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Reduced matrix equations for direct nuclear reactions

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A reduction procedure for the construction of matrix equations applicable to direct nuclear reactions is described, where the direct channel Green's functions are used to simplify the many-particle problem. The Faddeev-type matrix equations are reduced to simpler forms so that the partial connectivity of the iteration kernels is maintained for those direct channels which are explicitly coupled. Auxiliary channels are then introduced to improve the connectivity property, while direct channel distortion potentials and simple channel projection operators are incorporated to facilitate the solution of the coupled equations. Using the reduction procedure, the multiple scattering equations of Watson are extended to include explicitly the rearrangement channels which are strongly coupled.

[NUCLEAR REACTIONS Reduction of the Faddeev equations; coupled equations for direct interactions; auxiliary channels.]

I. INTRODUCTION

Since the pioneering work of Watson[1,2] on the multiple scattering theory for many-particle systems and the mathematically rigorous study of Faddeev[3] on three-particle scattering, there have appeared many alternative formulations[4-6] and various extensions[7-13] of these theories. While the Watson formulation is essentially equivalent to the Faddeev theory for the three-particle system, it is quite distinct for scattering systems with more than three particles. Due to the important role played by the channel Green's functions, to be discussed in detail, the multiple scattering theory is much simpler in structure and more readily applicable to complex physical systems. In the present paper, we will stress this feature of the theory in simplifying the formulation to suit the direct nuclear processes.

The formal theoretical difficulties of the many-particle scattering systems are associated[14,15] mainly with the particle rearrangements among different clusters. For systems with more than two particles, channels with different particle rearrangements are possible, each with a different channel Hamiltonian. The channel wave functions generated by these Hamiltonians are generally not orthogonal to each other. Thus the customary procedure of expanding the total wave function in terms of more than one set of channel states should be carried out with special attention paid to the question of overcompleteness. Furthermore, the usual $(i\epsilon)$ prescription for the boundary conditions on the Green's function in the Lippmann-Schwinger scattering equation may not be sufficient to uniquely specify a solution.[14,15] This difficulty is resolved by imposing all the open channel boundary conditions simultaneously. A multicomponent wave function may be used for this purpose, where each component assumes a specific boundary condition asymptotically.

When one tries to solve by iteration a set of coupled equations derived using the multicomponent wave function, the iteration kernels may contain terms which describe processes in which some of the particles are left as spectators and thus not interacting. These terms are associated with the additional energy-conserving δ functions[5] which, unlike the δ function corresponding to the overall energy conservation, cannot be factored out. This gives rise to an ill-defined iteration kernel. Thus, the problem is identified with all the partially connected processes in which one or more spectator particles are involved. Much of the complicated formal manipulations required in the many-particle theories[7-13] are to systematically isolate these troublesome terms from the completely connected part. The full connectivity is obtained by including the interactions of all the particles in the system at least once.

Although the partially connected parts of the iteration kernels lead naturally to the multicomponent form of the wave function, the multicomponent extension alone would not remove the difficulties completely. As will be shown below, there are many different ways of deriving the fully connected kernels and of introducing the multicomponent wave functions, all of which depend crucially on the role of the particular channel Green's functions used. Both the full connectivity and the multicomponent feature are essential in correctly formulating the many-particle

scattering theory.

As is well known, the conventional distorted wave Born approximation[16] (DWBA) for the direct reactions bypasses the nonorthogonality problem altogether, while the coupled-channel method[17,18] (CCM) employs only the fully connected kernels constructed with a small set of bound state wave functions for each channel cluster. Thus, the nonorthogonality problem among the rearrangement channels is treated only approximately in these approaches, while an attempt to improve them requires again a careful treatment of the nonorthogonality and related problems.

In several previous studies,[19,20] we have examined the possibility of simplifying the rigorous many-particle theory by relaxing the requirement of the full connectivity for some of the rearrangement channels. For a direct reaction of the type

$$\underset{\text{channel } \alpha}{a+A} \rightarrow \underset{\text{channel } \beta}{b+B}\ , \tag{1.1}$$

we may require that the appropriate iteration kernels be connected only in the channels α and β, while all the other rearrangement channels are to be treated noniteratively in some approximations. In fact, the original formulation of Watson[1] treats explicitly only those "external" interactions among the pairs which belong to *different* clusters while the "internal" interactions among the particles within the *same* clusters are included in the channel Green's functions. This makes the formalism simpler and more readily applicable to physical problems. Throughout this paper, we *emphasize* the simplifying role of the channel Green's functions and maintain the minimal connectivity property for the direct channels. The sets of coupled equations we derive should reflect these features.

II. REDUCED MATRIX EQUATIONS

The reduction procedure we have developed previously[19,20] is designed specifically to reduce the number of rearrangement channels which can appear in an N-particle scattering problem and to provide a partial connectivity property for the direct channel iteration kernels. As will become clear, the method is especially suited in treating the direct reactions where only a few rearrangement channels are strongly coupled. To illustrate the method, we consider the three-particle ($N=3$) Faddeev equations[3] for the two-cluster channels

$$\begin{bmatrix} H_1-E & v_{23} & v_{23} \\ v_{13} & H_2-E & v_{13} \\ v_{12} & v_{12} & H_3-E \end{bmatrix} \begin{bmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \tag{2.1}$$

where v_{ij} are the two-particle pair interactions and the channel Hamiltonians H_α are given by

$$H_\alpha = H_0 + V^\alpha\ . \tag{2.2}$$

In (2.2), H_0 is the three-particle kinetic energy operator and V^α is the internal interaction for the channel α, with $V^i = v_{jk}$ for $\alpha = i$ ($i,j,k = 1$, 2, and 3, cyclic). The total wave function is given by the sum[3,19]

$$\Psi = \sum_{\alpha=1}^{3} \Psi_\alpha\ . \tag{2.3}$$

It is of course well known that the iteration kernels of (2.1) are completely connected. [As N increases, the structure of the equations analogous to (2.1) becomes very complicated. For $N=4$, there are seven two-cluster channels[10] which are explicitly coupled.]

