence of Coulomb interactions. This is because t_n is the t matrix that comes into a Faddeev type formulation of three-body problems in nuclear physics in the presence of Coulomb interactions.[5]

Because the expansion functions $g_{nL}(r)$ used in this calculation are similar for different n, at some stage, the D^{-1} matrix becomes too singular for the purpose of matrix inversion. In the present calculation (carried out in single precision—to about 13 significant figures on a CDC CYBER series 70 model 72 computer) numerical difficulties with the inversion of D^{-1} appeared only for values of N greater than 8. On the other hand it is always possible to use a nearly orthogonalized set of functions as in Ref. 1 and in some of the articles of Ref. 9, so that we can easily go to higher N, but all the integrals are to be evaluated numerically.

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[13]See the first article in Ref. 3.