Calculation of the Λ-particle binding energy in nuclear matter with a simple functional variation method

S. Ali* and M. E. Grypeos†
International Centre for Theoretical Physics, Trieste, Italy

B. Kargas
Department of Theoretical Physics, University of Thessaloniki, Greece
(Received 19 February 1976)

Results of calculations based on a first order functional variation method for the binding energy D of a Λ particle in infinite nuclear matter are reported. Some additional details of the formal part of the problem which had been previously discussed are also given. In particular, the "stability condition" is studied and the analytic solution in the case of the exponential with hard core potentials is also derived. Various nucleon-nucleon correlation functions and central Λ-nucleon potentials are used in order to investigate the dependence of D on them. The results, which are obtained by including in the computations the second order terms (in the density ρ of nuclear matter) show, for some potentials, large overbinding of the Λ particle, while for other potentials the overbinding is considerably reduced. A discussion of the results is finally made.

[NUCLEAR STRUCTURE Binding energy of Λ particle, Λ-nucleon interaction.]

I. INTRODUCTION

A number of authors have performed calculations in order to obtain theoretically the binding (or separation) energy of a Λ particle in (infinite) nuclear matter: D, which is known empirically to be $D \lesssim 27-35$ MeV.[1] The complexity of the problem makes it necessary to use in the various approaches approximations, which together with the uncertainties pertaining the Λ-nucleon interaction, lead to ambiguities regarding to the reliability of the results. A review of calculations based on reaction matrix techniques has been given in Ref. 2. Variational techniques have also been employed.[3-7] Both methods lead to overbinding of the Λ particle, if s-wave central Λ-nucleon potentials are used.

Various efforts have been made in order to reduce the theoretical value of D.[8] These consist mainly in suppressing the Λ-nucleon interaction in p waves, including the coupling to the ΣN channel, a tensor component, three-body forces, etc. Although it is possible with the inclusion of these effects in the Λ-nucleon interaction to obtain values of D in the range which is compatible with the empirical estimates, it should be clear, as it was already pointed out, that a part of the discrepancy should be due to the approximations made in the calculation. The overbinding appears both in the reaction matrix and in the variational approach, even if correlation functions containing parameters are used in the latter. It appears therefore desirable to further investigate these methods with the aim of detecting possible causes of inaccuracies.

In the present paper we give results for D, based on the variational approach but using a correlation function which is determined by functional variation of the first order term $E_\Lambda^{(1)}$ in the cluster expansion, as it has been previously described.[5] In the second section we give a summary of the formalism and we also provide details of the derivation of the "stability condition." It is shown that the Λ-nucleon correlation function f_0, determined by functional variation minimizes indeed $E_\Lambda^{(1)}$. In the third section, we give the analytic solution of the Euler equation for f_0, in the case of the exponential potential with hard core, since all the potentials which are used are taken from Refs. 9-11 and are of this shape. It is shown that, in this case, the correlation function can be expressed in terms of new transcendental functions which are given as series expansions of well known functions. After giving some details about the nucleon-nucleon correlation functions and the Λ-nucleon potentials in the fourth section, we present our results. In the final section, we discuss and compare these results with those obtained with other methods.

II. SUMMARY OF THE FORMALISM AND THE STABILITY CONDITION

The Hamiltonian of the system (Λ particle + nuclear matter) is

$$H = H_\Lambda + H_N , \tag{1}$$

where H_N is the Hamiltonian of the (infinite and uniform) nuclear matter and H_Λ the Hamiltonian of the Λ particle. This is taken to be of the form[3]

$$H_\Lambda = -\frac{\hbar^2}{2M_\Lambda}\nabla_\Lambda{}^2 + \sum_{i=1}^{N} V(r_{\Lambda i}) . \tag{2}$$

The trial wave function of the system is taken to be

$$\Psi^{\text{tr}} = \Phi_N \prod_{i=1}^{N} f(r_{\Lambda i}) , \tag{3}$$

where Φ_N is the ground state wave function of nuclear matter, corresponding to energy E_N and f the Λ-nucleon correlation function.

The energy of the Λ particle $E_\Lambda = -D$ is then given by the expression

$$E_\Lambda \leq E_\Lambda^{\text{tr}} = \frac{\langle\Psi|H|\Psi\rangle}{\langle\Psi|\Psi\rangle} - E_N . \tag{4}$$

Following the procedure described in Ref. 3, the expression for the trial energy may be written

$$E_\Lambda = E_\Lambda^{(1)} + E_\Lambda^{(2)} + \mathcal{O}(\rho^3) . \tag{5}$$

The upperscript tr has been omitted for simplicity. The trial energy should be close to the actual one, provided that the trial function is good enough and the approximations made, reasonable.

The $E_\Lambda^{(1)}$ and $E_\Lambda^{(2)}$ in expression (5) are given as follows:

$$E_\Lambda^{(1)} = \rho \int [f(r_{\Lambda 1}) H_{\Lambda 1} f(r_{\Lambda 1})] d\vec{r}_{\Lambda 1} , \tag{6}$$

$$E_\Lambda^{(2)} = \rho^2 \left\{ \frac{1}{\Omega}\int [f(r_{\Lambda 1})H_{\Lambda 1}f(r_{\Lambda 1})]F(r_{\Lambda 2})K(r_{12})d\vec{r}_\Lambda d\vec{r}_1 d\vec{r}_2 - \frac{\hbar^2}{2M_N}\frac{1}{2\Omega}\int F(r_{\Lambda 2})[\vec{\nabla}_1 F(r_{\Lambda 1})\cdot\vec{\nabla}_1 K(r_{12})]d\vec{r}_\Lambda d\vec{r}_1 d\vec{r}_2 \right.$$
$$\left. - \frac{\hbar^2}{2M_\Lambda}\frac{1}{4\Omega}\int [\vec{\nabla}_\Lambda F(r_{\Lambda 1})\cdot\vec{\nabla}_\Lambda F(r_{\Lambda 2})]K(r_{12})d\vec{r}_\Lambda d\vec{r}_1 d\vec{r}_2 \right\} , \tag{7}$$

where $H_{\Lambda 1}$ is the Hamiltonian of the relative motion of a Λ-nucleon pair

$$H_{\Lambda 1} = -\frac{\hbar^2}{2\mu_{\Lambda N}}\nabla_{\Lambda 1}{}^2 + V(r_{\Lambda 1}) . \tag{8}$$

Also $F = f^2 - 1$ and $K(r_{12})$ is given by

$$K(r_{12}) = (g_{(r_{12})}{}^2 - 1) - g_{(r_{12})}{}^2\, l_{(K_F r_{12})}{}^2/4 , \tag{9}$$

where g is the nucleon-nucleon correlation function and $l(x) = 3j_1(x)/x$, K_F being the Fermi momentum, which is related to the density ρ.

In order to obtain a value for E_Λ, given the Λ-nucleon interaction and the properties of the "host medium" (nuclear matter), it is necessary to know the Λ-nucleon correlation function. As it was stated in the Introduction this function is determined here as in Ref. 5 by varying functionally the first order contribution $E_\Lambda^{(1)}$. The omission of the higher terms is compensated by imposing the following integral constraint

$$\rho \int (f-1)^2 d\vec{r}_{\Lambda 1} = I_1 (= \text{finite constant}) . \tag{10}$$

This is referred to as the "healing condition" and has the effect of introducing a Lagrange multiplier λ^2 into the variational problem.

The Euler equation for the correlation function f_0 which makes $E_\Lambda^{(1)}$ stationary is

$$\frac{d^2 f_0}{dr^2} + \frac{2}{r}\frac{df_0}{dr} - \frac{2\mu_{\Lambda N}}{\hbar^2}V_{\Lambda N}(r) f_0 - \beta^2 f_0 = -\beta^2 , \tag{11}$$

$$c \leq r < \infty ,$$

where $r = r_{\Lambda 1}$ and $\beta^2 = (2\mu_{\Lambda N}/\hbar^2)\lambda^2$.