For the direct reaction of the type

$$1+(2+3) \rightarrow 2+(1+3)\ , \tag{2.4}$$

for example, we may *simplify* (2.1) by keeping only the channels 1 and 2 explicitly. This can be accomplished by noting that the elements in each column in (2.1) add up to $H-E$, where the external interactions $V_\alpha = H - H_\alpha$ are given by $V_i = v_{ij} + v_{ik}$, for each $\alpha = i$ and ijk cyclic. Thus, by first adding the contents of the third row to any of the other two rows and then dropping that third row and column, we obtain

$$\begin{bmatrix} H_1-E+v_{12} & v_{23} \\ v_{13} & H_2-E+v_{12} \end{bmatrix} \begin{bmatrix} \Psi_1' \\ \Psi_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{2.5}$$

or

$$\begin{bmatrix} H_1-E & V_2 \\ V_1 & H_2-E \end{bmatrix} \begin{bmatrix} \Psi_1'' \\ \Psi_2'' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{2.6}$$

with

$$\Psi = \Psi_1' + \Psi_2' = \Psi_1'' + \Psi_2''\ . \tag{2.7}$$

(An analysis of Ψ_α'' is given in the Appendix.) Equations (2.5) and (2.6) are still exact, as the channel 3 is now implicitly contained in Ψ_α' or Ψ_α''. Note that the full connectivity is retained in (2.5) but lost in (2.6). On the other hand, (2.6) retains the channel Hamiltonians H_α in the diagonal positions and the off-diagonal parts contain the external interactions V_α. The contents of (2.5) and (2.6) have been analyzed in detail previously[20] using the Green's function decomposition, and (2.6) was applied to simple three-particle problems.[21,22]

Now consider the general case when the number of particles N is large. Equations analogous to (2.5) and (2.6) can be obtained directly from the following reduction procedure,[19,20] rather than starting with a complicated set of equations with

full connectivity.

(i) Choose a small number I of direct reaction channels whose asymptotic Hamiltonians H_α are mutually distinct. This will result in an $I\times I$ matrix equation, with

$$\Psi=\sum_{\alpha=1}^{I}\Psi_\alpha\,. \tag{2.8}$$

The size I should be large enough to include some of the strongly coupled channels, but small so that the coupled equations can readily be solvable. The limitation here to a few explicit channels is *not* an approximation, however.

(ii) For each channel α, $H_\alpha-E$ is put in the diagonal positions of the $I\times I$ matrix. The channel interactions $V_\alpha=H-H_\alpha$ are then distributed among the rows of the α column such that as much of the internal interactions V^β should appear in the β row of the α column. The solution of the resulting matrix equation is then guaranteed to be the *same* as the original solution of the problem so long as the sums of each column equal $H-E$.

(iii) Since the number of explicit channels I chosen in (i) is rather arbitrary, the $I\times I$ matrix equation can be reduced to an $I'\times I'$ form, where $I'<I$. To eliminate the γ component, for example, we add the contents of the γ row to any of the other $(I-1)$ rows and then simply drop the γ row and the γ column.

(iv) Within each column, we can add or subtract some arbitrary interactions Y's without affecting the overall solution Ψ, but with often drastic modifications of the individual Ψ_α. This provides a nontrivial way of introducing direct distortion potentials, which are absent in the original Faddeev formulation. A judicious choice for Y will facilitate the convergence of the iteration series.

(v) The multicomponent form of the matrix equations allows one to introduce simple channel projection operators[23] for different rearrangement channels, without the complication of the nonorthogonality problem. The intermediate state contributions for each channel α can then be isolated and treated separately in terms of the optical potentials.

We illustrate the above reduction procedure by examining the cases with $N=3$ and 4. For $N=3$ and $I=3$, (2.1) provides a framework for the three two-cluster channels. Using (iv), we can introduce the distortion potentials Y in (2.1) as

$$\begin{bmatrix} H_1-E+Y_1 & v_{23}-Y_2^1 & v_{23}-Y_3^1 \\ v_{13}-Y_1^2 & H_2-E+Y_2 & v_{13}-Y_3^2 \\ v_{12}-Y_1^3 & v_{12}-Y_2^2 & H_3-E+Y_3 \end{bmatrix}\begin{bmatrix}\Psi_1\\ \Psi_2\\ \Psi_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}, \tag{2.9}$$

where $Y_\alpha=Y_\alpha^\beta+Y_\alpha^\gamma$. According to (iii), (2.1) is of course not the only possible form; we may have instead the operators[24]

$$\begin{bmatrix} H_1-E & V_2 & 0 \\ 0 & H_2-E & V_3 \\ V_1 & 0 & H_3-E \end{bmatrix} \tag{2.10}$$

or

$$\begin{bmatrix} H_1-E & 0 & V_3 \\ V_1 & H_2-E & 0 \\ 0 & V_2 & H_3-E \end{bmatrix}.$$

The first of these operators was considered recently by Kouri and Levin[25] and later by Tobocman[26] using the channel coupling array W. The role of W in their formalism is equivalent to the procedure (ii) of the reduction method, if we restrict the V_α to appear in single rows as a whole, rather than being distributed among the rows. They showed that[25] when all the two-cluster channels are included in the form (2.10), the resulting kernels are completely connected, and emphasized the use of the channel interactions V_α in constructing the coupled equations. By contrast, the reduction procedure emphasizes the role of the channel Green's function $(E+i\epsilon-H_\alpha)^{-1}$, and the channel interactions V_α are treated either as a whole [as in (2.10)] or in parts [as in (2.1)].

