PHYSICAL REVIEW C VOLUME 14, NUMBER 6 DECEMBER 1976

Exact solution of the quadrupole surface vibration Hamiltonian in body-fixed coordinates*

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(Received 28 June 1976)

Exact, closed-form eigenfunctions for the harmonic, quadrupole surface vibration model of Bohr are developed. These angular momentum labeled, multiplicity-resolved functions of β, γ, and the Euler angles are valid for an arbitrary number of phonons.

[NUCLEAR STRUCTURE Exact γ-vibration solutions, multiplicity resolved; theory.]

INTRODUCTION

The quadrupole surface oscillation model of Bohr[1] has become the natural starting point for the collective description of the positive parity states of even-even nuclei. The harmonic approximation in which the nuclear fluid executes quadrupole surface vibrations about a spherical equilibrium shape is of course entirely too restrictive and a wide variety of calculations have been performed which include various anharmonic terms.[2] Even in such cases it is necessary to have the solutions to the Bohr Hamiltonian as a starting basis. One has a choice of working in laboratory coordinates or in body-fixed coordinates oriented along principal axes. In either set of coordinates the energy is labeled by a principal quantum number N which is the number of quadrupole surface phonons. Since the creation and destruction operators for these phonons obey boson commutation rules, the basis functions are symmetric functions of the boson coordinates. In the laboratory frame the natural basis to use is pseudo-Cartesian and one is faced with the problem of constructing states of good angular momentum, J, and resolving the multiplicity which occurs as soon as $N=4$. In that case there are two $J=2$ states and two $J=4$ states. This multiplicity increases rapidly with increasing N. A partial resolution of this multiplicity is effected by using the "seniority scheme" which is merely an alternative way of describing the transformation properties of the basis states under the orthogonal group in five dimensions. Even this additional quantum number fails to resolve the multiplicity of J values as soon as $N=6$. A complete formal resolution of this multiplicity problem was given several years ago[3] and in the second of those papers a technique familiar from the work of Elliott[4,5] was used to obtain projective wave functions given a set known as intrinsic states. This technique is most applicable to working in the body-fixed coordinates. In these coordinates the problem is one of finding an explicit realization of the so-called γ part of the problem. A partial set of detailed solutions to this problem was presented by Bes[6] and by Jankovic[7] in 1959 and the problem has received renewed interest more recently.[8,9] The Yrast solutions have also been given earlier[10] but they are a special case of the largest angular momentum for a given seniority. The projective technique has been used for this problem by Holzwarth[11] but the calculation differs considerably from ours; for example, it starts with a variational function rather than a group theoretic one. Our solution resolves multiplicities in a natural way. In Ref. 8 the multiplicity resolution of Ref. 3 was used, but no closed form of the γ part of the wave functions was given. In this paper we utilize the techniques of Ref. 3 to present a closed form solution to this 24 year old γ-vibration problem.

In the first section we shall review the entire surface oscillation problem to establish the notation we shall use. In Sec. II we shall make connections between the physical problem and the relevant group theoretic problem, and in Sec. III we shall present the solution. Finally, in Sec. IV we shall give normalization and overlap factors and make comparisons with earlier work.

I. PHYSICAL PROBLEM

A. Model Hamiltonian

The usual starting point is the expansion of the nuclear surface (as seen by a laboratory observer) in spherical harmonics which introduces the collective coordinates $\alpha_{\lambda\mu}$. This expansion is[12]

$$R(\theta,\phi)=R_0\left[1+\sum_{\lambda\mu}(\alpha_{\lambda\mu})^*Y_{\lambda\mu}(\theta,\phi)\right]. \tag{1}$$

Changes in α_{00} and $\alpha_{1\mu}$ correspond to changes in the nuclear volume and center of mass, respectively[13]; hence to lowest order one has interest first in the quadrupole ($\lambda = 2$) coordinates. Throughout this paper our interest is only in these $\lambda = 2$ coordinates. We shall drop this extra label, which describes the transformation property of these coordinates under three-dimensional rotations. The nuclear surface may similarly be described in body-fixed coordinates by

$$R(\theta', \phi') = R_0\left[1 + \sum_{\mu}(a_\mu)^* Y_{2\mu}(\theta', \phi')\right]. \tag{2}$$

If the Euler angles $(\theta_1, \theta_2, \theta_3) \equiv (\theta_i)$ describe the laboratory to body transformation, then[12]

$$a_\nu = \sum_\mu D^2_{\mu\nu}(\theta_i)\alpha_\mu \,. \tag{3}$$

The reality of the surface insures that

$$(\alpha_\mu)^* = (-1)^\mu \alpha_{-\mu} \tag{4a}$$

and similarly

$$(a_\nu)^* = (-1)^\nu a_{-\nu}\,. \tag{4b}$$

It will prove convenient to define upper indexed coordinates

$$a^\nu \equiv (a_\nu)^* = (-1)^\nu a_{-\nu} \tag{5}$$

which are then seen to transform as

$$a^\nu = \sum_\mu D^{2*}_{\mu\nu}(\theta_i)\alpha^\mu \,. \tag{6}$$

For our purposes it is very useful to clearly classify quantities as to their transformation properties under spatial rotations. If the components of an object transform according to Eq. (3) or Eq. (6) we say that these transform contragradiently or cogradiently, respectively, and that the corresponding components are contravariant or covariant. In terms of the general surface expansion, the contravariant components, $a_{\lambda,\nu}$, of a vector $\underline{a}_\lambda$ (the underline here indicates a vector with $2\lambda+1$ components) form a basis for the $[\lambda]$ irreducible representation (IR) of R(3), while the covariant components a^ν_λ are a basis for the (equivalent) adjoint representation $[\lambda]^*$.

For small oscillations about a spherical equilibrium shape the classical Hamiltonian assumes the form

$$H = T + V = \tfrac{1}{2}B\sum_\mu \dot{\alpha}^\mu \dot{\alpha}_\mu + \tfrac{1}{2}C\sum_\mu \alpha^\mu\alpha_\mu \,, \tag{7}$$

where the dot means d/dt. The scalar constants B and C depend upon details of the model.

The momenta conjugate to these generalized coordinates are defined by

$$\pi^\mu = \frac{\partial T}{\partial \dot{\alpha}_\mu} = B(\dot{\alpha}_\mu)^* = B\dot{\alpha}^\mu \,. \tag{8}$$

One readily finds that under rotations these covariant components of $\underline{\pi}$ transform as

$$\bar{\pi}^\mu = \sum_\nu D^{2*}_{\nu\mu}(\theta_i)\pi^\nu \,, \tag{9}$$

where $\bar{\pi}$ is the body-frame generalized momenta. Hence the π^ν are properly labeled. The corresponding contravariant components are

$$\pi_\mu = (-1)^\mu \pi^{-\mu} \,. \tag{10}$$

The Hamiltonian then assumes the usual form

$$H = \frac{1}{2B}\sum_\mu \pi^\mu\pi_\mu + \tfrac{1}{2}C\sum_\mu \alpha^\mu\alpha_\mu \,. \tag{11}$$

By employing the convention of summation on repeated upper and lower indices, and by introducing a metric tensor, we may rewrite Eq. (11) as

$$H = \frac{1}{2B}\pi_\mu g^{\mu\nu}\pi_\nu + \tfrac{1}{2}C\alpha_\mu g^{\mu\nu}\alpha_\nu \tag{12a}$$

or as

$$H = \frac{1}{2B}\pi^\mu g_{\mu\nu}\pi^\nu + \tfrac{1}{2}C\alpha^\mu g_{\mu\nu}\alpha^\nu \,, \tag{12b}$$

where

$$g^{\mu\nu} = g_{\mu\nu} = (-1)^\mu \delta^\mu_{-\nu} \tag{13}$$

and δ^μ_ν is the usual Kroneker δ

$$\delta^\mu_\nu = \begin{cases} 1, & \mu = \nu \\ 0, & \mu \neq \nu \end{cases} . \tag{14}$$

Thus the metric tensor elements are the derived δ's

$$g_{\mu\nu} = \delta_{\mu\nu} = g_{\mu\sigma}\delta^\sigma_\nu \tag{15a}$$

and

$$g^{\mu\nu} = \delta^{\mu\nu} = g^{\mu\sigma}\delta^\nu_\sigma \,. \tag{15b}$$

It is usually convenient to cast the Hamiltonian into dimensionless form by writing

$$\underline{\alpha}' = \left(\frac{B\omega}{\hbar}\right)^{1/2}\underline{\alpha}\,, \tag{16a}$$

$$\underline{\pi}' = \left(\frac{1}{\hbar B\omega}\right)^{1/2}\underline{\pi}\,, \tag{16b}$$

$$H' = \frac{1}{\hbar\omega}H\,, \tag{16c}$$

where

$$\omega = \left(\frac{C}{B}\right)^{1/2} . \tag{17}$$

Hereafter we shall use the scaled quantities and omit the primes unless otherwise explicitly stated.

B. Quantization

Some care must be taken with quantization since the classical coordinates are not real and lead therefore to non-Hermitian operators. In fact, if we use † to denote Hermitian conjugation, one has

$$(\alpha_\mu)^\dagger = \alpha^\mu, \tag{18a}$$

$$(\pi^\mu)^\dagger = \pi_\mu. \tag{18b}$$

The coordinates and momenta do satisfy the expected commutations rules, however, which may be expressed as

$$[\underline{\alpha}, \underline{\pi}] = i\underline{\delta}; \quad [\underline{\alpha}, \underline{\alpha}] = [\underline{\pi}, \underline{\pi}] = 0 \tag{19a}$$

which encompasses, for example,

$$[\alpha_\mu, \pi^\nu] = i\delta^\nu_\mu \tag{19b}$$

and

$$[\alpha_\mu, \pi_\nu] = i\delta_{\mu\nu}. \tag{19c}$$

These quantum conditions allow two means for solving the problem and we shall utilize both. In the first instance, one introduces raising and lowering operators whose components transform under spatial rotations as angular-momentum-two objects and which become the surface phonon creation and destruction operators. That is, one defines

$$\underline{b}^+ \equiv \frac{1}{\sqrt{2}}(\underline{\alpha} - i\underline{\pi}), \tag{20a}$$

$$\underline{b} \equiv \frac{1}{\sqrt{2}}(\underline{\alpha} + i\underline{\pi}) \tag{20b}$$

which have the transformation properties

$$\bar{b}^+_\nu \equiv [R(\theta_i)\underline{b}^+ R^{-1}(\theta_i)]_\nu = \sum_\mu D^2_{\mu\nu}(\theta_i) b^+_\mu, \tag{21a}$$

$$\bar{b}^{+\nu} = \sum_\mu D^{2*}_{\mu\nu}(\theta_i) b^{+\mu}, \tag{21b}$$

and similarly for b_ν and b^ν. The bar refers to operators in the body frame. The creation and destruction operators satisfy commutation relationships which may be written in the symbolic form of Eq. (19a) as

$$[\underline{b}, \underline{b}^+] = \underline{\delta}, \quad [\underline{b}, \underline{b}] = [\underline{b}^+, \underline{b}^+] = 0. \tag{22}$$

Furthermore, one finds

$$(b^\mu)^\dagger = b^+{}_\mu, \quad (b^+{}_\mu)^\dagger = b_\mu \tag{23}$$

and

$$H = \tfrac{1}{2}(b^+{}_\mu b^\mu + b^\mu b^+{}_\mu) = b^+{}_\mu b^\mu + \tfrac{5}{2}. \tag{24}$$

The form of Eq. (24) quite clearly indicates that H is the Hamiltonian for the five-dimensional isotropic harmonic oscillator. This is often referred to as the quadrupole vibrator since the one-phonon states carry angular momentum 2.