The boundary conditions are

$$f_0(c) = 0 ; \quad f_0(\infty) = 1 . \tag{12}$$

The corresponding expression of $E_\Lambda^{(1)}$ takes the form

$$E_\Lambda^{(1)} = 4\pi\rho \frac{\hbar^2}{2\mu_{\Lambda N}} \beta^2 \int_c^\infty f_0(r)[1 - f_0(r)] r^2 dr . \tag{13}$$

The choice of the Lagrange multiplier is discussed in Secs. IV and V. Here we would like to investigate the "stability condition." It is a rather interesting feature of the present approach that a condition may be derived for $E_\Lambda^{(1)}(f_0)$ to be a minimum. It is instructive to consider this in some detail. The procedure, which will be described is similar to that developed in an analogous case for finite nuclei.[12] In that case, the variation is with respect to two-body correlated relative wave functions Ψ_{nls} in the oscillator shell model, which appear in the two-body matrix elements of the energy expression. There is also an additional Lagrange multiplier, due to the "normalization condition."

We first remark that expression (6) may also be written in the form

$$E_\Lambda^{(1)}(f) = 4\pi\rho \int_c^\infty \left[\left(\frac{\hbar^2}{2\mu_{\Lambda N}}\right)\left(\frac{df}{dr}\right)^2 + V(r) f^2(r)\right] r^2 dr \tag{14}$$

(provided of course, that df/dr tends to zero, as $r \to \infty$, sufficiently rapidly). Expression (14) can be easily shown by partial integration.

We may now proceed to calculate $\Delta E_\Lambda^{(1)} = E_\Lambda^{(1)}(f_1) - E_\Lambda^{(1)}(f_0)$, where

$$f_1 = f_0 + \eta . \tag{15}$$

Since f_1 too must satisfy the required boundary conditions, it follows that

$$\eta(c) = 0 \quad \text{and} \quad \eta(\infty) = 0 . \tag{16}$$

By substituting (15) into $E_\Lambda^{(1)}(f_1)$ [expression (14)] and taking into account the boundary conditions on η, we arrive after some algebra at the following expression

$$E_\Lambda^{(1)}(f_1) = E_\Lambda^{(1)}(f_0) + 4\pi\rho\left\{2\int_c^\infty \eta(r)\left[\left(-\frac{\hbar^2}{2\mu_{\Lambda N}}\right)\left(\frac{d^2 f_0}{dr^2} + \frac{2}{r}\frac{df_0}{dr}\right) + V(r) f_0\right] r^2 dr + \int_c^\infty \eta(r)\left[\left(-\frac{\hbar^2}{2\mu_{\Lambda N}}\right)\left(\frac{d^2\eta}{dr^2} + \frac{2}{r}\frac{d\eta}{dr}\right) + V(r)\eta\right] r^2 dr\right\} . \tag{17}$$

In order to eliminate f_0 from the above integrals, two things may be taken into account. The first is that f_0 satisfies the Euler equation (11) and the second that the variation is performed within the class of functions which comply with constraint (10). The healing integral I_1 for f_1 and f_0 is therefore the same:

$$\int_c^\infty (f_1 - 1)^2 r^2 dr = \int_c^\infty (f_0 - 1)^2 r^2 dr . \tag{18}$$

Taking into account that f_1 is given by (15), this expression is equivalent to

$$-2\int_c^\infty \eta(r)[f_0(r) - 1] r^2 dr = \int_c^\infty \eta^2(r) r^2 dr . \tag{19}$$

In view of the above remarks, the first integral in expression (17), with the factor 2 included, may be written successively as

$$-\lambda^2 2\int_c^\infty \eta(f_0 - 1) r^2 dr = \lambda^2 \int_c^\infty \eta^2(r) r^2 dr . \tag{20}$$

The expression for $\Delta E_\Lambda^{(1)}$ takes therefore the final form

$$\Delta E_\Lambda^{(1)} = 4\pi\rho \int_c^\infty \eta(r)\left[\left(-\frac{\hbar^2}{2\mu_{\Lambda N}}\right)\left(\frac{d^2\eta}{dr^2} + \frac{2}{r}\frac{d\eta}{dr}\right) + V(r)\eta(r) + \lambda^2\eta(r)\right] r^2 dr . \tag{21}$$

It is seen from this expression that $\Delta E_\Lambda^{(1)}$ is positive and therefore $E_\Lambda(f_0)$ is a minimum unless the eigenvalue problem

$$\left(-\frac{\hbar^2}{2\mu_{\Lambda N}}\right)\left(\frac{d^2\eta}{dr^2} + \frac{2}{r}\frac{d\eta}{dr}\right) + [V_{\Lambda N}(r) + \lambda^2]\eta = k\eta , \tag{22}$$

$$\eta(c) = 0 , \quad \eta(\infty) = 0 , \quad \lambda^2 \neq 0$$

has a nonpositive eigenvalue. It is indeed the case that the eigenvalue k is positive. Even for $\lambda^2 = 0$, there is no negative eigenvalue, because this would mean that the Λ-nucleon system could be in a bound state and it is well known that such a system has not been found and it is not expected to be found.

III. ANALYTIC SOLUTION OF THE EULER EQUATION FOR THE EXPONENTIAL WITH HARD CORE POTENTIAL

One feature of the simple functional variation approach we are discussing is the possibility of obtaining analytic solutions for some potentials. The analytic solutions for the square well with hard core potential were obtained in Ref. 5(a). In this paper we shall give some details of the derivation of the analytic solution in the case of a Λ-nucleon potential of exponential shape with hard core. This appears to be interesting, because the potentials, which are used in the present investigation as well as in many others, are of this type.

It is convenient to perform the transformation

$$f_0 = 1 - \frac{\Theta(r)}{r} . \tag{23}$$

In the differential equation for Θ there is no first derivative. This equation may be written as follows:

$$\frac{d^2\Theta(r)}{dr^2} + \left(-\frac{2\mu_{\Lambda N}}{\hbar^2}\right) V(r)\Theta(r) - \beta^2\Theta(r) = \left(-\frac{2\mu_{\Lambda N}}{\hbar^2}\right) r V(r) , \quad c \leq r < \infty . \tag{24}$$

The boundary conditions for $\Theta(r)$ are

$$\Theta(c) = c , \quad \Theta(\infty) = 0 . \tag{25}$$

The Λ-nucleon potential is assumed to be of the form

$$V(r) = -V_0 e^{-\mu(r-c)} = -V_0 e^{\mu c}\cdot e^{-\mu r} . \tag{26}$$

Equation (24) may therefore be written

$$\frac{d^2\Theta(r)}{dr^2} + V_0' e^{-\mu r}\Theta(r) - \beta^2\Theta(r) = V_0' r e^{-\mu r} , \tag{27}$$

where $V_0' = (2\mu_{\Lambda N}/\hbar^2) V_0 e^{\mu c}$.

The general solution of the homogeneous equation which corresponds to (27) can be expressed in terms of Bessel's functions of first kind

$$\Theta_{\text{homog.}} = C_1 J_\nu\left(\frac{2\sqrt{V_0'}}{\mu} e^{-\mu r/2}\right) + C_2 J_{-\nu}\left(\frac{2\sqrt{V_0'}}{\mu} e^{-\mu r/2}\right) \equiv C_1\Theta_1 + C_2\Theta_2 \tag{28}$$

provided that the parameter $\nu \equiv 2\beta/\mu$ is not an integer.[13]

A particular solution may be found in the usual way:

$$\Theta_{\text{part.}} = -\Theta_1(r)\int_c^r \frac{h(r')\Theta_2(r')dr'}{W(\Theta_1,\Theta_2)} + \Theta_2(r)\int_c^r \frac{h(r')\Theta_1(r')}{W(\Theta_1,\Theta_2)}dr', \tag{29}$$

where $h(r)$ is the inhomogeneous part of Eq. (27) and $W(\Theta_1, \Theta_2)$ the Wronskian of the functions Θ_1 and Θ_2, which is given by

$$W(\Theta_1,\Theta_2) = \left[J_\nu(z)\frac{dJ_{-\nu}(z)}{dz} - J_{-\nu}(z)\frac{dJ_\nu}{dz}\right]\frac{dz}{dr'} = \frac{\mu \sin\nu\pi}{\pi} \tag{30}$$

with $z = (2\sqrt{V_0'}/\mu)e^{-\mu r/2}$.