Now consider the reduction $I=3\to I'=2$. We have already seen that (2.1) reduces to (2.5), (2.6), or some other analogous forms. (2.9) also reduces for example to

$$\begin{bmatrix} H_1-E+Y_1 & V_2-Y_2 \\ V_1-Y_1 & H_2-E+Y_2 \end{bmatrix}\begin{bmatrix}\Psi_1\\ \Psi_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}. \tag{2.11}$$

Comparison of the iteration kernels for (2.6) and (2.11),

$$K_{12}=G_1V_2G_2V_1\,,\quad G_\alpha=(E+i\epsilon-H_\alpha)^{-1} \tag{2.12}$$

and

$$K_{12}^Y=G_1^Y(V_2-Y_2)G_2^Y(V_1-Y_1)\,,$$

$$G_\alpha^Y=(E+i\epsilon-H_\alpha-Y_\alpha)^{-1}\,,$$

indicates that we may have $\|K_{12}\|>\|K_{12}^Y\|$ with some judicious choices for Y_α, thus improving the convergence of the iteration solution.

In addition to the Y's, we can also introduce a simple set of projection operators for the reduced matrix equations (RME) we have derived above. Thus, for (2.6) for example, we have

$$P=\begin{bmatrix}P_1 & 0\\ 0 & P_2\end{bmatrix}\text{ and }Q=\begin{bmatrix}Q_1 & 0\\ 0 & Q_2\end{bmatrix}=I-P\,, \tag{2.13}$$

where

$$P_\alpha = \psi_0^\alpha \rangle\langle \psi_0{}^\alpha , \quad Q_\alpha = 1_\alpha - P_\alpha , \quad Q_\alpha P_\alpha = 0 . \qquad (2.14)$$

The bound states of the pair in the channel α are denoted by ψ_n^α. Note that $P_\alpha Q_\alpha = 0$, while $P_1 P_2 \neq 0$ and $P_1 Q_2 \neq 0$. A convenient choice for Y_α for example is $Y_\alpha = P_\alpha V_\alpha P_\alpha$.

The simplifications we gain by reducing the number I of direct reaction channels become important in applying the formalism to cases with $N \geq 4$. For $N = 4$, for example, Sloan[10] constructed a set of seven coupled equations for all the two-cluster channels. The coupling interactions in these equations are obtained in turn by the iterations of two or more different pairwise potentials. Construction of such coupling terms is in itself a major computational undertaking and often requires drastic approximations. In the present approach, however, we simplify the problem by limiting the number of explicit channels and by requiring a partial connectivity. Thus, for the process with $I = 2$ and $N = 4$,

$$1 + (2 + 3 + 4) - 2 + (1 + 3 + 4) , \qquad (2.15)$$

for example, we have simply (2.6) again, with

$$H_1 = H_0 + V^1 , \quad V^1 = v_{23} + v_{24} + v_{34} , \qquad (2.16)$$
$$V_1 = H - H_1 = v_{12} + v_{13} + v_{14} , \quad \text{etc.}$$

The iteration kernels $K_{12} = G_1 V_2 G_2 V_1$ and $K_{21} = G_2 V_1 G_1 V_2$ show only the partial connectivities (12), (123), (124), (13)-(24), (14)-(23), where (13)-(24) implies the connectivity between the pairs (13) and (24) separately. The particles 1 and 2 are completely connected, while those particles inside the clusters are not fully connected. Ways to improve the latter will be discussed in Sec. IV.

Recently, there have been some attempts[27] to derive a set of coupled equations for the multicomponent scattering wave functions, starting from the rigorous amplitude equations[5-12] with the full connectivity. For an N-particle system, there are altogether I_2 two-cluster channels, so that a set of I_2 coupled equations is obtained in the form

$$(E - H_0 - \mathcal{U}_\alpha)\tilde{\Psi}_\alpha = \mathcal{U}_\alpha \sum_{\gamma \neq \alpha} \tilde{\Psi}_\gamma , \qquad (2.17)$$

where $\mathcal{U}_\alpha G_\alpha \equiv \mathcal{W}_\alpha$ is the sum of all the iterated interactions with the connectivity α. It was then shown that $\tilde{\Psi}_\alpha$ contains the outgoing waves only in the α (two-cluster) channel. This is precisely the same essential property that the RME are designed to contain; however, as will be shown in the next section, the wave functions in the RME can also have the components of the other two-cluster channels which are included in the theory only implicitly.

When N is large, $\mathcal{U}_\alpha$ in (2.17) are difficult to construct in practice. Approximations are invariably introduced to simplify them, but it is not clear then whether the essential features of the original set are retained. By contrast, the RME discussed here are trivial to construct and yet exact; certain desirable features are included from the beginning for the particular direct reaction of interest. The price one pays for this simplicity is the requirement that the channel Green's functions G_α should be evaluated with care before the coupling with the other channels is included.

III. ASYMPTOTIC BOUNDARY CONDITIONS

One of the principal advantages of introducing a multicomponent wave function in the RME is that the asymptotic boundary conditions can readily be imposed on the rearrangement channel wave functions. In adopting the reduction procedure, the contributions from different channels are shifted around among the Ψ_α, so that it is *a priori* not obvious that Ψ_α should carry the physical amplitude for the channel α. We examine in this section the relationship between the asymptotic behavior of Ψ_α and the physical amplitudes for the differential cross sections.

For definiteness we discuss the problem using (2.6) for $N = 3$, but most of the result we obtain here should be applicable to other RME. By construction, H_1 lacks the internal interaction $V^2 = v_{13}$ for the channel 2 so that Ψ_1 cannot provide the asymptotic states of channel 2, where the pair 1 and 3 is interacting (or bound). The coupling interactions are arranged so that $V_2 G_2 V_1$ in K_{12} vanishes in channel 2. Thus, the Green's function

$$\mathcal{G}_1 \equiv [E + i\epsilon - H_1 - V_2 G_2 V_1]^{-1} \qquad (3.1)$$

cannot generate a branch cut corresponding to the cluster structure of channel 2. The asymptotic contribution to channel 1 is therefore carried entirely by Ψ_1, and Ψ_2 cannot contribute to channel 1 asymptotically, and vice versa.