One also has

$$[H, b^+{}_\mu] = b^+{}_\mu, \quad [H, b_\mu] = -b_\mu. \tag{25}$$

From Eqs. (24) and (25) the natural form for the eigenstates of H is a (necessarily) symmetric function of the $b^+{}_\mu$ operating on a vacuum state $|0\rangle$ defined by

$$b_\mu |0\rangle = 0, \quad \text{all } \mu . \tag{26}$$

The vacuum state carries angular momentum 0, or

$$R(\theta_i)|0\rangle = |0\rangle . \tag{27}$$

The one-phonon state $b^+{}_\mu|0\rangle$ carries angular momenta 2 and projection μ, or

$$R(\theta_i)[b^+{}_\mu|0\rangle] = \sum_\nu D^2_{\nu\mu}(\theta_i)[b^+{}_\nu|0\rangle] . \tag{28}$$

These properties follow directly from Eqs. (21) and allow one to deduce the form of the physical angular momentum operators without recourse to approximation or details of the model (such as one would have if the hydrodynamic form were invoked). The components of $\vec{J}$ must be of the form of linear combinations of the $b^+{}_\mu b_\nu$ and must transform under R(3) as a vector. Furthermore the laboratory components of $\vec{J}$ must satisfy

$$J_\sigma b^+{}_\mu|0\rangle = (-1)^\sigma \sqrt{6}\, C(212; \mu+\sigma, -\sigma) b^+{}_{\mu+\sigma}|0\rangle$$

which is sufficient to show that

$$J_\sigma = \sqrt{10}\sum_{\mu\nu} C(221; \mu\nu\sigma) b^+{}_\mu b_\nu \equiv \sqrt{10}\,[\underline{b}^+\underline{b}]^{[1\sigma]} , \tag{29}$$

where in the second part of Eq. (29) we have introduced the tensorial coupling notation

$$[\underline{b}^+\underline{b}]^{[Jm]} \equiv \sum_{\mu\nu} C(22J; \mu\nu m) b^+{}_\mu b_\nu. \tag{30}$$

A second approach to a quantization scheme satisfying the commutation relations of Eq. (19a) involves the realization for the operators α_μ and π^μ as

$$\alpha_\mu \to \alpha_\mu ,$$

$$\pi^\mu \to -i\frac{\partial}{\partial\alpha_\mu} . \tag{31}$$

Because of the nonidentity metric, some care must be exercised in writing out the Hamiltonian,

$$H = -\frac{1}{2}\frac{\partial^2}{\partial\alpha_\mu\partial\alpha^\mu} + \tfrac{1}{2}\alpha^\mu\alpha_\mu . \tag{32}$$

The requirement that the body-fixed axes lie along principal inertial axes places restrictions on the a_ν of Eq. (3). There are in fact 48 possible choices of orientation which amount to relabeling the body axes. Half of these result in left handed coordinate systems and are not of major interest. The remaining choices manifest themselves as symmetries of the solutions to the body axis form of Eq. (32). The body coordinates of Eq. (3) are required to satisfy[12]

$$a_1 = a_{-1} = 0 \ ,$$
$$a_2 = a_{-2} \ (\text{real}) \ ,$$
$$a_0 \ (\text{real}) \ .$$

The usual choice of generalized body-frame coordinates is $\{\beta, \gamma, \theta_i\}$ where the θ_i are as before the Euler angles of orientation of the body frame relative to the fixed laboratory frame (that is, they specify the rotation necessary to take the lab to the body frame). Then β and γ are defined by

$$a_0 = \beta \cos\gamma \ ,$$
$$a_{\pm 2} = \frac{\beta}{\sqrt{2}} \sin\gamma \ . \tag{33}$$

This choice results in the simplest possible form for the Hamiltonian, and β plays then the role of a radiallike variable since

$$\beta^2 = \alpha^\mu \alpha_\mu = a^\mu a_\mu \ . \tag{34}$$

In terms of these coordinates the Hamiltonian of Eq. (32) becomes

$$H = \frac{1}{2}\left\{-\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} - \frac{1}{\beta^2 \sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma} + \frac{1}{4\beta^2}\sum_{k=1}^{3}\frac{L_k^{\,2}}{\sin^2[\gamma - (2\pi/3)k]} + \beta^2\right\} \tag{35}$$

in which the L_k are the (dimensionless) body-frame components of the angular momentum operator and are realized as functions of the θ_i and $\partial/\partial\theta_i$; their specific form will not be required.

The Schrödinger equation to be solved then is

$$H\Psi(\beta, \gamma, \theta_i) = E\Psi(\beta, \gamma, \theta_i) \tag{36}$$

and one readily finds that the β part of this problem separates from the (γ, θ_i) part. That is, the radial part of the five-dimensional oscillator separates from the angular part. As is well known[15] Eq. (36) may be rewritten as two equations, viz.,

$$\left(-\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} + \beta^2 + \frac{\Lambda}{\beta^2} - 2E\right)F(\beta) = 0 \tag{37a}$$

and

$$\left\{-\frac{1}{\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma} + \frac{1}{4}\sum_{k=1}^{3}\frac{L_k^{\,2}}{\sin^2[\gamma - (2\pi/3)k]} - \Lambda\right\}\Phi(\gamma, \theta_i) = 0 \ . \tag{37b}$$

Equation (37b) is the angular part of the five-dimensional Laplace equation and is known as the γ part of the problem. It is the portion of the problem that we shall solve in closed analytic form. We shall append the well-known solution to the β part of the problem. The angular quantum number Λ takes on the values

$$\Lambda = l(l+3) \ , \quad l = 0, 1, 2, \ldots \tag{38}$$

and the quantum number l is called the "seniority."

The volume element follows from the form of the metric tensor appropriate to Eq. (35) and is

$$dV = \beta^4 d\beta \,|\sin 3\gamma|\, d\gamma\, d\Omega$$

in which $d\Omega$ symbolizes the Euler angle volume element.[14] The variable β which is the radial coordinate of the five-dimensional space ranges from 0 to ∞. The variable γ ranges from 0 to 2π, and the Euler angles take on their usual ranges.[14]

II. GROUP THEORY OF THE QUADRUPOLE VIBRATOR

A. Symmetry group of H

The model Hamiltonian given by Eqs. (24), (32), or (35) commutes with each of the 25 operators

$$A_{\mu\nu} = b^{+}{}_{\mu} b_\nu, \quad \mu, \nu = -2, -1, \ldots, 2 \ . \tag{39}$$

One may choose as independent symmetry operators the particular linear combinations

$$S_{\mu\nu} = \tfrac{1}{2}[b^{+\mu} b_\nu + b^{+\nu} b_\mu] \tag{40}$$

which are Hermitian and are closed under the commutation operation

$$[S_{\mu\nu}, S_{\sigma\gamma}] = \delta^\nu_\sigma S_{\mu\gamma} - \delta^\mu_\gamma S_{\nu\sigma} \ . \tag{41}$$

These may be cast into a Cartan-Weyl[16] format by defining

$$H_\mu = S_{\mu\mu} \tag{42}$$

in which case

$$[H_\mu, H_\nu] = 0 \ ,$$
$$[H_\mu, S_{\gamma\sigma}] = (\delta^\mu_\gamma - \delta^\mu_\sigma) S_{\gamma\sigma} \ . \tag{43}$$

The Lie group generated by the $S_{\mu\nu}$, that is the symmetry group of the Hamiltonian, is U(5). Degenerate eigenfunctions of H will then form a basis for an IR of U(5). One must then determine which IR are represented by the model states.

B. Symmetric IR of U(5)

It is convenient to work in a basis in which the H_μ have simultaneous eigenstates

$$|(\underline{\Lambda})\rangle\!\rangle = |(\Lambda_2, \Lambda_1, \ldots, \Lambda_{-2})\rangle\!\rangle \quad , \tag{44}$$

where the state labels are simply the eigenvalues of the commuting generators

$$H_\mu |(\underline{\Lambda})\rangle\!\rangle = \Lambda_\mu |(\underline{\Lambda})\rangle\!\rangle \quad . \tag{45}$$

The ordered 5-tuple $(\underline{\Lambda})$ is termed the *weight*[17] of the state $|(\underline{\Lambda})\rangle\!\rangle$. Any IR of U(5) is uniquely labeled by the (numerically) highest weight among the states that comprise its basis. We will label a U(5) IR by enclosing its associated highest weight $(\underline{\Omega})$ in square brackets, as $[\Omega_2, \Omega_1, \ldots, \Omega_{-2}]$, a notation which is identical to the usual partition labeling scheme when all of the Ω_μ are non-negative integers.

Eigenfunctions of H are to be constructed by operating on the vacuum with a succession of creation operators. The general state

$$|(\underline{v})\rangle\!\rangle = (b^+{}_2)^{v_2}(b^+{}_1)^{v_1}\cdots(b^+{}_{-2})^{v_{-2}}|0\rangle \quad , \tag{46}$$

where the v_α are non-negative integers, is a member of the basis (44) with energy E,

$$E = (N + \tfrac{5}{2}) \tag{47a}$$

with

$$N = \sum_\alpha v_\alpha \quad . \tag{47b}$$

We ask to what IR of U(5) does $|(\underline{v})\rangle\!\rangle$ belong? Every state in this IR, including the highest weight state, may be reached by operating upon $|(\underline{v})\rangle\!\rangle$ with an appropriate linear combination of the $S_{\mu\nu}$, and only states within the IR can be created in this manner. A little algebra leads to the conclusion that all states of the form (46) having the same value of N can be obtained from $|(\underline{v})\rangle\!\rangle$. The highest weight is clearly (N0000), labeling the so-called *symmetric tensor* IR of U(5) by the single non-negative integer N.

C. Generators of U(5) in the physical chain

The next task is to investigate the subgroup structure of U(5). When a nested chain of subgroups exists, it is possible to construct operators which transform as irreducible tensor components under each subgroup in the chain.[17] We have in mind forming and identifying the generators of the subgroups in the physical chain

$$\mathrm{U}(5) \supset \mathrm{SU}(5) \supset \mathrm{R}(5) \supset \text{physical R}(3) \tag{48}$$

by the appropriate coupling of the products $\underline{b}^+ \times \underline{b}$.

The generators of any Lie group form a tensorial set under the *regular* representation, which is irreducible if and only if the group is simple.[18] All of the groups appearing in (48) are simple with the exception of U(5), and their "generator IR" can be unambiguously identified by their dimensions. Adopting the usual partition labeling scheme, we list these: SU(5), 24 generators, [2111]; R(5), 10 generators, [11]; and R(3), 3 generators, [1].

For U(5) itself the situation is a bit more complicated, as the generators comprise two irreducible tensors. This may be seen in the following manner. The $\underline{b}^+$ form an irreducible tensor under U(5) as well as R(3). The same can be said for the $\underline{b}$. One method of identifying an IR tensor operator $\underline{\mathcal{O}}$, is through its commutation relations with the group generators. In particular, when a Cartan-Weyl basis is employed

$$[H_\mu, \mathcal{O}(\underline{\Omega}, \underline{\Lambda})] = \Lambda_\mu \mathcal{O}(\underline{\Omega}, \underline{\Lambda}) \,, \tag{49}$$

where as before $\underline{\Omega}$ is the highest $\underline{\Lambda}$ in the set of weights, and labels the IR to which the tensor belongs, while the $\underline{\Lambda}$ label its components. From

$$[H_\mu, b^+{}_\nu] = \delta^\nu_\mu b^+{}_\nu \text{ (no summation)} \tag{50a}$$

we find that the operators $\underline{b}^+$ are labeled by the five weights

(10000)(01000)$\cdots$(00001)

and hence belong to the [10000] IR of U(5). Similarly

$$[H_\mu, b_\nu] = -\delta^\nu_\mu b_\nu \text{ (no summation)} \tag{50b}$$

and $\underline{b}$ belongs to [0000 −1], the IR contragradient to [10000].[19] With this information we now note that the generators of U(5) may be taken to be the members of the direct product $\underline{b}^+ \times \underline{b}$, and may be classified into irreducible parts by the reduction of the direct product of the representations [10000] and [0000 −1]. We find

$$\underset{5}{[10000]} \times \underset{5}{[0000\,-1]} = \underset{24}{[1000\,-1]} + \underset{1}{[00000]} \,,$$

where the dimensions of the IR have also been listed. It is the simple combination of generators

$$H = \sum H_\mu$$

which commutes with all of the U(5) generators and hence transforms under the one-dimensional identity IR, [00000].

Under restriction to the subgroups in the physical chain, the U(5) generator IR decompose according to the branchings summarized in Table I.[20] With each of the downward paths stemming from [1000 −1] there are associated several components of the U(5) irreducible tensor $[\underline{b}^+\underline{b}]^{[1000\,-1]}$. These

TABLE I. Decomposition of the U(5) generator IR.

[10000] × [0000 − 1] =	[1000 − 1]	+ [00000]	U(5)
	↓	↓	
	[2111]	[0000]	SU(5)
	↓	↓	
	[11] + [20]	[00]	R(5)
	↓ ↓	↓	
	[1] + [3] + [2] + [4]	[0]	R(3)

components are labeled by the subgroup IR under which they transform and are listed in Table II. The generators of the subgroups have been identified there by noting the appearance of the generator IR labels as listed.