The expression for $\Theta(r)$ becomes

$$\Theta(r) = \left[C_1 - \frac{\pi V_0'}{\mu \sin\nu\pi}\int_c^r r' e^{-\mu r'} J_{-\nu}\left(\frac{2\sqrt{V_0'}}{\mu}e^{-\mu r'/2}\right)dr'\right] J_\nu\left(\frac{2\sqrt{V_0'}}{\mu}e^{-\mu r/2}\right) + \left[C_2 + \frac{\pi V_0'}{\mu\sin\nu\pi}\int_c^r r'e^{-\mu r'}J_\nu\left(\frac{2\sqrt{V_0'}}{\mu}e^{-\mu r'/2}\right)dr'\right]J_{-\nu}\left(\frac{2\sqrt{V_0'}}{\mu}e^{-\mu r/2}\right). \tag{31}$$

The constants are determined by the boundary conditions. We find

$$C_2 = -\frac{\pi V_0'}{\mu \sin\nu\pi}\int_c^\infty r' e^{-\mu r'} J_\nu\left(\frac{2\sqrt{V_0'}}{\mu}e^{-\mu r'/2}\right)dr' \tag{32}$$

and

$$C_1 = \left[c - C_2 J_{-\nu}\left(\frac{2\sqrt{\overline{V}_0}}{\mu}\right)\right]\Big/ J_\nu\left(\frac{2\sqrt{\overline{V}_0}}{\mu}\right), \tag{33}$$

where $\overline{V}_0 = (2\mu_{\Lambda N}/\hbar^2)V_0$.

The solution may therefore be written in the following form

$$\Theta(r) = C_1 J_\nu\left(\frac{2\sqrt{V_0'}}{\mu}e^{-\mu r/2}\right) - \frac{\pi V_0'}{\mu\sin\nu\pi}\left[J_\nu\left(\frac{2\sqrt{V_0'}}{\mu}e^{-\mu r/2}\right)\int_c^r r'e^{-\mu r'}J_{-\nu}\left(\frac{2\sqrt{V_0'}}{\mu}e^{-\mu r'/2}\right)dr' + J_{-\nu}\left(\frac{2\sqrt{V_0'}}{\mu}e^{-\mu r/2}\right)\int_r^\infty r'e^{-\mu r'}J_\nu\left(\frac{2\sqrt{V_0'}}{\mu}e^{-\mu r'/2}\right)dr'\right]. \tag{34}$$

If we take into account the series expansion of Bessel's functions and perform the integration we can express Θ, in a rather simple form, in terms of new transcendental functions, which are defined by series of well known functions. After some algebra we arrive at the following expression:

$$\Theta(r) = c\frac{S_+(r)}{S_+(c)}e^{-\beta(r-c)} + \left(\frac{\nu}{\mu}\right)\frac{2\mu_{\Lambda N}}{\hbar^2}V_0\left\{e^{-\beta(r-c)}\frac{S_+(r)}{S_+(c)}[S_+(c)E_-(c) - S_-(c)E_+(c)] + e^{-\mu(r-c)}[S_-(r)E_+(r) - S_+(r)E_-(r)]\right\}, \tag{35}$$

where the functions $S_\pm(r)$ and $E_\pm(r)$ are defined as follows:

$$S_\pm(r) = \sum_{k=0}^\infty \frac{(-1)^k(\overline{V}_0/\mu^2)^k e^{-k\mu(r-c)}}{k!(\pm\nu)(\pm\nu+1)\cdots(\pm\nu+k)}, \tag{36}$$

$$E_\pm(r) = \sum_{k=0}^\infty \frac{(-1)^k(\overline{V}_0/\mu^2)^k e^{-k\mu(r-c)}}{k!(\pm\nu)(\pm\nu+1)\cdots(\pm\nu+k)}\left\{\frac{r}{[\pm\nu\mu/2 + \mu(k+1)]} + \frac{1}{[\pm\nu\mu/2 + \mu(k+1)]^2}\right\}. \tag{37}$$

It is worth mentioning that somehow similar in structure is the s-wave solution for the same potential of the differential equation of the reference spectrum method for nuclear matter.[14] One of the main differences is the lack of trigonometric functions in the solution of the present equation.

The usefulness of the analytic solution (35) we have derived is that we can obtain the asymptotic behavior of the correlation function and also that we can check the accuracy of the numerical solution.

TABLE I. Parameters of the nucleon-nucleon potentials.

Potential	c (fm)	μ_s (fm^{-1})	V_s^{even} (MeV)	μ_t (fm^{-1})	V_t^{even} (MeV)
OMY-I	0.6	2.6272	397.31	3.6765	947.02
OMY-II	0.4	2.0344	235.41	2.5214	475.04

IV. NUMERICAL RESULTS FOR VARIOUS NUCLEON- NUCLEON CORRELATION FUNCTIONS AND Λ-NUCLEON POTENTIALS

In this section we shall give the results of our calculations.[15]

The first-order energy was computed directly from expression (6) by numerical integration. For the second-order energy, expression (7) may be used. This is easily written in the following form, which can be used immediately in the computations:

$$E_\Lambda^{(2)} = 4\pi\rho \int_0^\infty \Bigg\{ M(r_{\Lambda 1}) f(r_{\Lambda 1}) \left[\left(-\frac{\hbar^2}{2\mu_{\Lambda N}}\right)\left(\frac{d^2 f(r_{\Lambda 1})}{dr_{\Lambda 1}^2} + \frac{2}{r_{\Lambda 1}}\frac{df(r_{\Lambda 1})}{dr_{\Lambda 1}}\right) + V(r_{\Lambda 1}) f(r_{\Lambda 1})\right] + [R(r_{\Lambda 1}) + S(r_{\Lambda 1})] f(r_{\Lambda 1}) \frac{df(r_{\Lambda 1})}{dr_{\Lambda 1}} \Bigg\} dr_{\Lambda 1}\,, \tag{38}$$

where

$$M(r_{\Lambda 1}) = 2\pi\rho r_{\Lambda 1} \int_0^\infty r_{\Lambda 2}[f^2(r_{\Lambda 2}) - 1]dr_{\Lambda 2} \int_{|r_{\Lambda 1}-r_{\Lambda 2}|}^{r_{\Lambda 1}+r_{\Lambda 2}} r_{12} K(r_{12}) dr_{12}\,, \tag{39}$$

$$R(r_{\Lambda 1}) = \pi\rho\left(-\frac{\hbar^2}{2M_N}\right) \int_0^\infty r_{\Lambda 2}[f^2(r_{\Lambda 2}) - 1]dr_{\Lambda 2} \int_{|r_{\Lambda 1}-r_{\Lambda 2}|}^{r_{\Lambda 1}+r_{\Lambda 2}} \frac{dK(r_{12})}{dr_{12}} (r_{\Lambda 1}^2 - r_{\Lambda 2}^2 + r_{12}^2) dr_{12} \tag{40}$$

and

$$S(r_{\Lambda 1}) = \pi\rho\left(-\frac{\hbar^2}{2M_\Lambda}\right) \int_0^\infty f(r_{\Lambda 2}) \frac{df(r_{\Lambda 2})}{dr_{\Lambda 2}} dr_{\Lambda 2} \int_{|r_{\Lambda 1}-r_{\Lambda 2}|}^{r_{\Lambda 1}+r_{\Lambda 2}} r_{12} K(r_{12})(r_{\Lambda 1}^2 + r_{\Lambda 2}^2 - r_{12}^2) dr_{12}\,. \tag{41}$$

The expression in the first brackets [ ] in (38) may also be substituted from Eq. (11) since the correlation function f_0 is used in the present analysis.

