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# Pi-nucleon scattering*

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In preparation for a field-theoretic treatment of $\pi$-nucleus scattering, the corresponding $\pi$-nucleon problem is examined. In order to include two-body input in the many-body problem it is useful to have a linear wave equation which describes $\pi$-nucleon scattering. It is shown that the nonlinear Low equation, in the one-meson, one-nucleon truncation, may be expressed as a linear wave equation with an energy-dependent driving term interpretable as a potential. This potential includes the usual driving term of the Low equation plus an infinite set of higher order terms which incorporate the effects of crossing. A discussion of these terms and an iterative prescription for calculating them is given. In the static limit, our solution is shown to be equivalent to that of Chew and Low.

[NUCLEAR REACTIONS Scattering theory, Low equation, $\pi$-nucleon scattering.]

## I. INTRODUCTION

There is a great deal of activity concerned with understanding the interactions of pions with nuclei. Many theoretical treatments use some form of multiple scattering theory[1,2] which gives the $\pi$-nucleus scattering an expansion in elementary off-shell $\pi$-nucleon amplitudes. Derivations of such theories assume the existence of a projectile-nucleon potential. Such an assumption is believed to be valid for nucleon-nucleon interactions. However, the corresponding evidence does not yet exist for meson-nucleon interactions because mesons interact by being singly absorbed by, or emitted from, nucleons. This leads to the possibility that effects, not contained in conventional theories occur.

In general, one would like to obtain a $\pi$-nucleus scattering theory which includes: (1) a justification (or disproof and improvement) of the use of a $\pi$-nucleon potential; (2) effects of meson annihilation reactions, such as $(\pi, p)$ and $(\pi, pp)$, on elastic and inelastic meson-nucleus scattering; (3) crossing symmetry; (4) relativistic effects; (5) consistent use of two-body input in the many-body problem (already in standard theories); (6) recognition that there are virtual mesons present in the nucleus and that they might be confused with the incident meson; and, (7) a clear relationship with the coventional theory so that specific correction terms may be isolated. In our view these are the problems which, in addition to problems in evaluating the standard theories, must be solved to understand meson-nucleus scattering. The first workers to consider such problems were Dover and Lemmer.[3] Since then there has been much interesting work[4-11] which deals with various aspects of the above problems.

One of the main difficulties with approaches to $\pi$-nucleon and $\pi$-nucleus scattering which use the dynamics of single absorption or emission is that one must solve nonlinear Low[12] equations for both systems. The present paper is a study of $\pi$-nucleon scattering in which an energy-dependent $\pi$-nucleon potential is obtained. This potential when used in a linear equation gives the same $\pi$-nucleon $T$ matrix as the Low equation.

The Low equation gives the scattering amplitude in terms of matrix elements of operators between eigenstates of a field-theoretic Hamiltonian. Such eigenstates are very complicated. For example, the wave function of the physical nucleon may be viewed as a bare nucleon plus an interacting pion cloud and even this simplest nucleus is a solution to a many-body problem. In a potential model, for $\pi$-nucleon scattering, one uses a massive point nucleon and massive point pion with the only interaction between them the potential that causes the scattering. One is working in a *different*, but simpler, Hilbert space and if the $T$ matrix for this problem is the same for on-shell and off-shell matrix elements one may be able to use the simpler dynamics to advantage in the many-body problem.

In Sec. II the derivation and properties of the Low equation are reviewed with emphasis on its renormalized and nonstatic nature. The "one-meson" truncation is used. Under this approximation the $\pi$-nucleon $T$ matrix is given as a sum of three terms. The first is the driving term which we interpret as a $\pi$-nucleon potential. The second term is quadratic in the $T$ matrix, arises from interations of the driving term, and includes the right-hand cut. The third term is quadratic in the $\pi$-nucleon $T$ matrix and includes the left-hand cut. This term insures the crossing symmetry of the solution. We call this last term the crossed term. Two approximations which simplify the solution of the equation are discussed. The first is the well-known static approximation in which the mass $m$ of the nucleon is taken to be infinite. The second is a scheme in which terms of order $1/m$ are kept. Both of these approximations lead to similar equations for the $p$-wave $T$ matrices.

In Sec. III the Low equation, with the neglect of the crossed term, is shown to be equivalent to a linear equation with an energy-dependent driving term interpretable as a potential. The proof proceeds by making a transformation on this linear equation which results in another linear equation with an energy-independent but non-Hermitian potential. Then a biorthogonal basis is set up and the derivation of the Low equation from the linear equation proceeds via conventional techniques.