Since each R(3) IR occurs only once, the $[\underline{b}^+\underline{b}]^{[1000\,-1]}$ and $[\underline{b}^+\underline{b}]^{[00000]}$ components may be uniquely specified by the two angular momentum labels as Q_{JM}. The method of isoscalar factors[17] (ISF) could now be used to explicitly construct the Q_{JM} in terms of the $b^+{}_\mu b_\nu$; this program is made trivial here by the simplicity of the branching rules in Table I which imply that the needed ISF's are unity. Alternatively we may notice that the construction of the standard form angular momentum tensor Q_{JM} from the two standard form angular momentum tensors $b^+{}_\mu$ and b_μ can only be achieved through the usual couplings

$$Q_{JM} = A(J)\sum_{\mu\nu} C(22J;\mu\nu M)b^+{}_\mu b_\nu\,. \tag{51}$$

The $A(J)$ are arbitrary multiplicative factors. The Q_{1M} generate the physical angular momentum subgroup, and can be made equal to the J_M of (29) by choosing $A(1)$ to be $\sqrt{10}$. We will define (dimensionless) generators by choosing $A(J)=\sqrt{10}$ for all $J=0,1,\ldots,4$.

D. State labeling problem in the physical chain

As noted previously, eigenstates of the model Hamiltonian with energy

$$E=(\tfrac{5}{2}+N)$$

will form a basis for a symmetric IR of U(5), $[N0000]$. Within such an IR, a basis may be selected in several ways, one of which was employed in Sec. III B. If eigenstates of physical angular momentum are desired, one may choose a basis to be simultaneous eigenstates of the Casimir operators of each subgroup in the physical chain (48). They may also be taken to eigenstates of J_0 as well. Since the eigenvalues of the Casimir operators are in one-to-one correspondence with the IR labels, the latter can serve to label the states. Hence a typical basis state would be labeled by

$$|[N0000],[\mathrm{SU}(5)\,\mathrm{IR}],[\mathrm{R}(5)\,\mathrm{IR}],[\mathrm{R}(3)\,\mathrm{IR}],M\rangle, \tag{52}$$

where the usual partition labels are used for the subgroup IR as well as for U(5) itself. The Casimir operator for R(3) is just $\vec{J}^2$, and the partition label is the usual angular momentum quantum number J; M is the eigenvalue of J_0.

The decomposition of the symmetric IR of U(5) under restriction to the SU(5) and R(5) subgroups

TABLE II. Generators of U(5) subgroups in the physical chain.

Q_{1M}	Q_{3M}	Q_{2M}	Q_{4M}	Q_{00}
[1000 − 1]	[1000 − 1]	[1000 − 1]	[1000 − 1]	[00000]
$[b^+b]$ [2111]	$[b^+b]$ [2111]	$[b^+b]$ [2111]	$[b^+b]$ [2111]	$[b^+b]$ [0000]
[11]	[11]	[20]	[20]	[00]
[1]M	[3]M	[2]M	[4]M	[0]

Generate physical R(3) (Q_{1M})
Generate R(5) (Q_{1M}, Q_{3M})
Generate SU(5) (Q_{1M}, Q_{3M}, Q_{2M}, Q_{4M})
Generate U(5) (all)

are summarized by the branching rules

$$[N0000]_{U(5)} \to [N000]_{SU(5)}$$
$$[N000]_{SU(5)} \to [N0]_{R(5)} + [N-2,0] + \cdots + [1,0] \text{ or } [0,0]_{R(5)} \quad (53)$$

which show that the SU(5) IR labels are redundant, and that a single number

$$l = N,\ N-2, \ldots, 1 \text{ or } 0$$

serves to label the R(5) IR contained in the model states. The R(5)→R(3) branching rule is considerably more complicated; for $l \geq 6$, the decomposition permits the possibility of multiple occurrences of R(3) IR. For example in

$$[6,0] \to [0]+[3]+[4]+2[6]+[7]+[8]+[9]+[10]+[12]$$

the 140 states that form a basis for the R(5) IR contain two independent sets which span the $J=6$ IR of R(3). Hence the labeling scheme (52) is, in general, insufficient to completely distinguish the model states; an extra label ν is required to distinquish independent states which belong to the same R(5) IR and have the same angular momentum labels. In the spirit of (52) we will label basis states of the physical chain by

$$|N, l, \nu, J, M\rangle \ . \quad (54)$$

In Ref. 3, the R(5)→R(3) branching multiplicity problem was solved by introducing the extra label ν in a rather empirical way, and the ranges of ν and J within a given IR of R(5) were determined. We now offer a brief review of the results of Ref. 3 that are necessary for our explicit construction of the states $\Phi(\gamma, \theta_i)$ of Eq. (37b).

E. R(5) natural basis, intrinsic states, projected states

R(5) is a rank two, semisimple Lie group which requires two labels to specify an IR and four labels to specify basis states within a given IR. The physical chain provides only two labels to distinguish different states within an IR. In the "natural basis"

$$R(5) \supset R(4) \sim SU(2)\times SU(2) \quad (55)$$

the decomposition to the R(4) subgroup which is isomorphic with SU(2)×SU(2) provides the full four labels. It must be noted that neither SU(2) subgroup is the covering group of the physical R(3).

In Ref. 3, it is shown that the 10 R(5) generators may be taken to be the set

$$[p_\mu, \mu = 1,0,-1; q_\nu, \nu = 1,0,-1; T^{[\frac{1}{2}\frac{1}{2}]}_{\alpha\beta}, \alpha,\beta = \pm\tfrac{1}{2}] \quad (56)$$

in which $[p_\mu]$ generates one SU(2) subgroup, $[q_\nu]$ generates the other, and the remaining four R(5) generators transform under the SU(2)×SU(2) operations as a bispinor. The states of this natural basis are labeled by

$$|(l,k)p\lambda q\mu\rangle \quad (57)$$

in which (l,k) labels the R(5) IR, and p, λ, q, and μ are the eigenvalue labels of p^2, p_0, q^2, and q_0, respectively. The IR labels l and k are two nonnegative integers or half integers satisfying $l \geq k$. For fixed l and k the rules for the ranges of p and q are given in Ref. 3; here we merely restate these for the symmetric tensor IR $[l,0]$, l an integer:

$p=q$ runs from 0 to $\frac{1}{2}l$ in steps of $\frac{1}{2}$,

λ and μ run independently from $-p$ to $+p$ (58)

in steps of 1.

For the symmetric tensor IR the label q is redundant and only three state labels are required. The IR labels are related to the eigenvalues of two R(5) Casimir operators A^2 and M^4 (given in Ref. 3) whose eigenvalues are, respectively, $\frac{1}{2}[l(l+3)+k(k+1)]$ and $(l+1)(l+2)k(k+1)$. For the symmetric tensor IR one needs only A^2 whose eigenvalue is $\frac{1}{2}\Lambda = \frac{1}{2}l(l+3)$. The operator A^2 is given by

$$A^2 = p^2 + q^2 - [T^{[\frac{1}{2}\frac{1}{2}]}, T^{[\frac{1}{2}\frac{1}{2}]}]^{[00]} . \quad (59)$$

In the physical chain the generators are the sets $[Q_{1\mu} \equiv J_\mu, \mu = 1,0,-1;\ Q_{3\nu} \equiv Q_\nu, \nu = +3,2,1,0,-1,-2,-3]$ where the Q_{JM} are given by Eq. (51). These generators satisfy the commutation relationships

$$\begin{aligned}
[J_\mu, J_\nu] &= -\sqrt{2}\,C(111;\mu,\nu)J_{\mu+\nu} ,\\
[J_\mu, Q_\nu] &= -2\sqrt{3}\,C(133;\mu,\nu)Q_{\mu+\nu} ,\\
[Q_\mu, Q_\nu] &= -2\sqrt{7}\,C(331;\mu,\nu)J_{\mu+\nu} + \sqrt{6}\,C(333;\mu,\nu)Q_{\mu+\nu} .
\end{aligned} \quad (60)$$

The connection between the physical basis set and the natural basis set is developed in Appendix A and results in

$$\begin{aligned}
p_{\pm1} &= \frac{1}{\sqrt{10}} Q_{\pm3}, \quad p_0 = \tfrac{1}{10}(3J_0 - Q_0),\\
q_{\pm1} &= \tfrac{1}{5}(J_{\pm1} + \tfrac{1}{2}\sqrt{6}\,Q_{\pm1}), \quad q_0 = \tfrac{1}{10}(J_0 + 3Q_0),\\
T^{[\frac{1}{2}\frac{1}{2}]}_{\pm\frac{1}{2},\pm\frac{1}{2}} &= \pm\frac{1}{\sqrt{5}} Q_{\pm2},\\
T^{[\frac{1}{2}\frac{1}{2}]}_{\pm\frac{1}{2},\mp\frac{1}{2}} &= \mp\tfrac{1}{5}(\sqrt{3}\,J_{\pm1} - \sqrt{2}\,Q_{\pm1}),
\end{aligned} \quad (61)$$

where upper or lower signs are to be taken consistently.

The physical basis states will be labeled as in Eq. (54) and will be eigenstates of H, A^2, J^2, and J_0 as

$$H\,|N l \nu J M\rangle = (N+\tfrac{5}{2})\,|N l \nu J M\rangle,$$
$$A^2\,|N l \nu J M\rangle = \tfrac{1}{2}l(l+3)\,|N l \nu J M\rangle,$$
$$J^2\,|N l \nu J M\rangle = J(J+1)\,|N l \nu J M\rangle, \qquad (62)$$
$$J_0\,|N l \nu J M\rangle = M\,|N l \nu J M\rangle.$$

The label ν is an empirical label which distinguishes the multiple occurrences of J, M within a given R(5) IR labeled by l. In Ref. 3 it is shown that ν takes on the values

$$\nu = 0, 1, 2, \ldots, [\tfrac{1}{3}l], \qquad (63)$$

where $[\frac{1}{3}l]$ denotes the integer part of $\frac{1}{3}l$. The R(5) basis vectors in the physical chain, $|l\nu JM\rangle$, were determined by projecting from a small subset of R(5) natural basis states; the members of this subset are called intrinsic states and are labeled by $|l, \nu\rangle$ where

$$|l, \nu\rangle \equiv |(l,0)\tfrac{1}{2}l, \tfrac{1}{2}l-\nu, \tfrac{1}{2}l, -\tfrac{1}{2}l\rangle \qquad (64a)$$

in the notation of Eq. (57). All of these intrinsic states may be constructed from the maximal state.

$$|l\rangle \equiv |(l,0)\tfrac{1}{2}l, \tfrac{1}{2}l, \tfrac{1}{2}l, \tfrac{1}{2}l\rangle \qquad (64b)$$

by use of p_{-1} and q_{-1} as

$$|l, \nu\rangle = p_{-1}^{\nu} q_{-1}^{l}\,|l\rangle. \qquad (65)$$

The physical states are then given by

$$|l\nu JM\rangle = \int d\Omega D_{MK}^{J*}(\Omega) \mathrm{R}(\Omega)\,|l, \nu\rangle \qquad (66)$$

in which $\mathrm{R}(\Omega)$ is a rotation operator (as defined for example by Rose[14]) and $D_{MK}^{J*}(\Omega)$ is the M, K element of the IR matrix for the $[J]$ IR of R(3). In Eq. (66) K takes the value $l-3\nu$ and for given K, J has the values

$$J = 2K, 2K-2, 2K-3, \ldots, K. \qquad (67)$$

Three points should be made concerning the states of Eq. (66). First of all, they are not normalized nor are they orthogonal on the multiplicity quantum label. Normalization and overlap factors are given in Ref. 3 but they will not be needed here. Secondly, these are abstract states whose properties have been deduced solely from the Lie algebra and group properties. A realization of the intrinsic states will lead to a realization of the physical basis states which is our goal. Thirdly, these states will represent only the angular part of the five-dimensional oscillator states; the radial part will have to be appended.

III. CONSTRUCTION OF THE WAVE FUNCTIONS FOR THE QUADRUPOLE VIBRATOR

We are now in position to construct the explicit realization of the five-dimensional oscillator wave functions from Eq. (66) which, apart from the radial part, constitutes a formal solution. We shall first outline the procedure and then fill in some details.

(i) Using Eqs. (51) and (61) the two sets of R(5) generators are expressed in terms of creation and destruction operators $\underline{b}^+$ and $\underline{b}$. The results are tabulated in Table III.