It is advisable to use in practice the following equivalent formula, which is obtained from expression (7) by applying Green's theorem (see also Ref. 4):

$$E_\Lambda^{(2)} = 4\pi\rho \int_0^\infty \Bigg(M(r_{\Lambda 1}) \Bigg\{ \frac{\hbar^2}{2\mu_{\Lambda N}} \left(\frac{df(r_{\Lambda 1})}{dr_{\Lambda 1}}\right)^2 + V(r_{\Lambda 1}) f^2(r_{\Lambda 1}) - \frac{\hbar^2}{4M_\Lambda}\left[\left(\frac{df(r_{\Lambda 1})}{dr_{\Lambda 1}}\right)^2 + f(r_{\Lambda 1})\left(\frac{d^2 f(r_{\Lambda 1})}{dr_{\Lambda 1}^2} + \frac{2}{r_{\Lambda 1}}\frac{df(r_{\Lambda 1})}{dr_{\Lambda 1}}\right)\right]\Bigg\}\Bigg) dr_{\Lambda 1}\,. \tag{42}$$

The main advantage in using the above expression is that only one two-dimensional integral (that corresponding to M) of a "product type integrand" has to be computed for each value of $r_{\Lambda 1}$. The computations in this case are less time consuming.

The results reported in this section were obtained with expression (42), although it has been checked that practically almost the same value is obtained if expression (38) is used instead.

The value of the density of the "host medium" (uniform and infinite nuclear matter) is taken to be $\rho = 0{,}172$ nucleons/fm^3. This is appropriate to the central density of heavy nuclei.

Several choices are made for the nucleon-nucleon correlation function $g(r_{12})$, in order to investigate the sensitivity of the results to this function.

Firstly, the following nucleon-nucleon correlation function is assumed

TABLE II. Parameters of the Λ-nucleon potentials.

Potential	c (fm)	μ (fm^{-1})	V_0 (MeV)
H	0.6	3.935	685.95
F'	0.6	4.427	851.7
E'	0.45	3.219	398.9
E	0.45	3.219	414.5
DW	0.4	3.219	330.9
HTS	0.4	5.059	1020.8
B'	0.3	3.935	544.6

TABLE III. Detailed results with correlation function g_2 and potential H.

β (fm^{-1})	$E_\Lambda^{(1)}$ (MeV)	$E_\Lambda^{(2)}$ (MeV)	$E_\Lambda^{(1)}+E_\Lambda^{(2)}$ (MeV)	$\lvert E_\Lambda^{(2)}/E_\Lambda^{(1)}\rvert$	I_1	I_2
1	−100.0	28.7	−71.3	0.29	0.443	4.794
2	−87.5	12.3	−75.2	0.14	0.256	0.890
3	−81.8	9.1	−72.7	0.11	0.230	0.246
4	−76.6	8.4	−68.2	0.11	0.219	0.032
5	−70.6	8.6	−62.0	0.12	0.211	0.063

$$g(r_{12})=1\,,\quad 0\le r_{12}<\infty\,. \tag{43}$$

This means that the dynamical nucleon-nucleon correlations are completely neglected. "Pauli correlations," however, which have their origin in the antisymmetry of the wave function of nuclear matter and enter in the variational expression through the l functions, are included.

Secondly, the following nucleon-nucleon correlation function is considered:

$$g_2(r_{12})=\begin{cases}0\,, & 0\le r_{12}\le c_{NN}\\ (1-e^{-a(r_{12}-c_{NN})})(1+be^{-a(r_{12}-c_{NN})})\,, & \end{cases}$$

$$c_{NN}\le r_{12}<\infty\,. \tag{44}$$

This correlation function has also been used in Ref. 6. The values of the parameters are

$$c_{NN}=0.6\ \text{fm}\,,\quad a=2.30\ \text{fm}^{-1}\,,\quad b=1.394\,.$$

The parameter a is determined by minimizing at the experimental density the first-order expression in the cluster expansion of the energy per particle in nuclear matter by using the potential of Ohmura, Morita, and Yamada,[16] which is of exponential shape with hard core. This potential will be denoted here as OMY-I. Its parameters are given in Table I.

The parameter b of g_2 is fixed for each value of a by the normalization condition.[6]

Finally, a nucleon-nucleon correlation function g_3 of the same shape as that of the previous one is considered, but with values of parameters as follows:

$$c_{NN}=0.4\ \text{fm}\,,\quad a=2.338\ \text{fm}^{-1}\,,\quad b=1.257\,.$$

The procedure in determining the parameters is quite analogous, but now the OMY-II potential is used.[17] The parameters for this potential too are given in Table I.

A variety of central Λ-nucleon potentials are used in the present calculation. All of them are of exponential shape with hard core

$$V_{N\Lambda}(r)=-V_0e^{-\mu(r-c_{N\Lambda})}\,, \tag{45}$$

where

$$V_0=\tfrac{1}{4}V_0^s+\tfrac{3}{4}V_0^t\,, \tag{46}$$

V_0^s and V_0^t being the depths of the potential is singlet and triplet states, respectively.

The notation for each potential and the corresponding values for the hard core radius $c_{N\Lambda}$, the range μ, and the average depth V_0 are listed in Table II. Most of these potentials have been used by other authors as well,[2-7,18,19] so that we may compare our results.

Potentials H, F', E', E, and B' have been determined by Herndon and Tang,[11] while DW and HTS are the older potentials of Downs and Ware[9] and of Herndon, Tang, and Schmid,[10] respectively.

Concerning the potentials of Herndon and Tang, it should be noted that the unprimed ones have been determined by fitting the three- and four-body hypernuclear data, while the primed potentials have been determined by fitting the three- and five-body hypernuclear data. The potentials, which have been used in the present calculation are without suppression in odd-parity states. It should be recalled, in connection with this, that the fitting of the Λ-p scattering data requires a reduction of the strength of the Λ-nucleon poten-

TABLE IV. Detailed results with correlation function g_2 and potential E'.

β (fm^{-1})	$E_\Lambda^{(1)}$ (MeV)	$E_\Lambda^{(2)}$ (MeV)	$E_\Lambda^{(1)}+E_\Lambda^{(2)}$ (MeV)	$\lvert E_\Lambda^{(2)}/E_\Lambda^{(1)}\rvert$	I_1	I_2
1	−79.8	16.8	−63.0	0.21	0.250	3.926
2	−71.7	7.3	−64.4	0.10	0.130	0.808
3	−67.8	5.2	−62.6	0.08	0.113	0.282
4	−64.2	4.6	−59.6	0.07	0.105	0.105
5	−60.1	4.5	−55.6	0.08	0.099	0.026

TABLE V. Detailed results with correlation function g_2 and potential B'.

β (fm^{-1})	$E_\Lambda^{(1)}$ (MeV)	$E_\Lambda^{(2)}$ (MeV)	$E_\Lambda^{(1)}+E_\Lambda^{(2)}$ (MeV)	$\lvert E_\Lambda^{(2)}/E_\Lambda^{(1)}\rvert$	I_1	I_2
1	−55.0	9.2	−45.8	0.17	0.129	2.753
2	−49.7	3.2	−46.5	0.07	0.050	0.600
3	−47.3	1.7	−45.6	0.04	0.040	0.236
4	−45.6	1.1	−44.5	0.02	0.036	0.114
5	−43.9	0.8	−43.1	0.02	0.034	0.058

tial in odd states. An additional parameter x, the reduction factor, is therefore introduced which determines the relative strength of the interactions in even- and odd-parity states. The analysis of the hypernuclear binding energy data is performed with $x=0$, while in the analysis of the Λ-p scattering data the reduction factor has to be taken different from zero. The introduction of this factor leads therefore to an inconsistency, which, however, is not expected to be too serious. This point is discussed by Tang in Ref. 20.

The Λ-nucleon correlation function is determined by solving the differential equation (11) numerically or analytically. Although numerical integration has been mostly employed, the analytic solution, which was derived in Sec. III, has also been used in checking, in some cases, the accuracy of the numerical solution. Very satisfactory agreement between the values of the two solutions was found.