In Sec. IV the crossed term is included. In this case the potential and Green's function depend on the $T$ matrix and the resulting equation, while linear in appearance, is implicitly nonlinear. An iterative procedure to determine these quantities is set up and a physical interpretation of the crossing term is obtained.

In Sec. V an explicit comparison with the Chew-Low[13] solution is obtained and it is found that our solution is essentially identical to theirs. The convergence of our iterative procedure is also discussed.

Section VI contains a summary of our principal results. Derivations of certain equations are given in the Appendix.

## II. LOW EQUATION

In this section the $\pi$-nucleon Low equation is reviewed. The original treatment of Low is followed closely. Emphasis is placed on those aspects, such as the renormalized and nonstatic nature, of Low's treatment which are relevant for present studies of $\pi$-nucleon and $\pi$-nucleus reactions. We start with the interaction Hamiltonian

$$H_I(t) = ig\int \bar{\psi}(x)\gamma_5\vec{\tau}\cdot\vec{\phi}\psi(x)d^3x - \delta m\int\bar{\psi}(x)\psi(x)d^3x - \tfrac{1}{2}(\delta\mu^2)\int\phi^2(x)d^3x + \tfrac{1}{4}\lambda\int\phi^2(x)\phi^2(x)d^3x$$
$$\equiv \int d^3x\,\mathcal{H}(\vec{x},t)\,. \tag{2.1}$$

The first term of Eq. (2.1) is the usual pseudoscalar $\pi$-nucleon interaction. The remaining terms are necessary for the renormalization procedure.

Low uses the formal expression for the $S$ matrix

$$S = \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int dt_1\cdots dt_n\,\langle\Phi_{p'}|a_j(q)P[H_I(t_1)\cdots H_I(t_n)]a_i^\dagger(k)|\Phi_p\rangle \tag{2.2}$$

for the scattering of a meson in a momentum-isospin state $ki$ to a meson in momentum-isospin state $qj$, while the nucleon goes from $p$ to $p'$. In Eq. (2.2) $\Phi_p$ and $\Phi_{p'}$ are noninteracting nucleon wave functions. The indices $p$ and $p'$ represent the charge and spin of the nucleon as well as its four-momentum.

Low obtains a nonperturbative equation from Eq. (2.2) by commuting $a_i^\dagger(k)$ to the left and $a_j(q)$ to the right. By using an expression derived by Gell-Mann and Low[14] and transforming the operators to the Heisenberg representation Low obtains $(q\neq k)$

$$S = -i\int d^4x d^4y\langle\Psi_{p'}|P[\hat{J}^\dagger_{qj}(y)\hat{J}_{ki}(x)]|\Psi_p\rangle + i\lambda\int d^4x\frac{e^{i(k-q)\cdot x}}{(4q_0q_0')^{1/2}}\langle\Psi_{p'}|[\delta_{ij}\hat{\phi}^2(x)+2i\hat{\phi}_i(x)\hat{\phi}_j(x)]|\Psi_p\rangle\,, \tag{2.3}$$

where $\Psi_p$ are full physical nucleons and the caret designates the Heisenberg representation. The currents are defined by

$$\hat{J}_{q,i}(\vec{x}) = [\mathcal{H}_I(x), a_i^\dagger(q)]_{x_0=0} = \frac{e^{i\vec{q}\cdot\vec{x}}}{(2q_0)^{1/2}}\,\mathcal{O}_i(\vec{x},x_0=0) = \frac{e^{i\vec{q}\cdot\vec{x}}}{(2q_0)^{1/2}}(\Box+\mu^2)\phi_i(x)\,|_{x_0=0}\,. \tag{2.4}$$

Equation (2.3) obeys the Gell-Mann–Goldberger[15] crossing relation

$$\langle\vec{p}\,',\vec{q}j|S|\vec{p},\vec{k}i\rangle = \langle\vec{p}\,',-\vec{k}i|S|\vec{p},-\vec{q}j\rangle\,. \tag{2.5}$$

It is convenient to define a scattering operator $T$

$$T(p'qj,pki) = \int d^4x\langle\Psi_{p'}^{(-)}(qj)|\hat{J}_{ki}(x)|\Psi_p\rangle e^{-ik_0x_0}\,, \tag{2.6}$$

where $\Psi_{p'}^{(-)}(qj)$ is an incoming wave scattering eigenstate with an asymptotic part consisting of a physical nucleon of momentum $p'$ and a meson in a momentum-isospin state $qj$. Low showed that by writing $\Psi_{p'}^{(-)}$ in terms of the time development operator acting on a noninteracting single-meson, single-nucleon state one could obtain

$$T(p'qj,pki)=iS(p'qj,pki) \tag{2.7}$$

for $qj \neq ki$.