(ii) A realization of the R(5) state $|l\rangle$ of Eq. (64b) in terms of creation operators operating on the vacuum will be found. This state will belong to the $[l0000]$ IR of R(5). Operations with an R(5) scalar formed from the $\underline{b}^+$ will not affect the R(5) content and will be used to produce a member of the $[N0000]$ IR of U(5) as required.

(iii) The results of (i) and (ii) together with Eq. (65) will lead to a realization of the intrinsic states $|Nl\nu\rangle$ of Eq. (64a) in terms of creation operators operating on the vacuum. These will also be members of the $[N0000]$ IR of U(5).

(iv) By using Eqs. (20) and (31) one may find the intrinsic states $|Nl\nu\rangle$ realized in terms of the α_μ and their derivatives. Similarly, a realization of the vacuum state may be obtained by using Eq. (26) together with Eq. (20).

(v) The laboratory to body transformation, Eq. (3), together with the defining equations for β and γ, Eq. (33), may then be used to express the intrinsic states in terms of $(\beta, \gamma, \theta_i)$. Finally, this expression is inserted into Eq. (66) to allow us to express the solution to Eq. (36) in the form

$$\Psi_{Nl\nu JM}(\beta, \gamma, \theta_i) = \langle r\,|Nl\nu JM\rangle = f_{nl}(\beta)\sum_K g_{l\nu JK}(\gamma) D_{MK}^{J*}(\theta_i).$$

Explicit forms of the $f_{nl}(\beta)$ and $g_{l\nu JK}(\gamma)$ may then be found.

We begin by noting that the intrinsic state $|Nl\rangle$ may be realized as

$$|Nl\rangle = \{[b^+ b^+]^{[0]}\}^{(N-l)/2}(b^+_2)^l\,|0\rangle. \qquad (68)$$

This state is an eigenstate of the following operators with listed eigenvalues:

$$H\colon\ N+\tfrac{5}{2}, \quad p^2\colon\ \tfrac{1}{2}l(\tfrac{1}{2}l+1), \quad p_0\colon\ \tfrac{1}{2}l,$$
$$A^2\colon\ \tfrac{1}{2}l(l+3), \quad q^2\colon\ \tfrac{1}{2}l(\tfrac{1}{2}l+1), \quad q_0\colon\ \tfrac{1}{2}l.$$

The validity of Eq. (68) is most easily established by expressing each of the operators above in terms of the $\underline{b}^+$ and $\underline{b}$ via the expressions of Appendix A and Table III. One first shows that $|l\rangle \equiv (b^+_2)^l\,|0\rangle$ satisfies all but the first condition. It is an eigenstate of H but with eigenvalue $l+\frac{5}{2}$ and hence belongs to the $[l0000]$ IR of U(5). Hence, one must modify this state by operating with an R(5) scalar which increases the number of phonons from l to N; such a scalar will not affect the R(5) content. Since $\underline{b}^+$ transforms under the $[10000]$ IR of U(5)

TABLE III. Physical and natural R(5) generators expressed in terms of the creation and destruction operators.

$J_\nu = \sqrt{10}\sum_{\alpha\beta} C(221;\alpha,\beta,\nu) b^{+}{}_{\alpha} b_{\beta}$, $\nu=-1,0,1$
$Q_\nu = \sqrt{10}\sum_{\alpha\beta} C(223;\alpha,\beta,\nu) b^{+}{}_{\alpha} b_{\beta}$, $\nu=-3,-2,-1,0,1,2,3$
$p_{\pm1} = \mp\frac{1}{\sqrt{2}}(b^{+}_{\pm1}b^{\mp2} + b^{+}_{\pm2}b^{\mp1})$
$p_0 = \frac{1}{2}(b^{+}{}_{2}b^{2} - b^{+}{}_{-2}b^{-2}) + \frac{1}{2}(b^{+}{}_{1}b^{-1} - b^{+}{}_{-1}b^{-1})$
$q_{\pm1} = \mp\frac{1}{\sqrt{2}}(b^{+}{}_{\pm2}b^{\mp1} + b^{+}{}_{\mp1}b^{\mp2})$
$q_0 = \frac{1}{2}(b^{+}{}_{2}b^{2} - b^{+}{}_{-2}b^{-2}) - \frac{1}{2}(b^{+}{}_{1}b^{1} - b^{+}{}_{-1}b^{-1})$
$T_{++} = b^{+}{}_{2}b^{0} - b^{+}{}_{0}b^{-2}$
$T_{--} = b^{+}{}_{-2}b^{0} - b^{+}{}_{0}b^{2}$
$T_{+-} = b^{+}{}_{1}b^{0} + b^{+}{}_{0}b^{-1}$
$T_{-+} = b^{+}{}_{-1}b^{0} + b^{+}{}_{0}b^{1}$

one may form products $\underline{b}^{+}\underline{b}^{+}$ classified in the physical chain by the reduction

$$
\begin{array}{lllll}
[10000]\times[10000] = & [20000] & + [11000] & & \mathrm{U}(5) \\
 & \downarrow & \downarrow & & \\
 & [2000] & + [1100] & & \mathrm{SU}(5) \\
 & \downarrow & \downarrow & & \\
 & [00] + [20] & + [11] & & \mathrm{R}(5) \\
 & \downarrow \quad\ \downarrow & \downarrow & & \\
 & [0] + [2] + [4] & + [1] + [3] & & \mathrm{R}(3).
\end{array}
$$

The proper R(5) scalar is the left most chain and is given by

$$[\underline{b}^{+}\underline{b}^{+}]^{[0]} \equiv \sum_{\mu\nu} C(220;\mu,\nu,0) b^{+}{}_{\mu} b^{+}{}_{\nu}. \tag{69}$$

When this is raised to $\frac{1}{2}(N-l)$ power and allowed to operate on $(b^{+}{}_{2})^{l}|0\rangle$ the state of Eq. (68) is produced. The state $|Nl\rangle$ differs from the intrinsic maximal state $|l\rangle$ only by an R(5) scalar part. The states $|Nl\nu\rangle$ will differ from the $|l\nu\rangle$ of Eq. (65) by this same R(5) scalar part, and finally the states $|Nl\nu JM\rangle$ will also differ from the $|l\nu JM\rangle$ of Eq. (66) by this same factor.

The formation of the intrinsic states $|Nl\nu\rangle$ from Eq. (68) is easily accomplished by expressing p_{-1} and q_{-1} in terms of $\underline{b}^{+}$ and $\underline{b}$, as in Table III. These are inserted into Eq. (65). By the binomial expansion one finds

$$(q_{-1})^{l} = \left(\frac{1}{\sqrt{2}}\right)^{l} \sum_{j} \binom{l}{j} (b^{+}{}_{1})^{l-j} (b^{+}{}_{-2})^{j} (b^{-1})^{j} (b^{2})^{l-j} \tag{70a}$$

and

$$(p_{-1})^{\nu} = \left(\frac{1}{\sqrt{2}}\right)^{\nu} \sum_{j} \binom{\nu}{j} (b^{+}{}_{-1})^{\nu-j} (b^{+}{}_{-2})^{j} (b^{2})^{\nu-j} (b^{1})^{j}. \tag{70b}$$

Since $[\underline{b}^{+}\underline{b}^{+}]^{[0]}$ commutes with all of the R(5) generators, one easily finds

$$
\begin{aligned}
|Nl\nu\rangle &= \{[b^{+}b^{+}]^{[0]}\}^{(N-l)/2} (p_{-1})^{\nu} (q_{-1})^{l} |l\rangle \\
&= \{[\underline{b}^{+}\underline{b}^{+}]^{[0]}\}^{(N-l)/2} \left(\frac{1}{\sqrt{2}}\right)^{l+\nu} \frac{(l!)^2}{(l-\nu)!} \\
&\quad \times (b^{+}{}_{-2})^{\nu} (b^{+}{}_{1})^{l-\nu} |0\rangle.
\end{aligned} \tag{71}
$$

Since we will attend to normalization at the end we shall drop the factors of $1/\sqrt{2}$, $l!$, and $(l-\nu)$

To proceed to step (iv) one needs

$$b^{+}{}_{\mu} = \frac{1}{\sqrt{2}}\left[\alpha_{\mu} - (-1)^{\mu}\frac{\partial}{\partial\alpha_{-\mu}}\right] \tag{72a}$$

and

$$b^{\mu} = \frac{1}{\sqrt{2}}\left[(-1)^{\mu}\alpha_{-\mu} + \frac{\partial}{\partial\alpha_{\mu}}\right]. \tag{72b}$$

Then $b^{\mu}|0\rangle = 0$ has a solution (again to within factors which only affect normalization)

$$|0\rangle = \exp\left[-\frac{1}{2}\sum_{\nu}(-1)^{\nu}\alpha_{-\nu}\alpha_{\nu}\right] = e^{-\beta^2/2}. \tag{73}$$

Then using Eqs. (72) and (73) one finds directly that

$$|l\nu\rangle = (b^{+}{}_{-2})^{\nu}(b^{+}{}_{1})^{l-\nu}|0\rangle = (\sqrt{2}\,\alpha_{1})^{l-\nu}(\sqrt{2}\,\alpha_{-2})^{\nu}|0\rangle. \tag{74}$$

Now, one also requires the result

$$[\underline{b}^{+}\underline{b}^{+}]^{[0]} = \frac{1}{\sqrt{3}}\left\{-H - \frac{5}{2} + \sum_{\mu}\alpha^{\mu}\alpha_{\mu} - \sum_{\mu}\alpha_{\mu}\frac{\partial}{\partial\alpha_{\mu}}\right\} \tag{75a}$$

or

$$[\underline{b}^{+}\underline{b}^{+}]^{[0]} = \frac{1}{\sqrt{3}}\left\{\beta^2 - \beta\frac{\partial}{\partial\beta} - H - \tfrac{5}{2}\right\}. \tag{75b}$$

This result is most easily established by expressing the right hand side of Eq. (69) using Eqs. (72), and also using Eq. (32) to recognize the Hamiltonian. If one defines

$$T(\beta) = \beta^2 - \beta\frac{\partial}{\partial\beta} - \tfrac{5}{2}$$

and

$$\Theta(\beta) = \beta^2 - \beta\frac{\partial}{\partial\beta} - l - \tfrac{5}{2},$$

then $[T(\beta), H] \neq 0$ prevents writing $|Nl\nu\rangle$ as $[\Theta(\beta)]^{n}$ times Eq. (74). One may, however, write

$$|Nl\nu\rangle = \left\{\sum_{k=0}^{n} A_{n,k}[\Theta(\beta)]^{k}\right\} \alpha_{1}^{l-\nu}\alpha_{-2}^{\nu}|0\rangle$$

with the A's determined by the recursion relationships

$$A_{n+1,k+1}=\sum_{m=1}^{n} A_{nm}\binom{m}{k}(-2)^{m-k}, \quad A_{nn}=1,$$

$$A_{n0}=0.$$

This establishes the form of $\langle \underline{r}| Nl\nu\rangle \equiv \Phi_{Nl\nu}(\beta,\gamma,\theta_i)$ as

$$\Phi_{Nl\nu}(\beta,\gamma,\theta_i)=f_{nl}(\beta)\,h_{l\nu}(\gamma,\theta_i), \tag{76a}$$

where

$$f_{nl}(\beta)=(-1)^n 2^n (l+\tfrac{5}{2},n)\,{}_1F_1(-n;\,l+\tfrac{5}{2};\,\beta^2)\beta^l e^{-\beta^2/2}, \tag{76b}$$

$$N=2n+l,$$

and

$$h_{l\nu}(\gamma,\theta_i)=\left(\frac{\alpha_1}{\beta}\right)^{l-\nu}\left(\frac{\alpha_{-2}}{\beta}\right)^{\nu}. \tag{76c}$$

The validity of this form is established by induction by noting first that apart from factors, $|\,l\nu\rangle$ of Eq. (74) yields the form

$$\langle \underline{r}|\, l\nu\rangle = h_{l\nu}(\gamma,\theta_i)\beta^l e^{-\beta^2/2}.$$

Thus Eqs. (76) are clearly valid for $n=0$. The induction proof follows easily if we recall that $H|\,Nl\nu\rangle=(N+\tfrac{5}{2})|\,Nl\nu\rangle=(2n+l+\tfrac{5}{2})|\,Nl\nu\rangle$.