For each nucleon-nucleon correlation function and each Λ-nucleon potential, the values of the

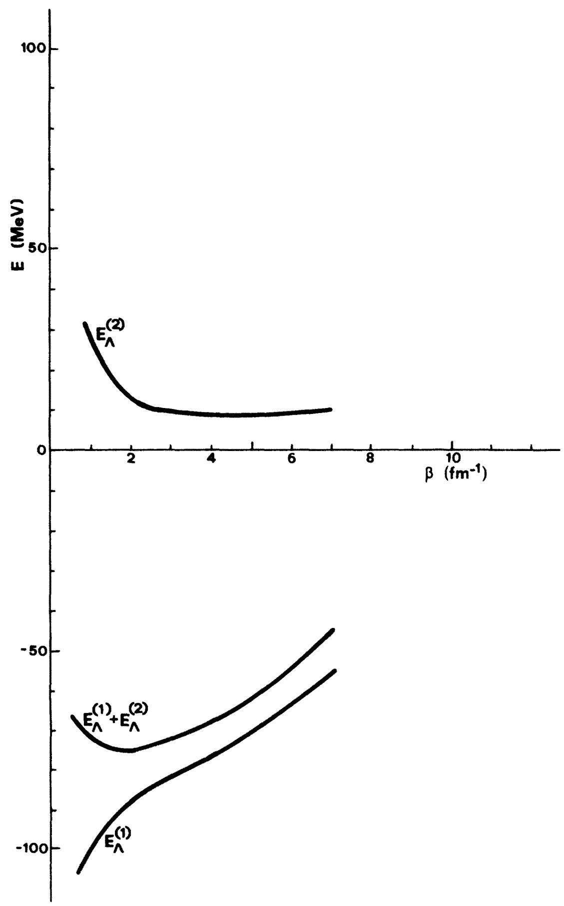

FIG. 1. The first- and second-order energies and their sum as functions of β. (Nucleon-nucleon correlation function g_2, potential H.)

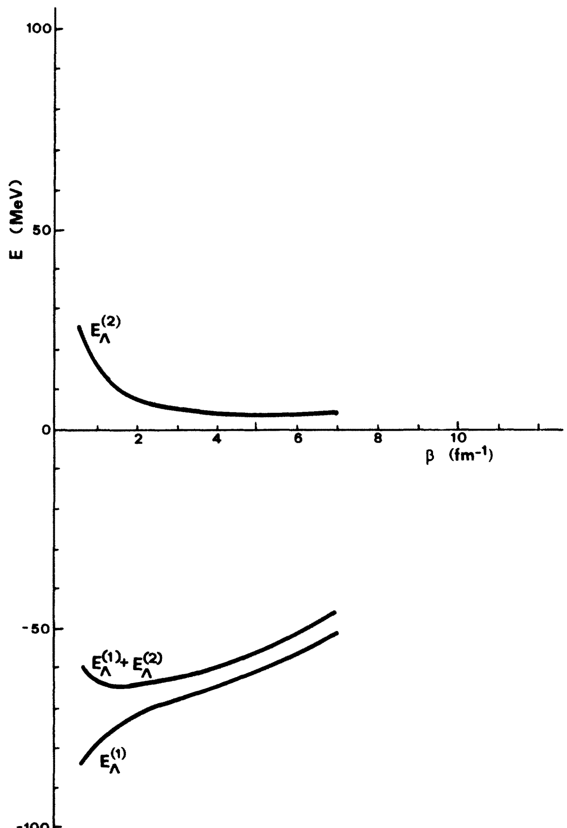

FIG. 2. The first- and second-order energies and their sum as functions of β. (Nucleon-nucleon correlation functions g_2, potential E'.)

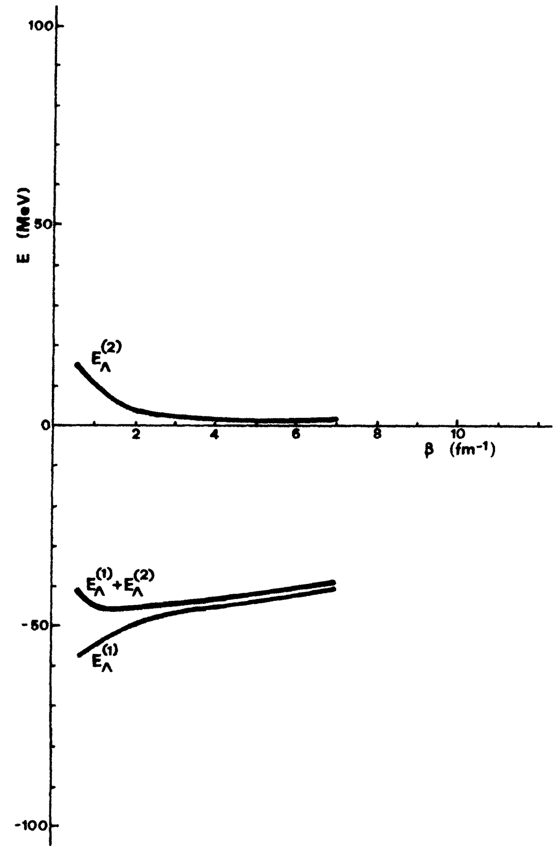

FIG. 3. The first- and second-order energies and their sum as functions of β. (Nucleon-nucleon correlation functions g_2, potential B'.)

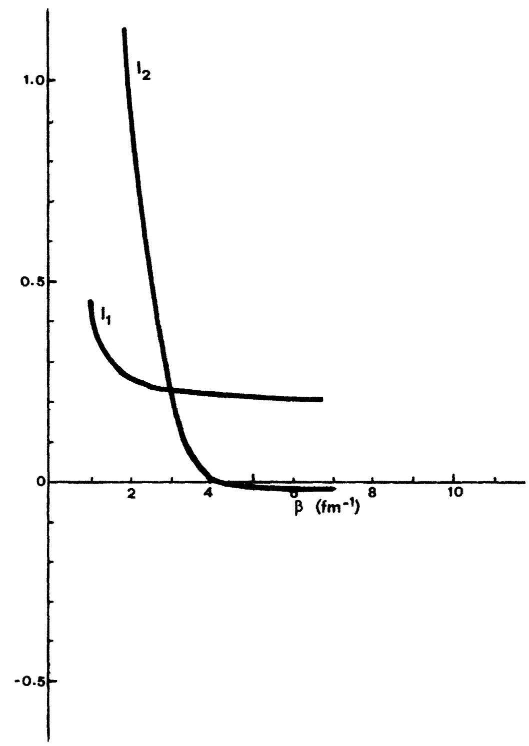

FIG. 4. The "healing integral" I_1 and the "normalization integral" I_2 as function of β. (Nucleon-nucleon correlation function g_2, potential H.)

first- and second-order energies $E_\Lambda^{(1)}$ and $E_\Lambda^{(2)}$, their sum $E_\Lambda^{(1)}+E_\Lambda^{(2)}$, and the absolute value of the ratio $E_\Lambda^{(2)}/E_\Lambda^{(1)}$, as well as the values of the "healing integral" I_1 and the "normalization integral" I_2 were computed for various values of the parameter β. This parameter is connected to the Lagrange multiplier λ^2, which appears because of the healing condition imposed in the variation, by the relation

TABLE VI. Values of $E_\Lambda^{(1)}$, $E_\Lambda^{(2)}$, $E_\Lambda^{(1)}+E_\Lambda^{(2)}$, I_1 and I_2 for values of β which minimize $E_\Lambda^{(1)}+E_\Lambda^{(2)}$ (ME case). The g_1 nucleon-nucleon correlation function was used.