Apparently, this is not the case with channel 3; recall that this channel in the original Faddeev equations (2.1) is eliminated by the reduction procedure and its effect is incorporated into the newly defined wave functions Ψ_1 and Ψ_2. In fact the iteration kernels K_{12} and K_{21} contain the terms $v_{12} G_2 v_{12}$ and $v_{12} G_1 v_{12}$, respectively, which vanish in channels 1 and 2 but not in channel 3.

Thus, $\mathcal{G}_1$ and $\mathcal{G}_2$ contain cuts corresponding to channel 3, with the outgoing waves in channel 3 if the energy E is sufficiently high. The asymptotic forms for Ψ_α can therefore be written as

$$\Psi_1 \xrightarrow[R_1\to\infty]{} \Phi_1 + \frac{\lambda_{11}}{(2\pi)^{3/2}}\psi_0^{(1)}(\vec{r}_{23})\frac{e^{ik_0^1 R_1}}{R_1}\mathcal{Y}_1 + \frac{\lambda'_{31}}{(2\pi)^{3/2}}\psi_0^{(3)}(\vec{r}_{12})\frac{e^{ik_0^3 R_3}}{R_3}\mathcal{Y}_3 + \begin{pmatrix}\text{other outgoing waves}\\ \text{in channels 1 and 3}\end{pmatrix} \tag{3.2}$$

and

$$\Psi_2 \xrightarrow[R_2\to\infty]{} \frac{\lambda_{21}}{(2\pi)^{3/2}}\psi_0^{(2)}(\vec{r}_{13})\frac{e^{ik_0^2 R_2}}{R_2}\mathcal{Y}_2 + \frac{\lambda''_{31}}{(2\pi)^{3/2}}\psi_0^{(3)}(\vec{r}_{12})\frac{e^{ik_0^3 R_3}}{R_3}\mathcal{Y}_3 + \begin{pmatrix}\text{other outgoing waves}\\ \text{in channels 2 and 3}\end{pmatrix}. \tag{3.3}$$

In (3.2), we have

$$(H_\alpha - E)\Phi_\alpha = 0,\qquad \Phi_\alpha = \psi_0^{(\alpha)}(\vec{r}_{\beta\gamma})e^{i\vec{k}_0^\alpha\cdot\vec{R}_\alpha}/(2\pi)^{3/2}, \tag{3.4}$$

and other notations are obvious, while

$$E = e_n^{(\alpha)} + \frac{\hbar^2}{2\mu_\alpha}(k_n^\alpha)^2, \tag{3.5}$$

where e_n^α is the $(\beta\gamma)$ pair energy.

For the elastic scattering in channel 1, we have

$$\frac{d\sigma_{11}}{d\Omega_1} = |f_{11}|^2,$$

with

$$f_{11} = -\frac{2\mu_1}{4\pi\hbar^2}(2\pi)^3\int d\vec{r}_{23}d\vec{R}_1\Phi_1^* V_1\Psi \equiv -\frac{4\pi^2\mu_1}{\hbar^2}T_{11} \tag{3.6}$$

and

$$T_{11} \equiv (\Phi_1 \mid V_1 \mid \Psi) = (\Phi_1 \mid V_1 \mid \Psi_1 + \Psi_2). \tag{3.7}$$

In order to relate the λ's in (3.2) and (3.3) to T_{11}, we use the first of the two equations in (2.6) to obtain

$$(\Phi_1 \mid H_1 - E \mid \Psi_1) = -(\Phi_1 \mid V_2 \mid \Psi_2). \tag{3.8}$$

Performing the partial integrations twice and using (3.2), the left hand side of (3.8) becomes

$$(\Phi_1 \mid H_1 - E \mid \Psi_1) = \frac{\hbar^2}{4\pi^2\mu_1}\lambda_{11}. \tag{3.9}$$

The right hand side of (3.8) can be converted using the second of the two equations in (2.6), which gives

$$(\Phi_1 \mid H_2 - E \mid \Psi_2) = -(\Phi_1 \mid V_1 \mid \Psi_1). \tag{3.10}$$

However, the left hand side of (3.10) can be modified using (3.2) and (3.3) to a form

$$(\Phi_1 \mid H_2 - E \mid \Psi_2) = (\Phi_1 \mid (H_2 + V_2 - V_1) + (V_1 - V_2) - E \mid \Psi_2) = (\Phi_1 \mid V_1 - V_2 \mid \Psi_2). \tag{3.11}$$

Thus, the right hand side of (3.8) can be written as

$$(\Phi_1 \mid V_2 \mid \Psi_2) = (\Phi_1 \mid V_1 \mid \Psi_1) + (\Phi_1 \mid V_1 \mid \Psi_2) = T_{11} \tag{3.12}$$

and finally

$$\lambda_{11} = f_{11}. \tag{3.13}$$

That is, the source term $V_2\Psi_2$ in (2.6) is such that Ψ_1 carries asymptotically the correct elastic amplitude.