It is now possible to define the conventional $T$ matrix $t$ via

$$S(p'qj,pki)=-2\pi i\,\delta(E_i-E_f)t(p'qj,pki)\,,$$
$$t(p'qj,pki)=\int d^3x\langle\Psi_{p'}(qj)|J_{ki}(\vec{x})|\Psi_p\rangle\,, \tag{2.8}$$

where $q\neq k$ and $E_i=k_0+p_0$, $E_f=q_0+p_0'$. The quantity $t$ is related to $T$ by performing the time integral of Eq. (2.6) and dividing by $2\pi$.

The two terms of Eq. (2.3) are separately divergent. However, Low uses the $\lambda$ term to cancel the divergence arising from the four-meson subgraph. Thus if the $\lambda$ term is dropped and renormalized coupling constants, vertex functions, and masses are used in the remaining terms, no divergences appear and the resulting equations are finite and physical. The finite pieces of the $\phi^2$ and $\phi^4$ terms which remain after renormalization are neglected in this treatment.

Low showed that a very elegant and simple-looking expression could be obtained by rewriting the operators $J$ in the Schrödinger representation and by explicitly performing the integrals over $x_0$ and $y_0$. Then

$$t(p'qj,pki)=\sum_n \frac{\langle\Psi_{p'}|J_{qj}^\dagger|\Psi_n^{(-)}\rangle\langle\Psi_n^{(-)}|J_{ki}|\Psi_p\rangle}{k_0+p_0-E_n+i\epsilon}+\sum_n \frac{\langle\Psi_{p'}|J_{ki}|\Psi_n^{(-)}\rangle\langle\Psi_n^{(-)}|J_{qj}^\dagger|\Psi_p\rangle}{p_0'-k_0-E_n+i\epsilon}\,, \tag{2.9}$$

where the matrix elements involve three-dimensional space integrals only. Each of the matrix elements contains an implicit momentum conserving $\delta$ function. The states $|\Psi_n^{(-)}\rangle$ are eigenstates of the *full* Hamiltonian with incoming boundary conditions. All quantities of (2.9) are in the Schrödinger representation.

The sum over intermediate states includes states which have asymptotically any number of pions plus the nucleon. The $T$ matrix for a process in which mesons are created and destroyed may be obtained by reducing the corresponding $S$-matrix element. The result is a set of coupled channel equations. In the present work we consider only the sum over nucleon and single-meson nucleon states. Inelastic channels are important[10] and will be included in a forthcoming paper by this author.

A more explicit form of Eq. (2.9) may be obtained. Consider the sum over single-nucleon states as a driving term and define it as $v$. The use of

$$\langle\Psi_{p'}|J_{qj}^\dagger|\Psi_n\rangle=\frac{\delta^{(3)}(\vec{p}_n-\vec{p}\,'-\vec{q})}{(2q_0)^{1/2}}\times\langle\Psi_{p'}|\mathcal{O}_j(0)|\Psi_{p'+q}\rangle\,, \tag{2.10}$$

where Hermitian isospin operators are used, gives

$$v(p'qj,pki)=\sum_{\sigma\tau}\frac{1}{(4k_0q_0)^{1/2}}\left(\frac{\langle\Psi_{\vec{p}\,'}|\mathcal{O}_j(0)|\Psi_{\vec{p}\,'+\vec{q}}(\sigma\tau)\rangle\langle\Psi_{\vec{p}\,'+\vec{q}}(\sigma\tau)|\mathcal{O}_i(0)|\Psi_p\rangle}{k_0+p_0-E(\vec{p}\,'+\vec{q})+i\epsilon}+\frac{\langle\Psi_{\vec{p}\,'}|\mathcal{O}_i(0)|\Psi_{\vec{p}-\vec{k}}(\sigma\tau)\rangle\langle\Psi_{\vec{p}-\vec{k}}(\sigma\tau)|\mathcal{O}_j(0)|\Psi_p\rangle}{p_0'-k_0-E(\vec{p}\,'-\vec{k})+i\epsilon}\right). \tag{2.11}$$