Potential	$E_\Lambda^{(1)}$ (MeV)	$E_\Lambda^{(2)}$ (MeV)	$E_\Lambda=E_\Lambda^{(1)}+E_\Lambda^{(2)}$ (MeV)	I_1	I_2
H	−97.9	10.6	−87.3	0.393	3.845
F'	−83.3	8.5	−74.8	0.368	3.589
E'	−79.1	7.7	−71.4	0.233	3.523
E	−86.8	9.0	−77.8	0.241	3.513
DW	−49.8	3.3	−46.5	0.150	2.742
HTS	−57.4	6.0	−51.4	0.167	2.559
B'	−54.6	5.5	−49.1	0.118	2.475

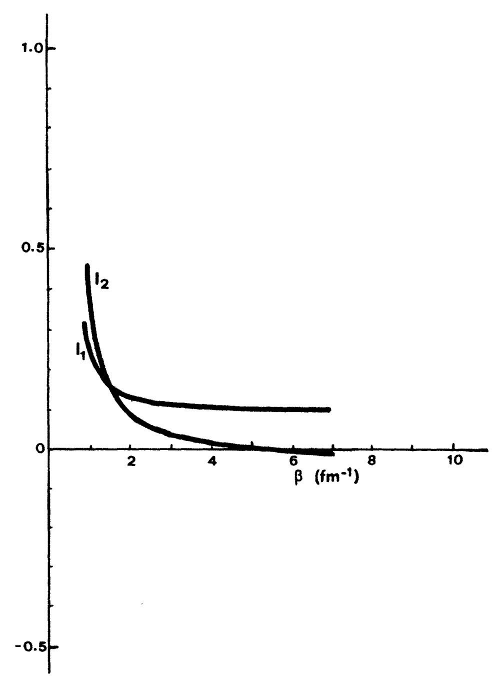

FIG. 5. The "healing integral" I_1 and the "normalization integral" I_2 as functions of β. (Nucleon-nucleon correlation function g_2, potential E'.)

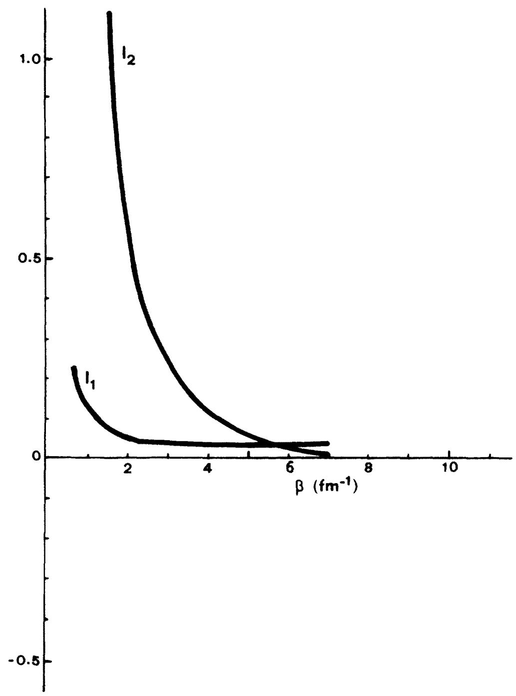

FIG. 6. The "healing integral" I_1 and the "normalization integral" I_2 as functions of β. (Nucleon-nucleon correlation function g_2, potential B'.)

$$\beta=\left(\frac{2\mu_{\Lambda N}}{\hbar^2}\lambda^2\right)^{1/2}. \tag{47}$$

The expression for I_2 is the following:

$$I_2=\rho\int[f^2(r)-1]d\vec{r}. \tag{48}$$

Detailed results obtained with the nucleon-nucleon correlation function g_2 and three "representative" Λ-nucleon potentials, H, E', and B' of hard core radii 0.6, 0.45, and 0.3, are exhibited in Tables III, IV, and V, respectively. The variation of the previously mentioned quantities with β for the other nucleon-nucleon correlation functions and Λ-nucleon potentials is rather similar.

In Figs. 1, 2, and 3 the energy quantities $E_\Lambda^{(1)}$, $E_\Lambda^{(2)}$, and $E_\Lambda^{(1)}+E_\Lambda^{(2)}$ are plotted as functions of β, for potentials H, E', and B', respectively, while in Figs. 4–6 the integrals I_1 and I_2 are plotted as functions of the same quantity.

It is seen that the sum of the first- and second-order energy $E_\Lambda=E_\Lambda^{(1)}+E_\Lambda^{(2)}$ has a minimum for a value of β, which is usually in the range $1\lesssim\beta\lesssim 2$ fm^{-1}. One possible choice of the value of this parameter would be the one for which E_Λ becomes minimum. We shall refer to this choice as ME (minimum energy) choice.

The results for the various quantities with such a choice for the value of β, for each nucleon-nucleon correlation function and Λ-nucleon potential are given in Tables VI–VIII. In Figs. 7–9 the Λ-nucleon correlation functions with the ME choice for β, the nucleon-nucleon correlation function g_2, and the Λ-nucleon potentials H, E', and B' are plotted.

Another choice of the value of β, which seems reasonable, is the one for which the ratio $|E_\Lambda^{(2)}/E_\Lambda^{(1)}|$ takes its minimum value. Such a choice of β might be considered quite attractive because the magnitude of $|E_\Lambda^{(2)}/E_\Lambda^{(1)}|$ is an indication (though not fully satisfactory) of how rapidly the energy cluster expansion converges. We aim

TABLE VII. Values of $E_\Lambda^{(1)}$, $E_\Lambda^{(2)}$, $E_\Lambda^{(1)}+E_\Lambda^{(2)}$, I_1 and I_2 for values of β which minimize $E_\Lambda^{(1)}+E_\Lambda^{(2)}$ (ME case). The g_2 nucleon-nucleon correlation function was used.

Potential	$E_\Lambda^{(1)}$ (MeV)	$E_\Lambda^{(2)}$ (MeV)	$E_\Lambda=E_\Lambda^{(1)}+E_\Lambda^{(2)}$ (MeV)	I_1	I_2
H	−89.0	13.7	−75.3	0.267	1.172
F'	−76.1	11.9	−64.2	0.258	1.043
E'	−73.6	9.0	−64.6	0.145	1.270
E	−80.4	9.8	−70.6	0.148	1.309
DW	−47.7	5.5	−42.2	0.107	1.081
HTS	−53.2	5.6	−47.6	0.101	0.923
B'	−51.4	4.8	−46.6	0.065	1.056

TABLE VIII. Values of $E_\Lambda^{(1)}$, $E_\Lambda^{(2)}$, $E_\Lambda^{(1)}+E_\Lambda^{(2)}$, I_1 and I_2 for values of β which minimize $E_\Lambda^{(1)}+E_\Lambda^{(2)}$ (ME case). The g_3 nucleon-nucleon correlation function was used.

Potential	$E_\Lambda^{(1)}$ (MeV)	$E_\Lambda^{(2)}$ (MeV)	$E_\Lambda=E_\Lambda^{(1)}+E_\Lambda^{(2)}$ (MeV)	I_1	I_2
H	−90.9	12.3	−78.6	0.284	1.575
F'	−77.6	10.6	−67.0	0.273	1.419
E'	−75.0	8.7	−66.3	0.161	1.707
E	−82.1	9.7	−72.4	0.164	1.745
DW	−48.2	4.9	−43.3	0.114	1.380
HTS	−54.0	6.0	−48.0	0.109	1.148
B'	−52.0	5.2	−46.8	0.072	1.260

TABLE IX. Values of $E_\Lambda^{(1)}$, $E_\Lambda^{(2)}$, $E_\Lambda^{(1)}+E_\Lambda^{(2)}$, I_1 and I_2 for values of β which minimize $|E_\Lambda^{(2)}/E_\Lambda^{(1)}|$ (BC case). The g_1 nucleon-nucleon correlation function was used.