Next, one must process the $h_{l\nu}(\gamma,\theta_i)$ further. To do so we note that from Eq. (33) and the inverse of Eq. (3) one has

$$\left(\frac{\alpha_1}{\beta}\right)=\cos\gamma D^{2*}_{10}(\theta_i)+\frac{\sin\gamma}{\sqrt{2}}\left[D^{2*}_{12}(\theta_i)+D^{2*}_{1-2}(\theta_i)\right]$$

and

$$\left(\frac{\alpha_{-2}}{\beta}\right)=\cos\gamma D^{2*}_{-20}(\theta_i)+\frac{\sin\gamma}{\sqrt{2}}\left[D^{2*}_{-22}(\theta_i)+D^{2*}_{-2-2}(\theta_i)\right].$$

Thus by the binomial expansion

$$h_{l\nu}(\gamma,\theta_i)=\sum_{j=0}^{\nu}\sum_{\rho=j}^{j+l-\nu}\sum_{\sigma=0}^{j}\sum_{\tau=0}^{\rho-j}\binom{l-\nu}{\rho-j}\binom{\rho-j}{\tau}\binom{\nu}{j}\binom{j}{\sigma}(\cos\gamma)^{l-\rho}\left(\frac{\sin\gamma}{\sqrt{2}}\right)^{\rho}$$
$$\times (D^{2*}_{-20})^{\nu-j}(D^{2*}_{-22})^{j-\sigma}(D^{2*}_{-22})^{\sigma}(D^{2*}_{10})^{l-\nu+j-\rho}(D^{2*}_{12})^{\rho-j-\tau}(D^{2*}_{1-2})^{\tau}. \tag{77}$$

From the form of Eq. (77) together with the completeness of the $D^{J*}_{MK}(\theta_i)$ it follows that

$$\Phi_{Nl\nu}(\beta,\gamma,\theta_i)=f_{nl}(\beta)\sum_{J'K'M'} C^{l\nu}_{J'K'M'}(\gamma)D^{J'*}_{M'K'}(\theta_i), \tag{78}$$

where we may determine the C's from integration over Euler angles as

$$C^{l\nu}_{JKM}(\gamma)=\int d\Omega(\theta_i)D^{J}_{MK}(\theta_i)h_{l\nu}(\gamma,\theta_i). \tag{79}$$

In terms of this realization, the projection relationship, Eq. (66), becomes

$$\Psi_{Nl\nu JM}(\beta,\gamma,\theta_i)=\int d\Omega D^{J*}_{MK}(\Omega)\mathrm{R}(\Omega)\Phi_{Nl\nu}(\beta,\gamma,\theta_i).$$

(It is important to keep in mind that the coordinates θ_i are not the same as the integration Euler angles symbolized by Ω.) However,

$$\mathrm{R}(\theta_i)\Phi_{Nl\nu}(\beta,\gamma,\theta_i)=f_{nl}(\beta)\sum_{J'K'M'}C^{l\nu}_{J'K'M'}(\gamma)\mathrm{R}(\Omega)D^{J'*}_{M'K'}(\theta_i)$$
$$=f_{nl}(\beta)\sum_{J'K'M'M''}C^{l\nu}_{J'K'M'}(\gamma)D^{J'}_{M''M'}(\Omega)D^{J'}_{M''K'}(\theta_i).$$

Thus

$$\Psi_{Nl\nu JM}(\beta,\gamma,\theta_i)=f_{nl}(\beta)\sum_K g_{l\nu JK}(\gamma)D^{J*}_{MK}(\theta_i), \tag{80}$$

where $f_{nl}(\beta)$ is given by Eq. (76b) and

$$g_{l\nu JK}(\gamma)=C^{l\nu}_{JKl-3\nu}(\gamma)=\int d\Omega(\theta_i)D^{J}_{l-3\nu,K}(\theta_i)h_{l\nu}(\gamma,\theta_i). \tag{81}$$

All that remains is to explicitly determine the $g_{l\nu JK}(\gamma)$ by evaluating the right hand side of Eq. (81). Since

$$\int d\Omega(\theta_i)=\frac{1}{8\pi^2}\int_0^{2\pi}d\theta_1\int_0^{\pi}\sin\theta_2 d\theta_2\int_0^{2\pi}d\theta_3,$$

we may use $D^J_{MK}(\theta_i)=e^{-iM\theta_1}d^J_{MK}(\theta_i)e^{-iK\theta_3}$ and dispose immediately of the integrals over θ_1 and θ_3 to find

$$g_{l\nu JK}(\gamma)=\sum_{j=0}^{\nu}\sum_{\rho=j}^{j+l-\nu}\sum_{\sigma=0}^{j}\sum_{\tau=0}^{\rho-j}\binom{l-\nu}{\rho-j}\binom{\rho-j}{\tau}\binom{\nu}{j}\binom{j}{\sigma}(\cos\gamma)^{l-\rho}\left(\frac{\sin\gamma}{\sqrt{2}}\right)^{\rho}\delta^K_{2\rho-4\sigma-4\tau}\mathcal{J}, \tag{82}$$

where

$$\mathcal{J}\equiv\int_0^{\pi}\sin\theta d\theta[d^2_{-20}]^{\nu-j}[d^2_{-22}]^{j-\sigma}[d^2_{-2-2}]^{\sigma}[d^2_{10}]^{l-\nu+j-\rho}[d^2_{12}]^{\rho-j-\tau}[d^2_{1-2}]^{\tau}d^J_{l-3\nu,K}. \tag{83}$$

One notes that the δ function forces K to be an even integer.

Now, the explicit integral over the d functions is performed by using

$$d^j_{m'm}(\theta)=[(j+m)!\,(j-m)!\,(j+m')!\,(j-m')!]^{1/2}\sum_{\nu}\frac{(-1)^{\nu}(\cos\frac{1}{2}\theta)^{2j+m-m'-2\nu}(-\sin\frac{1}{2}\theta)^{m'-m+2\nu}}{(j-m'-\nu)!\,(j+m-\nu)!\,(\nu+m'-m)!\,\nu!}$$

in Eq. (83). The integration is most easily carried out by changing the variable to $t=\sin^2\frac{1}{2}\theta$ and by recognizing the hypergeometric function

$${}_2F_1(a,b;c;z)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt,\quad \text{Real } c>\text{Real } b>0.$$

We then simplify the resulting expression for $g_{l\nu JK}(\gamma)$ somewhat by replacing the sum over $j\geq 0$ by a sum $c=\rho-j\geq 0$ and replacing also the σ sum by one over $f=\tau+\sigma\geq 0$. We find then,

$$\begin{aligned}g_{l\nu JK}(\gamma)=&\sum_{c\rho\tau fh}\delta^k_{2\rho-4f}2^c[\sqrt{6}\cos\gamma]^{l-\rho}[\sin\gamma/\sqrt{2}]^{\rho}(-1)^{h+c+\tau}\\&\times\binom{l-\nu}{c}\binom{\nu}{\rho-c}\binom{\rho-c}{f-\tau}\binom{c}{\tau}\left[\frac{(J+K)!\,(J-K)!}{(J+l-3\nu)!\,(J-l+3\nu)!}\right]^{1/2}\\&\times\frac{(3\tau+h+l-\nu-c)!\,(2\nu+2c-3\tau-h+J)!}{(\nu+c+J+l+1)!}\binom{J-l+3\nu}{h}\binom{J+l-3\nu}{l-3\nu-K+h}\\&\times{}_2F_1(-l+\nu+c,3\tau-c+h+l-\nu+1;\nu+c+J+l+2;2).\end{aligned}\tag{84}$$

This expression may be simplified even further by using the symmetry properties of the $g_{l\nu JK}(\gamma)$ discussed in Appendix B. The one we use here is $g_{l\nu J,-K}(\gamma)=(-1)^J g_{l\nu JK}(\gamma)$ to confine our attention to $K\geq 0$ in Eq. (84). We define integers m and p by

$$K=4m+2p\geq 0,\quad p=0,1,\quad m=0,1,2,\ldots.$$

Then the index f in Eq. (84) is found to run over the values

$$f=0,1,2,\ldots,[\tfrac{1}{2}(l-p)]-m.$$

The use of the δ function in Eq. (84) then yields the final results

$$\Psi_{Nl\nu JM}(\beta,\gamma,\theta_i)=\mathfrak{N}_{Nl\nu J}f_{nl}(\beta)\sum_{K\text{ even}}g_{l\nu JK}(\gamma)D^{J*}_{MK}(\theta_i), \tag{85a}$$

where for $K=4m+2p\geq 0$, $m=0,1,2,\ldots,$ $p=0,1,$

$$g_{l\nu JK}(\gamma)=[\cos\gamma]^l\sum_{f=0}^{[(l-p)/2]-m}A_f(l\nu JK)[\tan\gamma]^{2f+K/2} \tag{85b}$$

and the coefficients $A_f(l\nu JK)$ are given by

$$\begin{aligned}A_f(l\nu JK)=&(2)^{l/2-K/2-2f}(3)^{l/2-K/4-f}\left[\frac{(J+K)!\,(J-K)!}{(J+l-3\nu)!\,(J-l+3\nu)!}\right]^{1/2}\\&\times\sum_{c\tau h}(-1)^{h+c+\tau}2^c{}_2F_1(-l+\nu+c,3\tau-c+h+l-\nu+1;\nu+c+J+l+2;2)\\&\times\binom{\nu}{\frac{1}{2}K+2f-c}\binom{\frac{1}{2}K+2f-c}{f-\tau}\binom{l-\nu}{c}\binom{c}{\tau}\binom{J-l+3\nu}{h}\binom{J+l-3\nu}{l-3\nu-k+h}\\&\times\frac{(3\tau+h+l-\nu-c)!\,(2\nu+2c-3\tau-h+J)!}{(\nu+c+J+l+1)!}.\end{aligned}\tag{85c}$$

For $K<0$ one uses the above together with

$$g_{l\nu J,-K}(\gamma)=(-1)^J g_{l\nu JK}(\gamma)\,. \tag{86}$$

In Eq. (85a), $\mathfrak{N}_{Nl\nu J}$ is a normalization factor whose value is given in Sec. IV. The main result of this paper is that given in Eqs. (85b) and (85c) which are the closed form expression for the γ-vibration part of the quadrupole oscillator. All the sums of Eqs. (85b) and (85c) are finite and hence these expressions are easily prepared for use on modern computers. It is of interest to note in passing that separation of the problem into a β or radial part and angular parts is associated with introducing the direct product SO(2, 1) $\times$ R(5) into the problem. See Ref. 15. Chap. 20.

IV. NORMALIZATION AND OVERLAP FACTORS

It is now relatively straightforward to calculate the overlap of two wave functions of the form (85a). First we write the wave function to explicitly display the symmetry property which restricts the sum to $K\geq 0$:

$$\Psi_{Nl\nu JM}(\beta,\gamma,\theta_i)=\mathfrak{N}_{Nl\nu J}f_{nl}(\beta)\sum_{\substack{K\geq 0\\ \text{even}}}\frac{1}{[1+\delta_{K,0}]}g_{l\nu JK}(\gamma)[D^{J*}_{MK}(\theta_i)+(-1)^J D^J_{M,-K}(\theta_i)]$$

$$=\mathfrak{N}_{Nl\nu J}f_{nl}(\beta)\sum_{\substack{K\geq 0\\ \text{even}}}\frac{1}{1+\delta_{K0}}\left\{\frac{2[1+(-1)^J\delta_{K0}]}{2J+1}\right\}^{1/2}g_{l\nu JK}(\gamma)\psi_{JMK}(\theta_i)\,, \tag{87}$$

where the $\psi_{JMK}(\theta_i)$ are just the properly symmetrized wave functions for the symmetric rotor,[13]

$$\psi_{JMK}=\left\{\frac{2J+1}{2[1+(-1)^J\delta_{K0}]}\right\}^{1/2}[D^{J*}_{MK}+(-1)^J D^{J*}_{M,-K}]\,.$$

These are normalized, i.e.,

$$[\psi_{J'M'K'}(\theta_i),\psi_{JMK}(\theta_i)]=\int d\Omega\psi^*_{J'M'K'}(\theta_i)\psi_{JMK}(\theta_i)d\Omega=\delta_{JJ'}\delta_{MM'}\delta_{KK'}\,. \tag{88}$$

Thus

$$(\Psi_{N'l'\nu'JM},\Psi_{Nl\nu JM})=\mathfrak{N}_{N'l'\nu'J'}\mathfrak{N}_{Nl\nu J}(f_{n'l'}(\beta),f_{nl}(\beta))\sum_{\substack{K\geq 0\\ \text{even}}}\left\{\frac{2[1+(-1)^J\delta_{K,0}]}{(2J+1)(1+\delta_{K,0})^2}\right\}(g_{l'\nu'JK}(\gamma),g_{l\nu JK}(\gamma))\,. \tag{89}$$