For the rearrangement collision $1 \to 2$, we have[18]

$$\frac{d\sigma_{21}}{d\Omega_2} = |f_{21}|^2\left(\frac{\mu_1 k_0^2}{\mu_2 k_0^1}\right), \tag{3.14}$$

where

$$f_{21} = -\left(\frac{2\pi}{\hbar}\right)^2\mu_2 T_{21} \tag{3.15}$$

with

$$T_{21} = (\Phi_2 \mid V_2 \mid \Psi). \tag{3.16}$$

To relate the λ_{21} in (3.3) to T_{21}, we take the second of (2.6) and obtain

$$(\Phi_2 \mid H_2 - E \mid \Psi_2) = -(\Phi_2 \mid V_1 \mid \Psi_1) \tag{3.17a}$$

$$= \frac{\hbar^2}{4\pi^2\mu_2}\lambda_{21} \tag{3.17b}$$

after the partial integrations and using (3.3). The right hand side of (3.17a) can be rewritten in terms of the first equation in (2.6), as

$$(\Phi_2 \mid H_1 - E \mid \Psi_1) = -(\Phi_2 \mid V_2 \mid \Psi_2) = (\Phi_2 \mid H_1 + V_1 - V_2 - E + (V_2 - V_1) \mid \Psi_1) = (\Phi_2 \mid V_2 - V_1 \mid \Psi_1), \tag{3.18}$$

which gives for (3.17)

$$\frac{\hbar^2}{4\pi^2\mu_2}\lambda_{21} = -(\Phi_2 \mid V_2 \mid \Psi) = -T_{21}. \tag{3.19}$$

Therefore,

$$\lambda_{21} = f_{21}. \tag{3.20}$$

Now, consider the contribution of channel 3:

$$\frac{d\sigma_{31}}{d\Omega_3} = |f_{31}|^2\left(\frac{\mu_1 k_0^3}{\mu_3 k_0^1}\right), \tag{3.21}$$

where

$$f_{31} \equiv -\left(\frac{2\pi}{\hbar}\right)^2\mu_3 T_{31}, \tag{3.22}$$

$$T_{31} = (\Phi_3 \mid V_3 \mid \Psi). \tag{3.23}$$

From (3.2) and (3.3), and using (2.6), we have

$$(\Phi_3|H_1-E|\Psi_1)=-(\Phi_3|V_2|\Psi_2)$$
$$=(\Phi_3|H_3-E+(V_3-V_1)|\Psi_1)$$
$$=\frac{\lambda'_{31}}{4\pi^2\mu_3}+(\Phi_3|V_3-V_1|\Psi_1) \quad (3.24)$$

and

$$(\Phi_3|H_2-E|\Psi_2)=-(\Phi_3|V_1|\Psi_1)$$
$$=(\Phi_3|H_3-E+(V_3-V_2)|\Psi_2)$$
$$=\frac{\lambda''_{31}}{4\pi^2\mu_3}+(\Phi_3|V_3-V_2|\Psi_2). \quad (3.25)$$

Therefore, adding (3.24) and (3.25), we obtain

$$\frac{1}{4\pi^2\mu_3}(\lambda'_{31}+\lambda''_{31})=-T_{31} \quad (3.26)$$

and thus

$$f_{31}=\lambda'_{31}+\lambda''_{31}. \quad (3.27)$$

(3.27) shows that the physical contents of (2.6) and (2.1) are exactly the same. From a practical point, however, (2.6) should be more convenient if channel 3 is not strongly coupled to channels 1 and 2, in which case λ'_{31} and λ''_{31} are small. On the other hand, if λ_{31} is large, then it may be more appropriate to include channel 3 explicitly in the form (2.1) or (2.10). The mathematical contents of the wave functions Ψ''_1 and Ψ''_2 are analyzed explicitly in the Appendix, further clarifying their behavior in the asymptotic region for the third channel.

IV. AUXILIARY CHANNELS AND IMPROVED CONNECTIVITY

As is well known, the simple structure of (2.1) for $N=3$ cannot be extended readily to cases with $N>3$ because of new cluster structures which arise for larger systems. They in turn introduce many more partially connected parts into the iteration kernels, and, for a mathematically consistent treatment, elaborate methods[5,7-12] have been devised to isolate these partially connected parts and to treat them noniteratively. As a result, the formalism for $N>3$ becomes very complicated and impractical for most physical applications. We have seen in Sec. II that the RME are simpler to construct, but their iteration kernels are only partially connected. It is therefore of some interest to consider how the connectivity could be improved step by step within the context of the reduction procedure outlined in Sec. II. For this purpose, we introduce one or more auxiliary channels and go from the $I\times I$ equations to $I'\times I'$, where $I'>I$. As in the case $I'<I$, the procedure here is not unique and depends very much on the physical considerations.

As an example, we start with (2.6) for $N=3$, $I=2$ and introduce one auxiliary channel a, (consistent with the reduction procedure) as

$$\begin{bmatrix} H_1-E & 0 & V_{1a} \\ 0 & H_2-E & V_{2a} \\ V_1 & V_2 & H_a-E \end{bmatrix}\begin{bmatrix}\Psi_1\\ \Psi_2\\ \Psi_a\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}, \quad (4.1)$$

where

$$H=H_a+V_a, \quad V_a=V_{1a}+V_{2a},$$

and

$$\Psi=\Psi_1+\Psi_2+\Psi_a. \quad (4.2)$$

We can formally solve (4.1) for Ψ_a as

$$\Psi_a=G_aV_1\Psi_1+G_aV_2\Psi_2 \quad (4.3)$$

and obtain a set of coupled equations

$$\begin{aligned}(E-H_1)\Psi_1&=V_{1a}G_aV_1\Psi_1+V_{1a}G_aV_2\Psi_2,\\ (E-H_2)\Psi_2&=V_{2a}G_aV_1\Psi_1+V_{2a}G_aV_2\Psi_2.\end{aligned} \quad (4.4)$$

Equations (4.4) have roughly the same structure as the Faddeev equations (2.1), and some choices for V_a, such as $V_{1a}=v_{23}$ and $V_{2a}=v_{13}$, provide the same full connectivity when $N=3$. (For $N>3$, however, the freedom in the choice for V_{1a} and V_{2a} is not sufficient to obtain a full connectivity.)