The term involving the one-meson, one-nucleon states is simplified by noting that the matrix elements occurring on the left-hand side of Eq. (2.9) are precisely the same as those on the right-hand side. Thus

$$t(p'qj,pki)=v(p'qj,pki)+\sum_l\int\frac{d^3q_n}{(2\pi)^3}\frac{t^\dagger(\vec{p}\,'+\vec{q}-\vec{q}_n,\vec{q}_n l;\vec{p}\,',\vec{q}j)t(\vec{p}\,'+\vec{q}-\vec{q}_n,\vec{q},l;\vec{p},\vec{k}i)}{k_0+p_0-E(\vec{p}\,'+\vec{q}-\vec{q}_n)-\omega(\vec{q}_n)+i\epsilon}+\sum_l\int\frac{d^3q_n}{(2\pi)^3}\frac{t^\dagger(\vec{p}\,'-\vec{k}-\vec{q}_n,\vec{q}_n l;\vec{p},-\vec{k}j)t(\vec{p}\,'-\vec{k}-\vec{q}_n,\vec{q}_n l;\vec{p},-\vec{q}j)}{p_0'-k_0-E(\vec{p}\,'-\vec{k}-\vec{q}_n)-\omega(q_n)+i\epsilon}\,. \tag{2.12}$$

The potential $v$ is specified by the calculation of the matrix elements contained in Eq. (2.11). A theoretical calculation of such matrix elements is implied by our choice of interaction Hamiltonian (2.1). However, we do not address ourselves to this task. Explicit assumptions about Eq. (2.10) are discussed below.

In order to simplify the notation a symbolic crossing operator $C$ is introduced, i.e.

$$C[F(a,qj;b,ki)]=F(a,-ki;b,-qj)\,, \tag{2.13}$$

where $F$ is any function of incoming and outgoing meson lines, and $a,b$ represent any other coordinates of the problem. Application of $C$ is equivalent to crossing the *external* meson lines in Feynman diagrams. The Eq. (2.13) is a relation between four vectors.

From Eqs. (2.11) and (2.12)

$$\begin{aligned} v&=C[v]\,,\\ t&=C[t]\end{aligned} \tag{2.14}$$

so that Eq. (2.12) may be written as

$$t=v+t^{\dagger}Dt+C[t^{\dagger}Dt]\,, \tag{2.15}$$

where $D$ is the meson-nucleon propagator including the physical spectra and a schematic notation is used.

Given a model for the matrix elements of Eq. (2.10), the solution of Eq. (2.12) would result in a $\pi$-nucleon $T$ matrix which is crossing-symmetric and which includes nucleon recoil. Such a solution is difficult to obtain, and it has become customary to use various approximations. The static or no-recoil approximation involves the assumption that the mass of the nucleon is so large that the nucleon's kinetic energy may be neglected and the $\pi$-nucleon relative momenta is given by the momentum of the pion in the frame in which the nucleon is at rest. In this case the energies of the initial and final pion are equal and designated by the variable $z$, i.e., $k_0=q_0=z$. The energy denominators in the Eq. (2.11) for the potential are given by

$$\begin{aligned} &k_0+p_0-E(\vec{p}\,'+\vec{q})+i\epsilon\approx z+i\epsilon\,,\\ &p_0'-k_0-E(\vec{p}\,'-\vec{k})+i\epsilon\approx -z+i\epsilon\end{aligned} \tag{2.16}$$

and the energy denominators of Eq. (2.12) are given by

$$\begin{aligned} &k_0+p_0-E(\vec{p}\,'+\vec{q}-\vec{q}_n)-\omega(q_n)+i\epsilon\approx z-\omega(q_n)+i\epsilon\,,\\ &p_0'-k_0-E(\vec{p}\,'-\vec{k}-\vec{q}_n)-\omega(q_n)+i\epsilon\approx -z-\omega(q_n)+i\epsilon\,.\end{aligned} \tag{2.17}$$