The β-separated integral is easily found by noting that the ${}_1F_1$ is just a Laguerre polynomial. There will be a $\delta_{l'l}$ from the R(5) IR property of the wave functions, so one needs only consider

$$(f_{n'l}(\beta),f_{nl}(\beta))=\int_0^\infty \beta^4 f_{n'l}(\beta)f_{nl}(\beta)d\beta=2^{2n-1}n!\,\Gamma(n+l+\tfrac{5}{2})\delta_{n'n}\,. \tag{90}$$

Next we consider the γ integral and use Eq. (85b), plus $\delta_{l'l}$ from the IR property to find

$$(g_{l\nu'JK}(\gamma),\ g_{l\nu JK}(\gamma))=\int_0^{2\pi}g_{l\nu'JK}(\gamma)g_{l\nu JK}(\gamma)\,|\sin 3\gamma|\,d\gamma$$

$$=\sum_{f_1 f_2}A_{f_1}(l\nu'JK)A_{f_2}(l\nu JK)\int_0^{2\pi}|\sin 3\gamma|(\cos\gamma)^{2p-K-2f_1-2f_2}(\sin\gamma)^{K+2f_1+2f_2}d\gamma\,. \tag{91}$$

We define

$$\mathcal{J}(p,q)=\int_0^{2\pi}|\sin 3\gamma|(\cos\gamma)^{2p}(\sin\gamma)^{2q}d\gamma$$

$$=3I(p,q)-4I(p,q+1)\,, \tag{92a}$$

where

$$I(p,q)=\int_0^{2\pi}(\sin\gamma)^{2q+1}(\cos\gamma)^{2p}d\gamma\,. \tag{92b}$$

From symmetry one has for the integral of (92b)

$$\int_0^{2\pi}=4\int_0^{\pi/3}+2\int_{2\pi/3}^{\pi/3}\,. \tag{92c}$$

The indefinite integral is easily found to be

$$I(p,q)=\sum_{n=0}^{q}\binom{q}{n}\frac{(\cos\gamma)^{2n+2p+1}}{(2n+2p+1)}(-1)^n\,. \tag{93}$$

A little algebra then yields

$$\mathcal{G}(p,q)=4\sum_{n=0}^{q+1}\binom{q+1}{n}\frac{(-1)^{n+1}}{(2n+2p+1)}\times\left(\frac{3n+q+1}{q+1}\right)\left(\frac{2^{2n+2p}-1}{2^{2n+2p}}\right) \qquad (94)$$

so that

$$(g_{l\nu' JK}(\gamma),g_{l\nu JK}(\gamma)) = \sum_{f_1f_2}A_{f_1}(l\nu' JK)A_{f_2}(l\nu JK)\times\mathcal{G}(l-\tfrac{1}{2}K-f_1-f_2,\tfrac{1}{2}K+f_1+f_2)\,. \qquad (95)$$

We may now write normalization factors separately for the β and γ parts of the wave functions. We write

$$\mathfrak{N}_{nl\nu J}=N_{nl}N_{l\nu J}\,, \qquad (96)$$

where N_{nl} normalizes the β part and $N_{l\nu J}$ normalizes the γ part of Eq. (87). Then by writing the ${}_1F_1$ of $f_{nl}(\beta)$ as a Laguerre polynomial, one has

$$f_{nl}(\beta)=(-1)^n n!\beta^l e^{-\beta^2/2}L_n^{l+3/2}(\beta^2) \qquad (97a)$$

and

$$N_{nl}=\left[\frac{2n!}{\Gamma(n+l+\frac{5}{2})}\right]^{1/2}. \qquad (97b)$$

Then, the normalization factor for the γ part of the equation is just

$$N_{l\nu J}=\left(\sum_{\substack{K\geq 0\\ \text{even}}}\left\{\frac{2[1+(-1)^J\delta_{K0}]}{(2J+1)(1+\delta_{K,0})^2}\right\}(g_{l\nu JK},g_{l\nu JK})\right)^{-1/2} \qquad (98)$$

The normalized states of Eq. (87) agree with those previously given by Bes[6] for low values of l as well as with the Yrast states given in Ref. 10.

Finally, we note that very recently Chacón, Moshinsky, and Sharp[21] have also given exact solutions to the quadrupole vibration problem. Their resolution of the multiplicity problem is in terms of a quantum number $\mu=0,1,2,\ldots,[\frac{1}{3}l]$ such that n_2 ($n_2=2\mu$ for J even or $n_2=2\mu+1$ for J odd) is the number of zero angular momentum coupled boson triplets. This labeling was first suggested by Iachello and Arima.[22] That is, n_2 is the exponent of $[b^+b^+b^+]^{[00]}$ in their polynomial solutions to the problem. By combining Eqs. (4.13a) and (4.13b) of Ref. 21, one finds that their rule for multiplicity resolution is precisely the same as ours. That is, our multiplicity label ν and their label μ take on the values $0,1,2,\ldots,[\frac{1}{3}l]$ and the angular momentum (our J, their L) takes on the values $2l-6\nu, 2l-6\nu-2,\ldots,l-3\nu$. In the cases where no multiplicity exists our states and those of Ref. 21 will be at most different in sign when normalized; this includes all states for $l<6$. For $l\geq 6$ multiplicities occur and while the labels will be the same, one must not conclude that the states are identical; indeed they are likely not. The simplest multiplicity occurs for the two states $J=6$ for $l=6$. The two states in our solution are given quite simply by application of Eqs. (85). The states of Ref. 21 have to be constructed iteratively starting from the $l=0, J=0$ state. For this reason we have not computed the overlap between their states and ours. It would be an interesting piece of future research to compute the necessary transformation brackets in the general use.

APPENDIX A: R(5) GENERATORS

To make connection between the physical and natural basis generators for R(5) summarized by Eqs. (61), it is perhaps easiest to proceed via the Cartan-Weyl[16] formalism. Briefly, one needs to identify the commuting set H_1, H_2 and the stepping operators E_α such that $[H_i,E_\alpha]=\alpha_i E_\alpha$. If we start in the physical basis the R(5) generators are given by Eq. (51) and $J=1,3$ and $A(J)=\sqrt{10}$. Then, among the sets $J_\nu\equiv Q_{1\nu}$ and $Q_\nu\equiv Q_{3\nu}$ the commuting subset is J_0 and Q_0 which we temporarily call H_1 and H_2, respectively. Then, one has

$$[H_1,J_{\pm1}]=\pm J_{\pm1},\quad [H_1,Q_\nu]=\nu Q_\nu\,,$$
$$[H_2,J_{\pm1}]=\pm\sqrt{6}\,Q_{\pm1},\quad [H_2,Q_{\pm1}]=\pm(\sqrt{6}\,J_{\pm1}+Q_{\pm1})\,,$$
$$[H_2,Q_{\pm2}]=\pm Q_{\pm2},\quad\text{and}\quad [H_2,Q_{\pm3}]=\mp Q_{\pm3}\,.$$

It is therefore necessary to define four new operators $\xi_\pm,\eta_\pm$ which are linear combinations of $J_{\pm1}$ and $Q_{\pm1}$. If we define

$$\xi_\pm=AJ_{\pm1}+BQ_{\pm1}$$

and impose the condition

$$[H_2,\xi_\pm]=\pm\rho\xi_\pm$$

we find the roots $\rho=3$ and $\rho=-2$. Therefore

$$\xi_\pm=B\{\tfrac{1}{3}\sqrt{6}\,J_{\pm1}+Q_{\pm1}\}$$

and

$$\eta_\pm=B\{-\tfrac{1}{2}\sqrt{6}\,J_{\pm1}+Q_{\pm1}\}\,,$$

where B is a "normalization" factor and

$$[H_1,\xi_\pm]=\pm\xi_\pm,\quad [H_2,\xi_\pm]=\pm3\xi_\pm\,,$$
$$[H_1,\eta_\pm]=\pm\eta_\pm,\quad [H_2,\eta_\pm]=\mp2\eta_\pm\,.$$

Thus we have the root diagram shown in Fig. 1 from which it is clear that a simple rotation will make the two commuting SU(2) subalgebras manifest. This corresponds to a change of basis from H_1,H_2 to $\overline{H}_1,\overline{H}_2$. That is, $\overline{H}_1$ and $\overline{H}_2$ are to be linear combinations of H_1 and H_2 such that

$$[\bar{H}_1, Q_{\pm 3}] = [\bar{H}_2, \xi_\pm] = 0$$

and

$$[\bar{H}_1, \xi_\pm] = \pm \xi_\pm, \quad [\bar{H}_2, Q_{\pm 3}] = \pm Q_{\pm 3}.$$

These conditions yield

$$\bar{H}_1 = \tfrac{1}{10}(H_1 + 3H_2) = \tfrac{1}{10}(J_0 + 3Q_0) \equiv q_0$$

and

$$\bar{H}_2 = \tfrac{1}{10}(3H_1 - H_2) = \tfrac{1}{10}(3J_0 - Q_0) \equiv p_0 .$$

Now we must impose the condition on $\xi_\pm \equiv q_{\pm 1}$ such that $[q_{+1}, q_{-1}] = -q_0$. This yields $B = \sqrt{6}/10$ so that

$$q_{\pm 1} = \tfrac{1}{5}(J_{\pm 1} + \tfrac{1}{2}\sqrt{6}\, Q_{\pm 1}) .$$

Similarly, one finds

$$p_{\pm 1} = \frac{1}{\sqrt{10}} Q_{\pm 3} .$$

The two commuting SU(2) subgroups are then generated by the sets $\{p_{+1}, p_0, p_{-1}\}$ and $\{q_{+1}, q_0, q_{-1}\}$. The remaining generators $Q_{\pm 2}$ and $\eta_\pm$ are to be grouped into a bispinor $T^{[\frac{1}{2}\frac{1}{2}]}_{\alpha,\beta}$ under the two SU(2) subgroups. That is

$$[p_\nu, T^{[\frac{1}{2}\frac{1}{2}]}_{\alpha,\beta}] = (-1)^\nu C(\tfrac{1}{2}\,1\,\tfrac{1}{2}; \alpha+\nu, -\nu)\tfrac{1}{2}\sqrt{3}\, T^{[\frac{1}{2}\frac{1}{2}]}_{\alpha+\nu,\beta}$$

and

$$[q_\nu, T^{[\frac{1}{2}\frac{1}{2}]}_{\alpha,\beta}] = (-1)^\nu \tfrac{1}{2}\sqrt{3}\, C(\tfrac{1}{2}\,1\,\tfrac{1}{2}; \beta+\nu, -\nu) T^{[\frac{1}{2}\frac{1}{2}]}_{\alpha,\beta+\nu} .$$

The bispinor nature is evident from $[p_0, Q_{\pm 2}] = \pm\frac{1}{2} Q_{\pm 2}$, $[q_0, Q_{\pm 2}] = \pm \frac{1}{2} Q_{\pm 2}$, $[p_0, \eta_\pm] = \pm \frac{1}{2}\eta_\pm$ and $[q_0, \eta_\pm] = \mp \frac{1}{2}\eta_\pm$. These conditions of course remain valid when $Q_{\pm 2}$ and $\eta_\pm$ are multiplied by constants chosen to make the remaining equations involving $p_{\pm 1}$ and $q_{\pm 1}$ valid. Thus, one finds

$$T^{[\frac{1}{2}\frac{1}{2}]}_{\pm\frac{1}{2},\pm\frac{1}{2}} = \pm \frac{1}{\sqrt{5}} Q_{\pm 2}$$

and

$$T^{[\frac{1}{2}\frac{1}{2}]}_{\pm\frac{1}{2},\mp\frac{1}{2}} = \pm \frac{2}{\sqrt{3}} \eta_\pm = \mp\tfrac{1}{5}(\sqrt{3}\, J_{\pm 1} - \sqrt{2}\, Q_{\pm 1}) .$$

The root diagram corresponding to this choice of generator basis is shown in Fig. 2.

APPENDIX B: SYMMETRIES OF THE WAVE FUNCTION $\psi(\beta,\gamma,\theta_i)$

In Sec. II the collective coordinates $(\beta, \gamma, \theta_i)$ were defined. The θ_i are the Euler angles specifying the rotation from Laboratory (lab) axes to a set of body-fixed (BF) principal axes while β and γ are shape variables specifying the appearance of the nuclear surface in the BF frame, in accordance with Eqs. (2) and (33). In Fig. 3 we display a sampling of quadrupole shapes as they would appear to a BF observer. As is well known, β is a measure of the overall deviation from sphericity while γ is seen to influence the rotational asymmetry about the BF z axis.