The form (4.1) is also convenient in introducing certain types of approximations which can make the kernels connected. Thus, using the cluster decomposition procedure of Weinberg,[5] we can write

$$G_a=(E+i\epsilon-H_a)^{-1}=G_0+L_a+C_a, \quad (4.5)$$

where L_a is the sum of all the partially connected parts and C_a is the maximally connected part of G_a within the available $V^a\equiv H_a-H_0$. Thus, if we set[19]

$$G_a\approx C_a, \quad (4.6)$$

all four terms on the right hand side of (4.4) become fully connected. For the example (2.15) with $N=4$, $I=2$, we may choose for the auxiliary channel a the cluster structure

$$(1+2)+(3+4) \quad (4.7)$$

with

$$V^a=v_{12}+v_{34}.$$

Then, C_a of (4.6) is simply given by the convolution

$$C_a=C(12)*C(34), \quad (4.8)$$

where $C(12)$ is the part of the four-particle

Green's function with the pair (12) fully connected. It is a trivial matter to check that, with

$$V_{1a}=v_{23}+v_{24} \quad \text{and} \quad V_{2a}=v_{13}+v_{14},$$

all four terms in (4.4) are fully connected. Other combinations of H_a and V_a are also possible.

It is simple to show that (4.8) is related to the use of the channel projection operators P_a and Q_a for the channel a and retaining a finite set for P_a.

We can proceed further by introducing more than one auxiliary channel, thus relying on the channel interactions to produce improved connectivity rather than approximating G_a. Thus, with two auxiliary channels a and b: we have

$$\begin{bmatrix} H_1-E & 0 & V_a & 0 \\ 0 & H_2-E & 0 & V_b \\ 0 & V_2 & H_a-E & 0 \\ V_1 & 0 & 0 & H_b-E \end{bmatrix} \begin{bmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_a \\ \Psi_b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{4.9}$$

where

$$\Psi=\Psi_1+\Psi_2+\Psi_a+\Psi_b. \tag{4.10}$$

Formal elimination of Ψ_a and Ψ_b from (4.9) gives

$$\begin{aligned} (E-H_1)\Psi_1 &= V_aG_aV_2\Psi_2, \\ (E-H_2)\Psi_2 &= V_bG_bV_1\Psi_1. \end{aligned} \tag{4.11}$$

The structure of (4.11) is quite different from (4.4) and (2.1) in that the diagonal distortion terms are now absent. The iteration kernels are of the form $G_1V_aG_aV_2G_2V_bG_bV_1$, etc., and v_a and v_b can be chosen to improve the connectivity. This is, in certain respects, similar to the procedure of Kouri and Levin.[25] However, the equation such as (4.9) does not require the presence of all the two-cluster channels, and H_a and H_b are chosen rather arbitrarily, based mainly on the physical consideration. In particular, we may choose $H_a=H_b$, in which case the distinction between (4.4) and (4.9) becomes obvious. (4.10) then involves $2\Psi_a$, while (4.2) contains single Ψ_a. (Of course, these two Ψ_a's are different.)

V. MULTIPLE SCATTERING THEORY FOR REARRANGEMENT PROCESSES

In this section, we apply the reduction procedure of Sec. II to the multiple scattering theory of Watson[1] such that some of the strongly coupled rearrangement channels can be included explicitly. A brief review of the multiple scattering formalism for the elastic scattering should also clarify the essential role played by the channel Green's function in that theory. The equations we derive for the scattering wave functions will be expressed in terms of the two-particle amplitudes, insofar as the external interactions are concerned.

For the elastic scattering of single particles by a target nucleus with N_α nucleons, we have

$$\Psi=\Phi_\alpha+G_\alpha V_\alpha\Psi \tag{5.1}$$

with

$$\begin{aligned} &(H_\alpha-E)\Phi_\alpha=0, \\ &G_\alpha=(E+i\epsilon-H_\alpha), \end{aligned} \tag{5.2}$$

and

$$V_\alpha=H-H_\alpha=\sum_{i=1}^{N_\alpha} v_i, \tag{5.3}$$

where v_i is the interaction between the projectile and the ith target nucleon. Obviously, the internal interactions V^α for the α channel are contained in G_α, where $V^\alpha\equiv H_\alpha-H_0$. Following Watson, we set

$$\Psi=\Phi_\alpha+G_\alpha\sum_{i=1}^{N_\alpha} t_i^\alpha\varphi_i^\alpha, \tag{5.4}$$

where

$$t_i^\alpha=v_i+v_iG_\alpha t_i^\alpha. \tag{5.5}$$

The wave functions φ_i^α then satisfy the coupled equations[1,2]

$$\varphi_i^\alpha=\Phi_\alpha+G_\alpha\sum_{j\neq i}^{N_\alpha} t_j^\alpha\varphi_j^\alpha \tag{5.6}$$

or, in the differential form,

$$(E-H_\alpha)\varphi_i^\alpha=\sum_{j\neq i}^{N_\alpha} t_j^\alpha\varphi_j^\alpha. \tag{5.7}$$

Throughout the analysis, H_α appears only in G_α and Φ_α such that the complexity of H_α does *not* affect the formal manipulations involved in getting (5.6). Although the iteration kernels in (5.6) will not be fully connected for $N>3$, the formalism is simpler when G_α is used.