The neglect of the momentum of the nucleon in the expression for the matrix elements of the current operator results in the simplified expression

$$\langle\Psi_{p'}\,|J^{\dagger}_{\vec{q}j}\,|\Psi_{p'+q}\rangle=\frac{\vec{\sigma}\cdot\vec{q}}{(2\omega_q)^{1/2}}\,\tau_j f(q)\,\frac{\sqrt{4\pi}\,f}{\mu}\,, \tag{2.18}$$

where $\sqrt{4\pi}\,f/\mu=g/2m$ and $f(q)$ is an assumed form factor. An evaluation of the right-hand side of Eq. (2.18) between nucleon spin and isospin states is implied. Because the potential is constructed from these matrix elements, one finds

$$\begin{aligned} v(p'qj,pki)\equiv v_z(\vec{q}j,\vec{k}i)=\frac{f(q)f(k)}{(4\omega_q\omega_k)^{1/2}}\,\frac{4\pi f^2}{\mu^2}\Bigg(&\sum_{\sigma''\tau''}\frac{\langle\Psi_{\sigma\tau}\,|\vec{\sigma}\cdot\vec{q}\tau_j\,|\Psi_{\sigma''\tau''}\rangle\langle\Psi_{\sigma''\tau''}\,|\vec{\sigma}\cdot\vec{k}\tau_i\,|\Psi_{\sigma'\tau'}\rangle}{z}\\ &-\sum_{\sigma''\tau''}\frac{\langle\Psi_{\sigma\tau}\,|\vec{\sigma}\cdot\vec{k}\tau_i\,|\Psi_{\sigma''\tau''}\rangle\langle\Psi_{\sigma''\tau''}\,|\vec{\sigma}\cdot\vec{k}\tau_j|\,\Psi_{\sigma'\tau'}\rangle}{z}\Bigg).\end{aligned} \tag{2.19}$$

In Eq. (2.19) the spin and isospin of the initial and final nucleon are specifically indicated. In the following treatment the quantities $v$ and $t$ are to be understood as acting between nucleon states with such quantum numbers, but the labels $\sigma\tau$ are not specifically given.

The form of the driving term means that the solution is of the form

$$t(p'qj,pki)=t_z(\vec{q}j,\vec{k}i)\,. \tag{2.20}$$

The use of Eqs. (2.16)–(2.20) allows us to write the static Low equation as

$$t_z(\vec{q}j,\vec{k}i)=v_z(\vec{q}j,\vec{k}i)+\sum_l\int\frac{d^3p}{(2\pi)^3}\,\frac{t^{\dagger}_{\omega_p}(\vec{p}l,\vec{q}j)t_{\omega_p}(\vec{p}l,\vec{k}i)}{z-\omega_p+i\epsilon}+\sum_l\int\frac{d^3p}{(2\pi)^3}\,\frac{t^{\dagger}_{\omega_p}(\vec{p}l,-\vec{k}i)t(\vec{p}l,-\vec{q}j)}{-z-\omega_p} \tag{2.21}$$

which may be given in a schematic form which is identical to Eq. (2.15) except for the replacement of the full crossing operator $C$ by its static form $C_s$:

$$C_s[F_z(a,\vec{q}j;b,\vec{k}i)]=F_{-z}(a,-\vec{k}i;b,-\vec{q}j)\,. \tag{2.22}$$

Under Eqs. (2.19)–(2.22) it is explicitly true that

$$\begin{aligned} C_s[v_z]&=v_z\,,\\ C_s[t_z]&=t_z\,.\end{aligned} \tag{2.23}$$

However, one wishes to use the Low equation at energies up to and including the (3, 3) resonance which occurs at a lab energy of about 200 MeV. For this energy the lab momentum is 310 MeV/$c$ while the $\pi$-nucleon relative momentum is 234 MeV/$c$. Furthermore the center-of-mass kinetic energy of the target nucleon is about 30 MeV. In this energy region the $\pi$-nucleon $T$ matrix varies rapidly and one could make a substantial error by evaluating it at an energy which is wrong by the amount. It is therefore desirable to include nucleon recoil in some way. We keep terms of order $1/m$ in Eq. (2.12). Our most crucial assumption is about how to treat the matrix elements of the current operator between single-nucleon eigenstates. We work in the $\pi$-nucleon center of mass and *assume* that such matrix elements are well approximated by

$$\langle \Psi_p | J^\dagger_{\vec{q}_c, i} | \Psi_{p+q_c} \rangle = \frac{\vec{\sigma} \cdot \vec{q}_c}{(2\omega_{q_c})^{1/2}} \tau_i f(q_c) \sqrt{4\pi} \frac{f}{\mu} , \tag{2.24}$$

where $q_c$ is the relative $\pi$-nucleon momentum calculated to order $1/m$. Once this assumption has been made, one may further examine the energy denominators of Eqs. (2.11) and (2.12) to obtain expressions for the potential and $T$ matrix. Along with Eq. (2.24) it is desirable to work in the center-of-mass frame and to treat $z$ as the total pion-nucleon energy (minus the nucleon mass) in that frame.