Potential	$E_\Lambda^{(1)}$ (MeV)	$E_\Lambda^{(2)}$ (MeV)	$E_\Lambda=E_\Lambda^{(1)}+E_\Lambda^{(2)}$ (MeV)	I_1	I_2
H	−72.2	0.09	−72.1	0.213	−0.045
F'	−62.1	0.01	−62.1	0.211	−0.061
E'	−57.4	0.09	−57.3	0.097	−0.001
E	−61.8	0.06	−61.7	0.095	−0.004
DW	−38.7	0.12	−38.6	0.074	0.014
HTS	−45.1	0.00	−45.1	0.069	0.033
B'	−40.6	0.00	−40.6	0.031	0.016

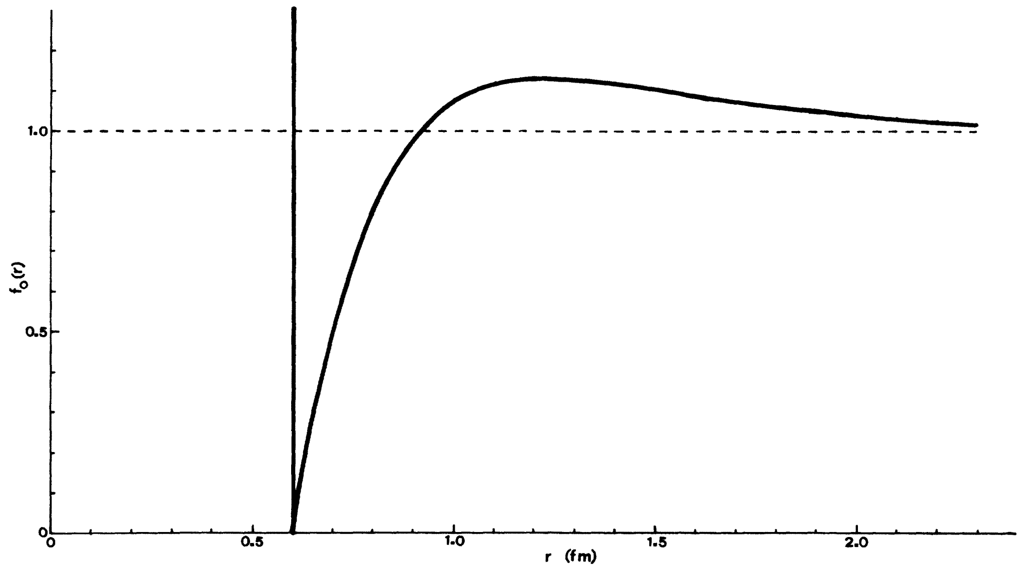

FIG. 7. The correlation function f_0 obtained with the ME choice for β. The g_2 nucleon-nucleon correlation function and potential H were used.

therefore, in this way, at a better convergence compared with that in the previous case (ME). For this reason we shall refer to this choice of β as BC ("better convergence") choice. It should be noted, that in this case the corresponding values of β are larger and vary usually between 3 and 7.

The results with the BC choice, for the value of β, are given in Tables IX–XI.

A discussion of the reported results is made in the following section.

V. DISCUSSION AND CONCLUSIONS

We may first remark that all the values of $D = -E_\Lambda$, which were calculated variationally and were reported in the previous section exceed the empirical value, in agreement with other calcu-

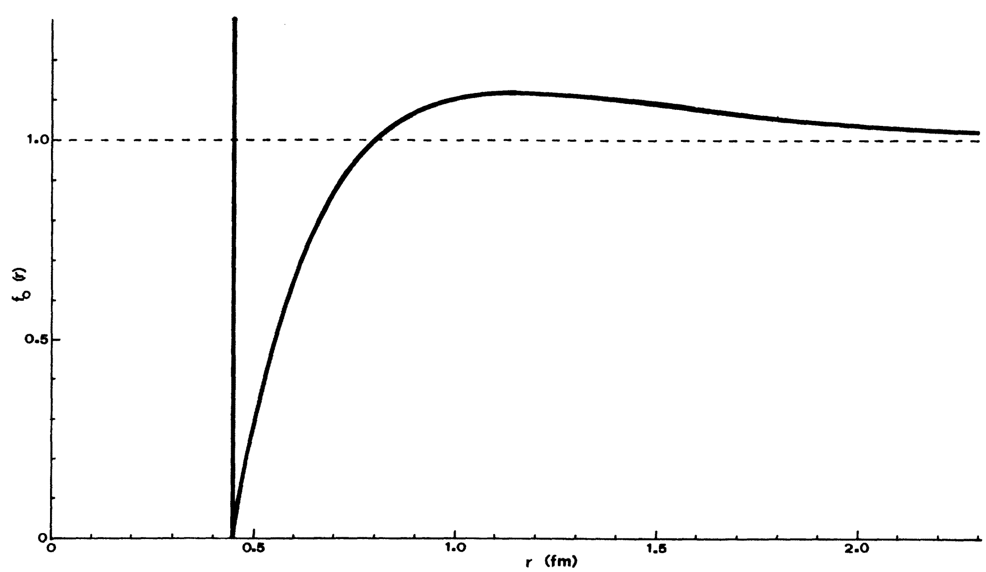

FIG. 8. The correlation function f_0 obtained with the ME choice for β. The g_2 nucleon-nucleon correlation functio- and potential E' were used.

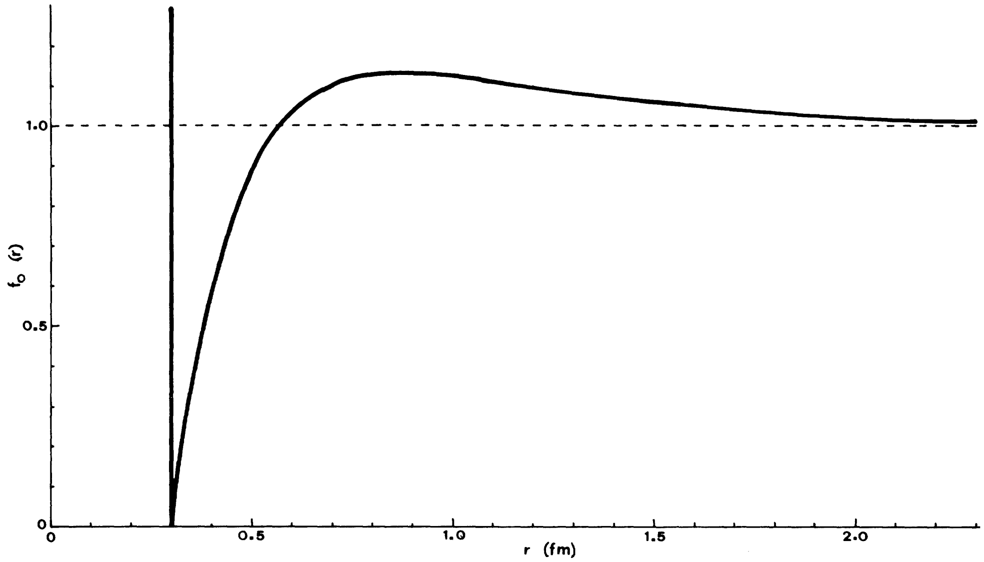

FIG. 9. The correlation function f_0 obtained with the ME choice for β. The g_2 nucleon-nucleon correlation function and potential B' were used.

lations. The observed overbinding, however, varies considerably. It depends on the nucleon-nucleon correlation function, the Λ-nucleon potential, and the choice of β.

The results obtained with the nucleon-nucleon correlation function g_1 and the ME choice for β are poorer. This indicates that it is not justifiable to neglect in this case the dynamical nucleon-nucleon correlations. On the other hand use of the correlation function g_2 or g_3 leads to improved results. The results with g_2 are a little better (closer to the empirical value).

The dependence of D on the Λ-nucleon potential is strong. For some potentials (H, F' and E', E), the overbinding is too large, even with the best nucleon-nucleon correlation function. These are the potentials which have the larger hard core radii and the values of the corresponding healing integrals are larger. For these potentials the values of $E_\Lambda^{(2)}$ are generally larger too and the convergence of the cluster expansion is not rapid. It is not therefore justifiable, particularly in these cases, to use an expansion truncated in the second term.