We now consider an extension of (5.4) to rearrangement processes, in which the strongly coupled channels are explicitly included. Using the result of Sec. II, we have, for example, for the α and β channels

$$\Psi=\Psi_\alpha+\Psi_\beta, \tag{5.8}$$

which satisfies equation (2.6). That is,

$$\begin{aligned} \Psi_\alpha &= a_\alpha\Phi_\alpha+G_\alpha V_\beta\Psi_\beta, \\ \Psi_\beta &= a_\beta\Phi_\beta+G_\beta V_\alpha\Psi_\alpha, \end{aligned} \tag{5.9}$$

where a_α and a_β are the initial boundary conditions, e.g. $a_\alpha=1$, $a_\beta=0$. We can introduce the two-par-

ticle amplitudes for the channels α and β, as

$$t_i^\alpha = v_i + v_i G_\alpha t_i^\alpha ,$$
$$t_j^\beta = v_j + v_j G_\beta t_j^\beta , \tag{5.10}$$

where i and j run over all the external interactions contained in the respective channels, i.e. $i \epsilon E(\alpha)$ and $j \epsilon E(\beta)$. Then, (5.9) may be written in the form

$$\Psi_\alpha = a_\alpha \Phi_\alpha + G_\alpha \sum_{j \epsilon E(\beta)} t_j^\beta \varphi_j^\beta ,$$
$$\Psi_\beta = a_\beta \Phi_\beta + G_\beta \sum_{i \epsilon E(\alpha)} t_i^\alpha \varphi_i^\alpha , \tag{5.11}$$

where φ_i^α and φ_j^β satisfy the coupled equations

$$\varphi_i^\alpha + G_\alpha t_i^\alpha \varphi_i^\alpha = a_\alpha \Phi_\alpha + G_\alpha \sum_{j \epsilon E(\beta)} t_j^\beta \varphi_j^\beta ,$$
$$\varphi_j^\beta + G_\beta t_j^\beta \varphi_j^\beta = a_\beta \Phi_\beta + G_\beta \sum_{i \epsilon E(\alpha)} t_i^\alpha \varphi_i^\alpha \tag{5.12}$$

or the differential form

$$(E - H_\alpha + t_i^\alpha)\varphi_i^\alpha = \sum_{j \epsilon E(\beta)} t_j^\beta \varphi_j^\beta ,$$
$$(E - H_\beta + t_j^\beta)\varphi_j^\beta = \sum_{i \epsilon E(\alpha)} t_i^\alpha \varphi_i^\alpha . \tag{5.13}$$

As in (5.6) and (5.7), (5.12) and (5.13) contain the channel Green's functions G_α and G_β in such a way that only the external interactions appear explicitly through the t's. Of course the total wave function obtained from (5.4) and (5.6) should be sufficient to determine all other reaction cross sections by the projections $\langle \Phi_\gamma | V_\gamma$. However, when approximations are introduced in (5.6), the β channel may not have been treated accurately, and (5.12) can be more efficient.

We can also extend the related formalism of Kerman, McManus, and Thaler[13] to rearrangement processes. For the elastic scattering, we have

$$\Psi = \Phi_\alpha + G_\alpha^A N_\alpha v_\alpha \Psi , \tag{5.14}$$

where

$$G_\alpha^A = A(E + i\epsilon - H_\alpha)^{-1} , \tag{5.15}$$

with the antisymmetrization operator A among the target nucleons. In (5.14), we kept the channel label on v_α for convenience. By introducing the projectile-target nucleon amplitude

$$t_\alpha = v_\alpha + v_\alpha G_\alpha^A t_\alpha , \tag{5.16}$$

(5.14) can be written in the form

$$\Psi = \Phi_\alpha + (N_\alpha - 1) G_\alpha^A t_\alpha \Psi \tag{5.17}$$

or, equivalently,

$$[E - H_\alpha - (N_\alpha - 1)t_\alpha]\Psi = 0 . \tag{5.18}$$

The form (5.17), which is specially suitable for the elastic scattering, may be generalized to direct reactions involving rearrangements. Thus, for channels 1 and 2, we have from (2.6),

$$\Psi = \Psi_1 + \Psi_2 \tag{5.19}$$

with

$$\Psi_\alpha = a_\alpha \Phi_\alpha + G_\alpha^A N_\beta v_\beta \Psi_\beta \quad (\alpha, \beta = 1, 2; \alpha \neq \beta) . \tag{5.20}$$

The potentials v_α and v_β may be eliminated in terms of the pair amplitudes (5.10); (5.20) becomes, for $\alpha \neq \beta$,

$$[E - H_\alpha + t_\beta + t_\beta G_\beta^A (G_\alpha^A)^{-1}]\Psi_\alpha = N_\beta t_\beta \Psi_\beta . \tag{5.21}$$

On the other hand, a slightly different definition for t_α in the form

$$t'_\alpha = v_\alpha + v_\alpha G_\beta^A t'_\alpha \quad (\alpha \neq \beta) \tag{5.22}$$

gives

$$[E - H_\alpha + t'_\beta]\Psi_\alpha = N_\beta t'_\beta \Psi_\beta . \tag{5.23}$$

Still another choice for t would be

$$t''_\alpha = v_\alpha + v_\alpha G_\beta^A t''_\beta , \tag{5.24}$$

which couples the t's in the α and the β channels. Substitution of (5.24) into (5.20) gives

$$[E - H_\alpha + t''_\alpha]\Psi_\alpha = N_\beta t''_\beta \Psi_\beta . \tag{5.25}$$

Equation (5.25) is more closely related in form to (5.11), and (5.24) related to (5.12).

VI. SUMMARY

When some of the two-particle (internal) interactions of an N-particle system are included in the channel Green's functions G_α, the resulting set of coupled equations simplifies greatly. By requiring the partial connectivity only among the direct reaction channels which appear explicitly in the coupled equations, much of the formal and complicated manipulations involved in the usual many-particle scattering theory can be avoided. If desired, the connectivity property of the iteration kernels of the RME can be improved by introducing one or more auxiliary channels. Thus, rather than constructing first the fully connected set of equations and then approximating them,[27] we have shown in this paper how the simple sets of RME can be derived, in which the channel Green's functions play an essential role. The introduction of the distortion potentials Y and the channel projection operators P and Q should further facilitate the solution of these equations, while the use of

the auxiliary channels and the approximations such as (4.6) should help improve the connectivity property.