The assumption (2.24) has powerful implications for the solution of the nonstatic Low equation. In particular it means that the *numerator* of the expression Eq. (2.11) for $v$ occurs for $\pi$-nucleon scattering in relative $p$ waves only. Thus the difficulties of large $s$-wave scattering caused by pair production, for example, has been thrown out.

Let us examine the energy denominators in the expression (2.11) for the potential more carefully. In the center-of-mass frame the total momentum is zero and the denominator of the first term is $z$. The denominator in the second term is more complicated. One has

$$p_0' - k_0 - E(\vec{p}' - \vec{k}) \approx \frac{\vec{p}'^2}{2m} - k_0 - \frac{(\vec{p}' - \vec{k})^2}{2m}$$
$$= -z + \frac{\vec{p}' \cdot \vec{k}}{m} \tag{2.25}$$

which differs from the corresponding static term of Eq. (2.16) by a term of order $1/m$. In (2.25) all momenta are evaluated in the center of mass and we have used the fact that $|\vec{p}| = |\vec{k}|$ in that frame. We now make a further restriction and consider $\pi$-nucleon scattering which occurs only in $p$ waves. That means that to order $1/m$ the $\vec{p}' \cdot \vec{k}/m$ term of Eq. (2.24) may be ignored. Thus the potential under the assumption of Eq. (2.24) and the restriction to $p$ waves has the same form as Eq. (2.19) except that all momenta are replaced by their relative values and $z$ is calculated as discussed in the above paragraph. This simplification occurs as a direct result of using Eq. (2.24).

An examination of the denominators of Eq. (2.12) reveals that the first energy denominator undergoes the replacement

$$k_0 + p_0 - E(\vec{p}' + \vec{q} - \vec{q}_n) - \omega(q_n) + i\epsilon$$
$$= z - \frac{q_n^2}{2m} - \omega(q_n) + i\epsilon . \tag{2.26}$$

One see that the cut occurs for the expected energies. The energy denominator in the crossed term of Eq. (2.12) involves

$$p_0' - k_0 - E(\vec{p}' - \vec{k} - \vec{q}_n) - \omega(q_n)$$
$$= -z - \omega(q_n) - \frac{q_n^2}{2m} + \frac{1}{m} (\vec{k} \cdot \vec{p} - \vec{k} \cdot \vec{q}_n + \vec{p}' \cdot \vec{q}_n) . \tag{2.27}$$

Under our approximation the driving term involves $p$-wave scattering only; hence, the physically reasonable soultions of the nonlinear equation are expected to have scattering in the $p$ wave only. This means that to order $1/m$ the fourth term on the right-hand side of Eq. (2.27) may be ignored. Under the conditions expressed by Eqs. (2.24)–(2.27) the Low equation is given by

$$t_z(\vec{q}j, \vec{k}i) = v_z(\vec{q}j, \vec{k}i) + \sum_l \int \frac{d^3q_n}{(2\pi)^3} \frac{t^\dagger(\vec{q}_n l; \vec{q}j) t(\vec{q}_n l, \vec{k}i)}{z - \omega(q_n) - q_n^2/2m + i\epsilon} + \sum_l \int \frac{d^3q_n}{(2\pi)^3} \frac{t^\dagger(\vec{q}_n l, \vec{k}i) t(\vec{q}_n l, \vec{q}j)}{-z - \omega(q_n) - q_n^2/2m} . \tag{2.28}$$

The result, Eq. (2.28) is similar in form to the static equation. The only differences are in the value of $z$, the use of relative momenta, and inclusion of the recoil kinematic energy of the nucleon in the intermediate state spectrum. One may write

$$t_z = v_z + t^\dagger D_z t + C[t^\dagger D_z t] , \tag{2.29}$$

where the crossing operator is given by

$$C[F_z(a\vec{q}_c i; b\vec{k}_c j)] = F_{-z}(a, -\vec{k}_c j; b - \vec{q}_c i) , \tag{2.30}$$

where the evaluation of the momenta in the center-of-mass frame is made explicitly by the subscript $c$.

These results follow directly from our assumption about the form of the current operator as well as our focus on $p$-wave scattering. The results seem reasonable in that one evaluates the on-shell $T$ matrix at the correct energy and momenta. Furthermore our procedure is very similar to the results of Chew, Goldberger, Low, and Nambu[16] and several other authors.[10] Note that the value of the momentum in the center of mass depends on the energy of the meson in the laboratory frame. In this prescription it is $z$ which changes sign under the crossing operator and not the lab energies of the meson.