There are 24 distinct ways of selecting sets of right handed axes fixed along principle directions in the body. If (x, y, z) and $(\bar{x}, \bar{y}, \bar{z})$ are the Cartesian coordinates employed by observers in the

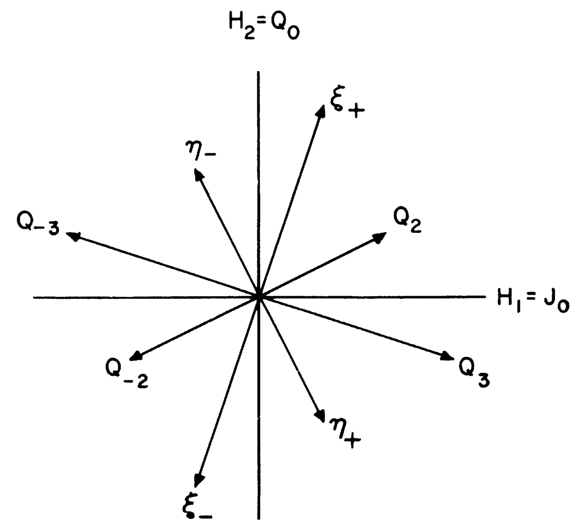

FIG. 1. Root diagram for R(5) in the Cartan-Weyl form of the physical basis generators.

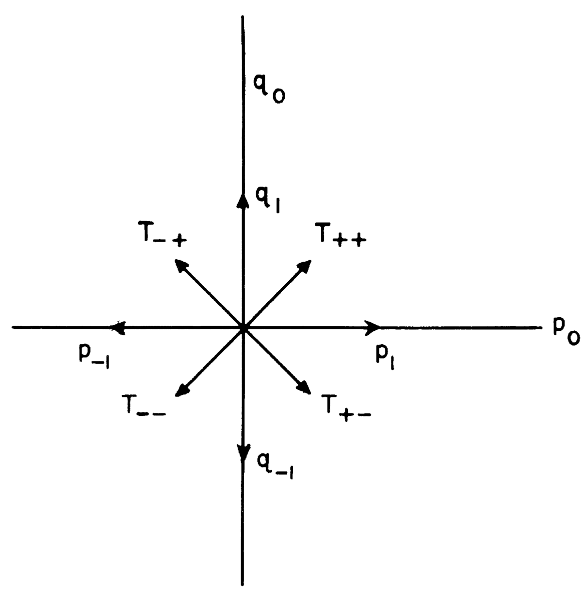

FIG. 2. Root diagram for R(5) in the natural basis which manifestly displays the R(4) ≈ SU(2) × SU(2) subgroup.

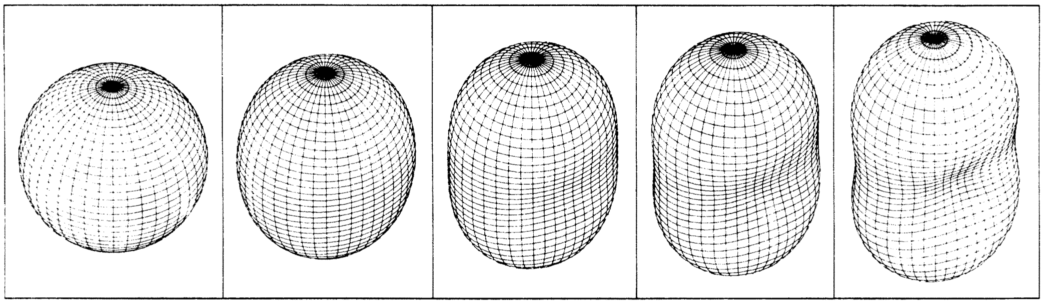

FIG. 3. Effect of increasing β. In this series of drawings γ is fixed at 20° and β takes on the values 0, 0.3, 0.6, 0.8, and 1.0, respectively, from left to right. In each case the surface is viewed from the point $(X_1, X_2, X_3) = (1, 2, 2)$ in the body-fixed (BF) frame, and the X_3 axis is upward.

two principal axes frames BF and $\overline{\mathrm{BF}}$, respectively, then these coordinates are related by one of the entries in Table IV which also gives the Euler angles (ϕ_1, ϕ_2, ϕ_3) which specify the BF $\rightarrow \overline{\mathrm{BF}}$ rotation in each case.

Let us consider using two sets of collective coordinates to describe the nuclear surface: $(\beta\gamma\theta_i)$ and $(\bar\beta\bar\gamma\bar\theta_i)$ referring, respectively, to the body-fixed frames BF and $\overline{\mathrm{BF}}$. The wave functions $\psi(\beta, \gamma, \theta_i)$ constructed in Sec. III are clearly single valued in the lab coordinates α_μ. Hence one must have

$$\Psi(\bar\beta, \bar\gamma, \bar\theta_i) = \Psi(\beta, \gamma, \theta_i) \tag{B1}$$

if both sets of collective coordinates give rise to the same α_μ, i.e., if

$$\alpha_\mu = \bar\alpha_\mu, \; \mu = -2, -1, \ldots, 2$$

or

$$\beta \cos\gamma D^{2*}_{\mu,0}(\theta_i) + \frac{\beta \sin\gamma}{\sqrt{2}}[D^{2*}_{\mu,2}(\theta_i) + D^{2*}_{\mu,-2}(\theta_i)] = \bar\beta \cos\bar\gamma D^{2*}_{\mu,0}(\bar\theta_i) + \frac{\bar\beta \sin\bar\gamma}{\sqrt{2}}[D^{2*}_{\mu,2}(\bar\theta_i) + D^{2*}_{\mu,-2}(\bar\theta_i)]. \tag{B2}$$

To write Eq. (B2) we have used the inverse of Eq. (3), namely,

$$\alpha_\mu = \sum_\nu D^{2*}_{\mu\nu}(\theta_i) a_\nu$$

together with Eqs. (33). The deformation is uniquely specified by the α_μ through Eq. (34) and the fact that β is a radiallike variable and ranges over 0 to ∞. Hence one has

$$\bar\beta = \beta. \tag{B3}$$

The representation property of the rotation matrices may be invoked

$$D^{2*}_{\mu,\nu}(\bar\theta_i) = \sum_m D^{2*}_{\mu,m}(\theta_i) D^{2*}_{m,\nu}(\phi_i), \tag{B4}$$

where $(\phi_i) = (\phi_1, \phi_2, \phi_3)$ are the BF $\rightarrow \overline{\mathrm{BF}}$ Euler angles of Table IV. When Eqs. (B4) and (B3) are substituted into Eq. (B2), one may equate the coefficients of the independent $D^{2*}_{\mu\nu}(\theta_i)$ to obtain the following:

$$(\nu = 0) \cos\gamma = \cos\bar\gamma D^{2*}_{0,0}(\phi) + \frac{\sin\bar\gamma}{\sqrt{2}}[D^{2*}_{0,2}(\phi) + D^{2*}_{0,-2}(\phi)], \tag{B5}$$

$$(\nu = 1) 0 = \cos\bar\gamma D^{2*}_{1,0}(\phi) + \frac{\sin\bar\gamma}{\sqrt{2}}[D^{2*}_{1,2}(\phi) + D^{2*}_{1,-2}(\phi)], \tag{B6}$$

$$(\nu = -1) 0 = \cos\bar\gamma D^{2*}_{-1,0}(\phi) + \frac{\sin\bar\gamma}{\sqrt{2}}[D^{2*}_{-1,2}(\phi) + D^{2*}_{-1,-2}(\phi)], \tag{B7}$$

$$(\nu = 2) \frac{\sin\gamma}{\sqrt{2}} = \cos\bar\gamma D^{2*}_{2,0}(\phi) + \frac{\sin\bar\gamma}{\sqrt{2}}[D^{2*}_{2,2}(\phi) + D^{2*}_{2,-2}(\phi)], \tag{B8}$$

$$(\nu = -2) \frac{\sin\gamma}{\sqrt{2}} = \cos\bar\gamma D^{2*}_{-2,0}(\phi) + \frac{\sin\bar\gamma}{\sqrt{2}}[D^{2*}_{-2,2}(\phi) + D^{2*}_{-2,-2}(\phi)]. \tag{B9}$$

One may verify that for all of the entries in Table IV

$$[D^{2*}_{2,2}(\phi) + D^{2*}_{2,-2}(\phi)] = [D^{2*}_{-2,2}(\phi) + D^{2*}_{-2,-2}(\phi)] \text{ (real)},$$

$$D^{2*}_{2,0}(\phi) = D^{2*}_{-2,0}(\phi) \text{ (real)}$$

and hence Eqs. (B8) and (B9) have identical content. Also

$$D^{2*}_{1,0}(\phi) = D^{2*}_{-1,0}(\phi) = [D^{2*}_{1,2}(\phi) + D^{2*}_{1,-2}(\phi)] = [D^{2*}_{-1,2}(\phi) + D^{2*}_{-1,-2}(\phi)] = 0$$

insure that Eqs. (B6) and (B7) are satisfied. The two independent equations (B5) and (B9) may be

TABLE IV. The octahedral group symmetries of the quadrupole surface expressed in terms of the group generators denoted by R_1, R_2, and R_3. The quantities x, y, z, and γ refer to one principal axis frame, BF, while $\bar{x}, \bar{y}, \bar{z}$, and $\bar{\gamma}$ refer to a second such frame, $\overline{\mathrm{BF}}$. The Euler angles ϕ_1, ϕ_2, and ϕ_3 specify the BF$\rightarrow\overline{\mathrm{BF}}$ rotation.

	$(\bar{x},\bar{y},\bar{z})$	(ϕ_1, ϕ_2, ϕ_3) BF$\rightarrow\overline{\mathrm{BF}}$	$\bar{\gamma}$	Expression in terms of generators
1	(x, y, z)	$(0, 0, 0)$	γ	$R_1^2 = R_2^4 = R_3^3$
2	$(x, z, -y)$	$(-\frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi)$	$-\gamma - \frac{2\pi}{3}$	$R_1R_2R_3^2$
3*	$(x, -y, -z)$	$(-\frac{1}{2}\pi, \pi, \frac{1}{2}\pi)$	γ	R_1
4	$(x, -z, y)$	$(\frac{1}{2}\pi, \frac{1}{2}\pi, -\frac{1}{2}\pi)$	$-\gamma - \frac{2\pi}{3}$	$R_2R_3^2$
5	$(-x, y, -z)$	$(0, \pi, 0)$	γ	$R_1R_2^2$
6	$(-x, z\ \ y)$	$(\frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi)$	$-\gamma - \frac{2\pi}{3}$	$R_2^3R_3^2$
7	$(-x, -y, z)$	$(\pi, 0, 0)$	γ	R_2^2
8	$(-x, -z, -y)$	$(-\frac{1}{2}\pi, \frac{1}{2}\pi, -\frac{1}{2}\pi)$	$-\gamma - \frac{2\pi}{3}$	$R_1R_2^3R_3^2$
9	$(y, x, -z)$	$(-\frac{1}{4}\pi, \pi, \frac{1}{4}\pi)$	$-\gamma$	R_1R_2
10*	(y, z, x)	$(0, \frac{1}{2}\pi, \frac{1}{2}\pi)$	$\gamma - \frac{2\pi}{3}$	R_3
11*	$(y, -x, z)$	$(\frac{1}{2}\pi, 0, 0)$	$-\gamma$	R_2
12	$(y, -z, -x)$	$(\pi, \frac{1}{2}\pi, -\frac{1}{2}\pi)$	$\gamma - \frac{2\pi}{3}$	R_1R_3
13	$(-y, x, z)$	$(0, 0, -\frac{1}{2}\pi)$	$-\gamma$	R_2^3
14	$(-y, z, -x)$	$(\pi, \frac{1}{2}\pi, \frac{1}{2}\pi)$	$\gamma - \frac{2\pi}{3}$	$R_1R_2^2R_3$
15	$(-y, -x, -z)$	$(\frac{1}{4}\pi, \pi, -\frac{1}{4}\pi)$	$-\gamma$	$R_1R_2^3$
16	$(-y, -z, x)$	$(0, \frac{1}{2}\pi, -\frac{1}{2}\pi)$	$\gamma - \frac{2\pi}{3}$	$R_2^2R_3$
17	(z, x, y)	$(\frac{1}{2}\pi, \frac{1}{2}\pi, \pi)$	$\gamma + \frac{2\pi}{3}$	R_3^2
18	$(z, y, -x)$	$(\pi, \frac{1}{2}\pi, \pi)$	$-\gamma + \frac{2\pi}{3}$	$R_1R_2R_3$
19	$(z, -x, -y)$	$(-\frac{1}{2}\pi, \frac{1}{2}\pi, \pi)$	$\gamma + \frac{2\pi}{3}$	$R_1R_3^2$
20	$(z, -y, x)$	$(0, \frac{1}{2}\pi, \pi)$	$-\gamma + \frac{2\pi}{3}$	R_2R_3
21	$(-z, x, -y)$	$(-\frac{1}{2}\pi, \frac{1}{2}\pi, 0)$	$\gamma + \frac{2\pi}{3}$	$R_1R_2^2R_3^2$
22	$(-z, y, x)$	$(0, \frac{1}{2}\pi, 0)$	$-\gamma + \frac{2\pi}{3}$	$R_2^3R_3$
23	$(-z, -x, y)$	$(\frac{1}{2}\pi, \frac{1}{2}\pi, 0)$	$\gamma + \frac{2\pi}{3}$	$R_2^2R_3^2$
24	$(-z, -y, -x)$	$(\pi, \frac{1}{2}\pi, 0)$	$-\gamma + \frac{2\pi}{3}$	$R_1R_2^3R_3$