By construction, the RME do not contain the nonorthogonality difficulty insofar as the direct reaction channels are concerned. The coupling potentials in the RME set do not have the usual nonorthogonality terms. Since all the other weakly coupled channels are treated in the RME only implicitly through the channel Hamiltonians H_α and the channel Green's functions G_α, one should be careful in obtaining the solution of the RME so as not to further decompose the cluster structures represented by H_α. In this connection, the use of the projection operators and the approximations such as (4.6) should be helpful. Some preliminary studies of the applicability of the RME have already been reported.[21,22] Further theoretical analysis of the RME formulation of this paper and the approach of Ref. 27 may provide an improved approximation procedure useful in treating the direct nuclear reaction problems.

APPENDIX

The reduction of the Faddeev equations (2.1) to the forms such as (2.5) and (2.6) involves certain rearrangements of the contribution of the eliminated channels among the different components Ψ_α of the total wave function Ψ. To examine explicitly how this is brought about in the course of the reduction, we can of course go back to the total Green's function $G=(E+i\epsilon-H)^{-1}$ and decompose it for the various rearrangement channels, as we have done in Ref. 20. Thus, after some algebra, we obtain

$$G=\bar{G}_3+G^{(1)}+G^{(2)} \tag{A1}$$

with

$$G^{(1)}=G_1V_2\bar{G}_3+G_1V_2G^{(2)},$$
$$G^{(2)}=G_2V_1\bar{G}_3+G_2V_1G^{(1)},$$

where

$$\bar{G}_3=(E+i\epsilon-H_0+v_{12})^{-1}.$$

Note that $\bar{G}_3$ differs from $G_3=(E+i\epsilon-H_0-v_{12})^{-1}$.

The decomposition (A1) gives

$$\Psi=\Phi_1+GV_1\Phi_1=\Psi_1''+\Psi_2'', \tag{A2}$$

where

$$\Psi_1''=\Phi_1+G^{(1)}V_1\Phi_1,$$
$$\Psi_2''=\bar{G}_3V_1\Phi_1+G^{(2)}V_1\Phi_1.$$

This result is exact, but the role of the third channel wave function is not obvious; its effect is carried implicitly by $\bar{G}_3$ and $G^{(\alpha)}$.

More directly, however, we can write

$$\Psi_3=\Psi_{31}+\Psi_{32}, \tag{A3}$$

where Ψ_3 satisfies

$$(H_3-E)\Psi_3=-v_{12}(\Psi_1+\Psi_2). \tag{A4}$$

Since H_3 may be rearranged as

$$H_3=H_1+(v_{12}-v_{23})=H_2+(v_{12}-v_{13}),$$

(A4) can be rewritten in the form

$$(H_1-E)\Psi_{31}+(v_{12}-v_{23})\Psi_{31}+(H_2-E)\Psi_{32}+(v_{12}-v_{13})\Psi_{32}=-v_{12}(\Psi_1+\Psi_2). \tag{A5}$$

The two functions Ψ_{31} and Ψ_{32} in (A3) are still undefined separately, so that we may *set* freely

$$(H_1-E)\Psi_{31}=-v_{12}\Psi_2-v_{12}\Psi_{32}+v_{23}\Psi_{31}, \tag{A6a}$$

$$(H_2-E)\Psi_{32}=-v_{12}\Psi_1-v_{12}\Psi_{31}+v_{13}\Psi_{32}, \tag{A6b}$$

which is certainly consistent with (A4). The step involved in getting (A6) from (A5) is equivalent to the reduction procedure of Sec. II; we recall[19] that (2.6) for example can be obtained simply by starting with the definition (2.7), which gives

$$(H_1-E)\Psi_1''+V_1\Psi_1''+(H_2-E)\Psi_2''+V_2\Psi_2''=0,$$

and *requiring* Ψ_1'' and Ψ_2'' to satisfy separately

$$(H_\alpha-E)\Psi_\alpha''=-V_B\Psi_B'', \quad \alpha,\beta=1,2 \text{ and } \alpha\neq\beta.$$

On the other hand, the Ψ_1 equation of (2.1) is, with (A3),

$$(H_1-E)\Psi_1=-v_{23}(\Psi_2+\Psi_{31}+\Psi_{32}). \tag{A7}$$

Adding (A6a) and (A7), we then have

$$(H_1-E)(\Psi_1+\Psi_{31})=-V_2(\Psi_2+\Psi_{32}) \tag{A8}$$

and similarly for Ψ_2 with (A6b). Therefore, comparing this with (2.6), we identify

$$\Psi_1''\equiv\Psi_1+\Psi_{31},$$
$$\Psi_2''\equiv\Psi_2+\Psi_{32}, \tag{A9}$$

where, from (A6),

$$\Psi_{31}=G_0v_{12}\Psi_2+G_0v_{12}\Psi_{32},$$
$$\Psi_{32}=G_0v_{12}\Psi_1+G_0v_{12}\Psi_{31}. \tag{A10}$$

Evidently, Ψ_1 and Ψ_2 contain the outgoing waves in channel 3 through the components Ψ_{31} and Ψ_{32}, and we have shown above precisely how they are distributed. (A9) is consistent with the asymptotic behavior (3.2) and (3.3), while (A10) shows that the outgoing waves in channel 3 can arise from the iteration kernel $G_0v_{12}G_0v_{12}$, in agreement with what one expects from (A4).

Similar analysis can be carried out for other sets of reduced equations, but the decompositions such as (A1) for the Green's function and (A6) for the

wave functions are not easy to obtain in general. The reduction procedure of Sec. II eliminates much of the formal manipulations involved, and allows one to construct a set of equations directly from the physical consideration. To understand the contents of the theory, however, one has to go through the decomposition such as that presented here.

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