We have ignored partial waves with $l \neq 1$. In theories which include recoil, crossing symmetry gives relations between the various partial waves. As dynamics, such as $\rho$ exchange, which contribute strongly to $s$-wave scattering have been ignored, we do not consider $s$-wave scattering. A treatment which addresses itself to such questions is found in Ref. 17.

Before proceeding it is necessary to make some observations and definitions. The $T$ matrices on the right-hand side of Eq. (2.28) are evaluated for half-shell kinematics because the energy of the intermediate state is not equal to the energy of the initial state. Thus the Eq. (2.28) defines the half-off-shell $T$ matrix

$$t_z(\vec{q}_c l;\vec{k}_c i) = \langle \Psi^{(-)}(\vec{q}_c l) \,|\, J_{ki} \,|\, \Psi\rangle \equiv t_{W_{q_c}}(\vec{q}_c l; ki) \quad (z = W_{q_c}) , \tag{2.31}$$

$$W_{q_c} = (q_c{}^2 + \mu^2)^{1/2} + q_c{}^2/2m . \tag{2.32}$$

The Low equation is specifically derived for the on-shell elements of the $T$ matrix. However, the equation (2.28) serves a definition of the off-shell $T$ matrix when the variable $z$ is chosen at values which are not associated with either the incident of final pion momentum.

### III. EQUIVALENT LINEAR EQUATION (NO CROSSING)

In the preceding section we have obtained a nonlinear equation which approximately includes nucleon recoil. Rather than attempting a direct solution of Eq. (2.28), we seek an equivalent problem. The wave functions $\Psi_p, \Psi_n^{(-)}$ of the preceding section are exceedingly complicated objects as they consist of superpositions of $n$-meson creation operators acting on the bare nucleon. Instead we work in a Hilbert space consisting of a massive point nucleon and massive point pion which interact via a potential. The idea is to find the potential, which when used in a suitable linear equation, gives a $T$ matrix which is the same for all momenta and energies as $t_z(\vec{q}_c i, \vec{k}_c j)$.

It is easiest to proceed by first ignoring the crossed term of Eq. (2.28) and to find a linear equation equivalent to the approximate Low equation

$$t_z(\vec{q}i,\vec{k}j) = \frac{1}{z}\, u(\vec{q}i,\vec{k}j) + \sum_l \int \frac{d^3p}{(2\pi)^3} \frac{t^\dagger_{W_p}(\vec{p}l,\vec{q}i)\, t_{W_p}(\vec{p}l,\vec{k}j)}{z - W_p + i\epsilon} , \tag{3.1}$$

where all momenta are evaluated in the center of mass and only $p$ waves are included. The full driving term which includes the crossed Born term is used in Eq. (3.1). One may write the channel decomposition

$$u(\vec{q}i,\vec{k}j) = \sum_\alpha \bar{u}_\alpha(\vec{q}i,\vec{k}_c j) P_\alpha(\vec{q}i,\vec{k}_c j) , \tag{3.1a}$$

where $\alpha$ are the spin-isospin channels and $P_\alpha$ the corresponding projection operators, and $\bar{u}_\alpha P_\alpha$ is assumed to be Hermitian. The driving term $u/z$ is completely specified by Eqs. (2.24) and (2.19). However, we are able to prove the assertions of this section for the slightly more general form of Eq. (3.1a).

In the remainder of this paper we allow $z$ to include an infinitesimal positive imaginary part. Thus the $i\epsilon$ term will no longer be specifically indicated.

It is also useful to use a simplified notation in which the index $k$ includes the vector $\vec{k}$ and isospin index $q$. An integral over $k$ then includes the sum over isospin. Then (3.1) may be given as

$$t_z(q,k) = \frac{u(q,k)}{z} + \int \frac{d^3p}{(2\pi)^3} \frac{t^\dagger_{W_p}(p,q)\, t_{W_p}(p,k)}{z - W_p} . \tag{3.1b}$$