Concerning the comparison of the results obtained with the ME and BC choices for the value of the parameter β, it is seen from Tables VI–VIII and IX–XI that the energy values corresponding to the latter choice are closer to the empirical value of E_Λ. The ME choice for the value of β implies that this parameter is considered as variational. Although such a possibility might not be *a priori* excluded, it appears more justifiable to attribute a different role to β. The present method is based on a first-order functional variation and the origin of β is the "healing constraint," which was introduced in order to "rem-

TABLE X. Values of $E_\Lambda^{(1)}$, $E_\Lambda^{(2)}$, $E_\Lambda^{(1)}+E_\Lambda^{(2)}$, I_1 and I_2 for values of β which minimize $|E_\Lambda^{(2)}/E_\Lambda^{(1)}|$ (BC case). The g_2 nucleon-nucleon correlation function was used.

Potential	$E_\Lambda^{(1)}$ (MeV)	$E_\Lambda^{(2)}$ (MeV)	$E_\Lambda = E_\Lambda^{(1)}+E_\Lambda^{(2)}$ (MeV)	I_1	I_2
H	−79.0	8.5	−70.5	0.223	0.105
F'	−68.1	7.7	−60.4	0.221	0.084
E'	−64.3	4.6	−59.7	0.105	0.109
E	−69.3	4.5	−64.8	0.103	0.098
DW	−42.9	3.4	−39.5	0.081	0.114
HTS	−44.1	1.5	−42.6	0.068	0.017
B'	−40.3	0.6	−39.7	0.031	0.014

TABLE XI. Values of $E_\Lambda^{(1)}$, $E_\Lambda^{(2)}$, $E_\Lambda^{(1)}+E_\Lambda^{(2)}$, I_1 and I_2 for values of β which minimize $|E_\Lambda^{(2)}/E_\Lambda^{(1)}|$ (BC case). The g_3 nucleon-nucleon correlation function was used.

Potential	$E_\Lambda^{(1)}$ (MeV)	$E_\Lambda^{(2)}$ (MeV)	$E_\Lambda = E_\Lambda^{(1)}+E_\Lambda^{(2)}$ (MeV)	I_1	I_2
H	−78.2	5.9	−72.3	0.222	0.077
F'	−67.5	5.4	−62.1	0.219	0.061
E'	−64.0	3.5	−60.5	0.104	0.100
E	−69.1	3.5	−65.6	0.103	0.093
DW	−42.9	2.6	−40.3	0.081	0.114
HTS	−44.4	1.6	−42.8	0.068	0.023
B'	−41.4	0.9	−40.5	0.031	0.022

TABLE XII. Values of D obtained with different methods.

Potential	Rote and Bodmer (Ref. 2)	Dabrowski and Hassan (Ref. 18)	Ram and Williams (Ref. 21)	Mueller and Clark (Ref. 6)	Present calculation (ME case with g_2)	Present calculation (BC case with g_2)
H	56.8	56.7	57.7	72	75.3	70.5
F'					64.2	60.4
E'	56.0	55.6	56.9	62	64.6	59.7
E	61.7	61.1	62.4	69	70.6	64.8
DW		36.3	36.4	42	42.2	39.5
HTS		42.4			47.6	42.6
B'					46.6	39.7

edy" for the omission of the higher terms. Consequently, it should be more appropriate to consider β as a parameter, which is at our disposal in order to guarantee, if this is feasible, good convergence of the energy cluster expansion. It is therefore reasonable to expect the results with the BC choice for the value of β to be improved.

In Table XII the values of D obtained with nucleon-nucleon correlation function g_2 and the two choices for the value of β are compared with the values produced with other methods. Most of the values for this comparison have been taken from Ref. 21 and it should be noted that the rearrangement energy correction in the reaction matrix methods has not been included.

It is seen from Table XII that the variational calculations give larger overbinding as it has also been pointed out in Ref. 19. It is encouraging, however, that the present calculations with the BC choice for the value of β lead to somehow less overbinding, compared with other variational calculations.[3-7] The gap between the values of D in the reaction matrix and variational methods is therefore a little narrowed.

In conclusion, it should be pointed out that it is important for a variational calculation like the present one to use appropriate Λ-nucleon potentials which lead to as small as possible higher-order terms in the energy cluster expansion in order to diminish the observed overbinding. A p-wave suppression or other effects like those commonly discussed in the literature may further reduce the theoretical value of the binding energy of the Λ particle.

The authors would like to thank very much Dr. L. Kok for his kind interest in this work, for useful comments, and for providing them with his computer program. Most of the computations were performed in the Computing Centers of the Universities of Trieste and Thessaloniki. Thanks are particularly due Dr. Ing. Hmeliak, Dr. Gregory, and Dr. Lazos and to Mr. K. Knight and Mr. Chrissos for programming and computational assistance and also to Mr. M. Hassan for a discussion.

Two of the authors (S.A. and M.E.G.) would like to thank Professor Abdus Salam and Professor Paolo Budini as well as the IAEA and UNESCO for making possible their visits to the International Centre for Theoretical Physics.

*Permanent address: Atomic Energy Centre, Dacca, Bangladesh.

†Permanent address: Department of Theoretical Physics, University of Thessaloniki, Thessaloniki, Greece.

[1]B. W. Downs and P. D. Kunz, in *Proceedings of the International Conference on Hypernuclear Physics, Argonne National Laboratory, 1969*, edited by A. R. Bodmer and L. G. Hyman (Argonne National Laboratory, Argonne, Illinois, 1969), Vol. II, p. 796.

[2]A. R. Bodmer and D. M. Rote, in *Proceedings of the International Conference on Hypernuclear Physics, Argonne National Laboratory, 1969* (see Ref. 1), p. 521; D. M. Rote and A. R. Bodmer, Nucl. Phys. A148, 97 (1970).

[3]B. W. Downs and M. E. Grypeos, Nuovo Cimento 44, 306 (1966).

[4]P. Westhaus and J. W. Clark, Phys. Lett. 23, 109 (1966).

[5](a) M. E. Grypeos, L. P. Kok, and S. Ali, Nucl. Phys. B4, 335 (1968); (b) M. E. Grypeos, L. P. Kok, and S. Ali, in *Proceedings of the International Conference on Hypernuclear Physics, Argonne National Laboratory, 1969* (see Ref. 1), p. 826.

[6]G. Mueller and J. W. Clark, Nucl. Phys. B7, 227 (1968).

[7]J. W. Clark and G. Mueller, Nuovo Cimento 64B, 217 (1969).

[8]R. H. Dalitz, in Proceedings of the Summer Study Meeting on nuclear and Hypernuclear Physics with Kaon beams, Brookhaven National Laboratory, July 1973, edited by H. Palevsky (unpublished), p. 1; A. R.

Bodmer, *ibid.*, p. 64.
[9]B. W. Downs and W. E. Ware, Phys. Rev. 133, B133 (1964).
[10]R. C. Herndon, Y. C. Tang, and E. W. Schmid, Phys. Rev. 137, B294 (1965).
[11]R. C. Herndon and Y. C. Tang, Phys. Rev. 159, 853 (1967).
[12]M. E. Grypeos, Phys. Lett. 34B, 481 (1971).
[13]P. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill, New York, 1953), p. 1670.
[14]M. E. Grypeos, J. of Math. Phys. 12, 1655 (1971).
[15]The results reported in this paper will be a part of a thesis to be submitted by one of the authors (B.K.) to the University of Thessaloniki.
[16]T. Ohmura, M. Morita, and M. Yamada, Prog. Theor. Phys. (Kyoto) 15, 222 (1956).
[17]M. E. Grypeos and E. Mavrommatis (private communication).
[18]J. Dabrowski and M. Y. M. Hassan, Phys. Rev. C 1, 1883 (1970).
[19]B. Ram and J. R. James, Phys. Lett. 28B, 372 (1969); 29B, 227 (1969).
[20]Y. C. Tang, in *Proceedings of the International Conference on Hypernuclear Physics, Argonne National Laboratory, 1969* (see Ref. 1), p. 276.
[21]B. Ram and W. Williams, Phys. Rev. C 3, 45 (1971).