rewritten as

$$\cos\gamma = \cos\bar{\gamma}[\tfrac{3}{2}\cos^2\phi_2 - \tfrac{1}{2}] + \frac{\sin\bar{\gamma}}{\sqrt{2}}[\tfrac{1}{2}\sqrt{6}\sin^2\phi_2\cos2\phi_3], \tag{B10}$$

$$\frac{\sin\gamma}{\sqrt{2}} = \cos\bar{\gamma}[\tfrac{1}{4}\sqrt{6}\sin^2\phi_2\cos2\phi_1] + \frac{\sin\bar{\gamma}}{\sqrt{2}}\left[\frac{(1+\cos^2\phi_2)}{2}\cos2\phi_3\cos2\phi_1 + \cos\phi_2\cos2\phi_3\sin2\phi_1\right]. \tag{B11}$$

These equations may be solved for $\bar{\gamma}$ in terms of γ for each ϕ of Table IV. Since only $\phi_2 = 0, \pi, \frac{1}{2}\pi$ appears, it is helpful to solve (B10) and (B11) for each of these cases separately:

$$\phi_2 = 0 \begin{cases} \sin\bar{\gamma} = \dfrac{\sin\gamma}{\cos2(\phi_1+\phi_3)}, \\ \cos\bar{\gamma} = \cos\gamma, \end{cases}$$

$$\phi_2 = \pi \begin{cases} \sin\bar{\gamma} = \dfrac{\sin\gamma}{\cos2(\phi_1-\phi_3)}, \\ \cos\bar{\gamma} = \cos\gamma, \end{cases} \tag{B12}$$

$$\phi_2 = \tfrac{1}{2}\pi \begin{cases} \sin\bar{\gamma} = \left[\dfrac{\sqrt{3}}{2\cos2\phi_3}\right]\cos\gamma + \left[\dfrac{1}{2\cos2\phi_1\cos2\phi_3}\right]\sin\gamma, \\ \cos\bar{\gamma} = [-\tfrac{1}{2}]\cos\gamma + \left[\dfrac{\sqrt{3}}{2\cos2\phi_3}\right]\sin\gamma. \end{cases}$$

The $\phi_2 = \frac{1}{2}\pi$ equations in turn imply that

$$\bar{\gamma} = \begin{cases} -\gamma + 2\pi/3, & \cos2\phi_1 = 1, \quad \cos2\phi_3 = 1, \\ \gamma - 2\pi/3, & \cos2\phi_1 = 1, \quad \cos2\phi_3 = -1, \\ \gamma + 2\pi/3, & \cos2\phi_1 = -1, \quad \cos2\phi_3 = 1, \\ -\gamma - 2\pi/3, & \cos2\phi_1 = -1, \quad \cos2\phi_3 = -1. \end{cases} \tag{B13}$$

At this point $\bar{\gamma}$ can be easily computed for each of the entries in the table and the results have been listed. If one defines the operator R by its effect upon an arbitrary function of the collective coordinates

$$Rf(\beta, \gamma, \theta_i) = f(\bar{\beta}, \bar{\gamma}, \bar{\theta}_i) \tag{B14}$$

then Table IV identifies 24 such operators which leave the $\alpha_\mu(\beta, \gamma, \theta_i)$ invariant. Only three of these are independent, however; they have been selected and labeled R_1, R_2, and R_3 in accordance with Bohr,[1] and are indicated by *. All others may be obtained from the successive application of the R as indicated in the table.

Let us now ascertain the consequences of Eq. (B1) which we may write as

$$R_i\Psi(\beta, \gamma, \theta_i) = \Psi(\beta, \bar{\gamma}, \bar{\theta}_i) = \Psi(\beta, \gamma, \theta_i), \quad i = 1, 2, 3. \tag{B15}$$

If we take the separated form of the wave function

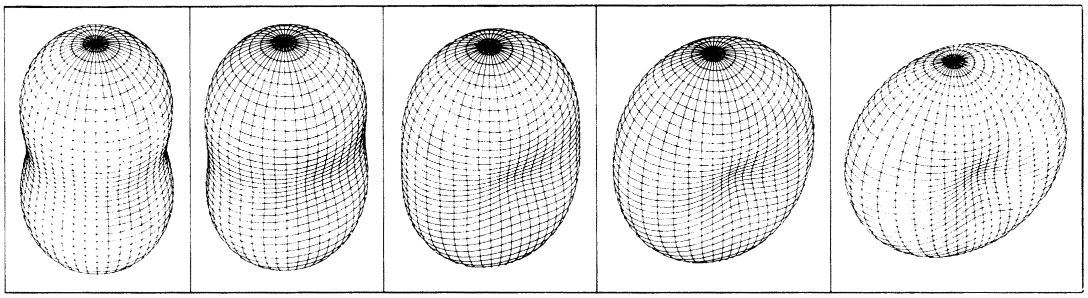

FIG. 4. Effect of increasing γ. Here β is fixed at 0.8 and, from left to right, γ runs through the values 0°, 15°, 30°, 45°, and 60°. Again the viewing point is $(X_1, X_2, X_3) = (1, 2, 2)$. Notice that the $\gamma = 0°$ surface has axial symmetry about the X_3 direction.

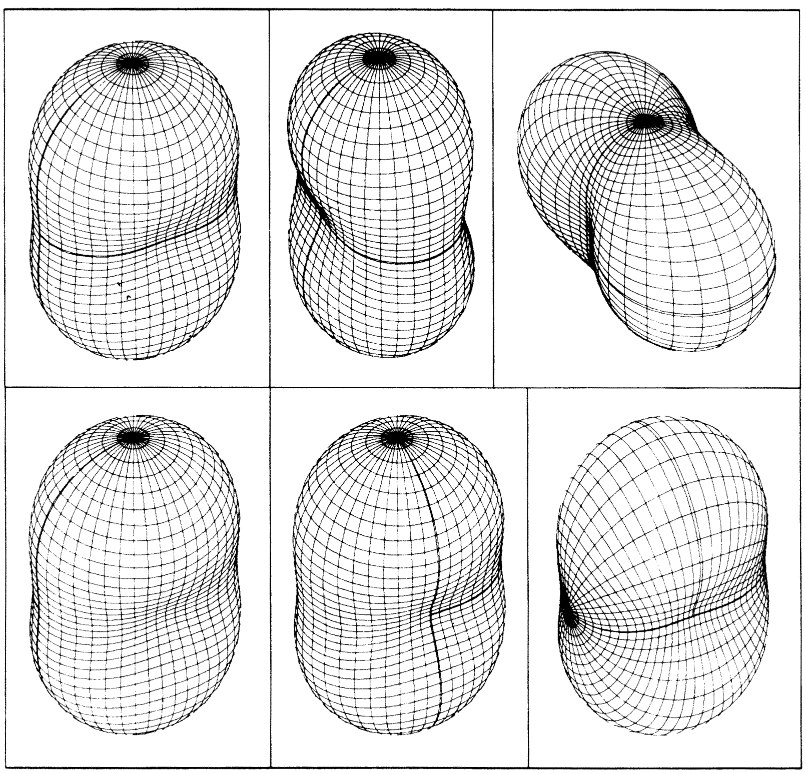

FIG. 5. Illustration of surface symmetries. From left to right, the top row of drawings depicts the surface (β, γ), $(\beta, -\gamma)$, and $(\beta, 120° - \gamma)$ as seen from the BF point $(X_1, X_2, X_3) = (1, 2, 2)$. Here $\beta = 0.9$ and $\gamma = 20°$. The prime meridian and semiequator have been drawn as double lines to aid orientation. Directly below each drawing is a second view of the same surface from a rotated vantage point (X_1', X_2', X_3'). The new vantage point is determined by $\vec{X}' = R_i^{-1}\vec{X}$, where R_i represents one of the three basic $\mathrm{BF} \rightarrow \overline{\mathrm{BF}}$ rotations in Table IV. From left to right R_1, R_2, and R_3 have been applied to obtain $(X_1' X_2' X_3') = (X_1, -X_2, -X_3)$, $(X_2, -X_1, X_3)$, and (X_2, X_3, X_1), respectively. The three surface views of the bottom row can be seen to be identical to the initial view at top left. Thus the surfaces of the top row are not distinct in the sense that they may be rotated into one another. Furthermore, the rotation R_1 takes the top left surface into itself.

described in Sec. III as Eq. (85a)

$$\Psi(\beta,\gamma,\theta_i)=\langle\underline{r}|Nl\nu JM\rangle = \mathfrak{N}_{Nl\nu J}f_{Nl}(\beta)\sum_K g_{l\nu JK}(\gamma)D^{J*}_{M,K}(\theta_i) \quad \text{(B16)}$$

we have from (B4) and (B15)

$$\sum_K g_{l\nu JK}(\gamma)D^{J*}_{M,K}(\theta_i) = \sum_{K'} g_{l\nu JK'}(\bar\gamma)\sum_K D^{J*}_{MK}(\theta_i)D^{J*}_{KK'}(\phi_i) \quad \text{(B17)}$$

implying that

$$g_{l\nu JK}(\gamma)=\sum_{K'} g_{l\nu JK'}(\bar\gamma)D^{J*}_{KK'}(\phi_i). \quad \text{(B18)}$$

For the three R_i, Eq. (B18) becomes

$$R_1:\ g_{l\nu JK}(\gamma) = \sum_{K'} g_{l\nu JK'}(\gamma)D^{J*}_{K,K'}(-\tfrac{1}{2}\pi,\pi,\tfrac{1}{2}\pi) = \sum_{K'} g_{l\nu JK'}(\gamma)[e^{-iK(\pi/2)}(-1)^{J+K}\delta_{K,-K'}e^{iK'(\pi/2)}] = (-1)^J g_{l\nu J-K}(\gamma), \quad \text{(B19)}$$

$$R_2:\ g_{l\nu JK}(\gamma)=\sum_{K'} g_{l\nu JK'}(-\gamma)D^{J*}_{K,K'}(\tfrac{1}{2}\pi,0,0) = \sum_{K'} g_{l\nu JK'}(-\gamma)[e^{i(\pi/2)}\delta_{K,K'}] = e^{iK(\pi/2)}g_{l\nu JK}(-\gamma), \quad \text{(B20)}$$

$$R_3:\ g_{l\nu JK}(\gamma)=\sum_{K'} g_{l\nu JK'}(\gamma-2\pi/3)D^{J*}_{K,K'}(0,\tfrac{1}{2}\pi,\tfrac{1}{2}\pi). \quad \text{(B21)}$$

Notice that the second symmetry applied twice implies that $g_{l\nu JK}(\gamma)$ vanishes if K is odd, a result which arises manifestly in our formalism.

Figure 5 displays the well-known symmetries of the surface which lead to these symmetries of the wave functions. We note that these are the well-known[1] symmetries of the quadrupole vibration problem. There are additional symmetries which manifest themselves as relationships among the A's of Eq. (85c). All these symmetries have their counterparts in the octapole surface vibration problem as well and we intend in a future publication to discuss these symmetries more fully.

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*Work performed for the U. S. Energy Research and Development Administration under Contract No. W-7405-eng-82.

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