It is our claim that a $T$ matrix equivalent to that of Eq. (3.1) is obtained by the solution of the following linear equation

$$t_z(q,k) = \frac{1}{z}\, u(q,k) + \int \frac{d^3p}{(2\pi)^3} \frac{u(q,p)}{z} \left(\frac{z}{W_p}\right)^2 \times \frac{1}{z - W_p}\, t_z(p,k) , \tag{3.2}$$

which is given in operator notation as

$$t_z = \frac{u}{z} + \frac{u}{z}\frac{z}{h_0} \frac{1}{z - h_0 + i\epsilon} \frac{z}{h_0}\, t_z , \tag{3.3}$$

where $h_0$ includes the kinetic energy of the nucleon and total energy of the pion. The linear equation (3.3) is equivalent to the Lippmann-Schwinger equation

$$t_z = v_1(z)\left(1 + \frac{1}{z - h_0}\, t_z\right) , \tag{3.4}$$

where $v_1(z)$ is obtained by using Eq. (3.3) in Eq. (3.4):

$$v_1(z) = \frac{u}{z}\left[1 + \frac{z + h_0}{h_0{}^2}\, v_1(z)\right] . \tag{3.5}$$

Equations (3.2)–(3.5) represent the situation of a massive point nucleon and a massive point pion interacting via a potential. The quantity $u/z$ is viewed as the value of a matrix element of an operator taken between plane wave states. The value of this matrix element is given by the value of the field-theoretic driving term which is of the form $u/z$. The next step is to show that the $t_z$ of Eq. (3.3) is the same as the $t_z$ of Eq. (3.1).

We prove our contention by starting from Eq. (3.3) and showing that the $T$ matrix defined there obeys Eq. (3.1). The first step is to define an auxiliary $T$ matrix $T_z$:

$$T_z = \frac{z}{h_0} t_z\,, \quad T_z(\vec{p}i, \vec{q}j) = \frac{z}{W_p}\, t_z(\vec{p}i, \vec{q}j)\,. \tag{3.6}$$

Note that if $z = W_p + i\epsilon$, as is the case for the matrix elements of Eq. (3.1), $T_z = t_z$. One has

$$T_z = \frac{1}{h_0} u + \frac{1}{h_0} u \frac{1}{h_0} \frac{z}{z-h_0} T_z\,. \tag{3.7}$$

It is useful to define an auxiliary potential $\tilde{u}$

$$T_z = \tilde{u} + \tilde{u} \frac{1}{z-h_0} T_z \tag{3.8}$$

which may be determined by using the $T_z$ of Eq. (3.7) in Eq. (3.8). Then one finds

$$\tilde{u} = \frac{1}{h_0} u + \frac{1}{h_0} u \frac{1}{h_0} \tilde{u} \tag{3.9}$$

or

$$\begin{aligned}\tilde{u} &= \frac{1}{h_0} u \left[ 1 \Big/ \left(1 - \frac{1}{h_0}\frac{1}{h_0} u\right)\right] \\ &= \left[1 \Big/ \left(1 - \frac{1}{h_0} u \frac{1}{h_0}\right)\right] \frac{1}{h_0} u\,. \end{aligned} \tag{3.10}$$

Thus $\tilde{u}$ is an energy-*in*dependent but non-Hermitian potential.

Low equations for non-Hermitian potentials may be derived by introducing the basis wave functions $|\psi_k\rangle$ and $|\overline{\psi}_k\rangle$ where

$$\begin{aligned}(W_k - h_0 - \tilde{u})\,|\psi_k\rangle &= 0\,, \\ (W_k - h_0 - \tilde{u}^\dagger)\,|\overline{\psi}_k\rangle &= 0\,. \end{aligned} \tag{3.11}$$

We assume that the scattering solutions are complete

$$\int \frac{d^3k}{(2\pi)^3}\, |\psi_k^{(-)}\rangle\langle\overline{\psi}_k^{(-)}| = 1\,, \tag{3.12}$$

where incoming boundary conditions are used and where the index $k$ includes the isospin state and the integral over $k$ includes a sum over such states. We also have the relations

$$\overline{T}_z = \tilde{u}^\dagger + \tilde{u}^\dagger \frac{1}{z-h_0} \overline{T}_z \tag{3.13}$$

and

$$\begin{aligned}T_{W_p}(\vec{p}, \vec{k}) &= \langle \overline{\psi}_p^{(-)} |\tilde{u}|\phi_k\rangle\,, \\ \overline{T}_{W_p}(\vec{p}, \vec{k}) &= \langle \psi_p^{(-)} |\tilde{u}^\dagger|\phi_k\rangle\,, \end{aligned} \tag{3.14}$$

where $|\phi_k\rangle$ is an eigenstate of $h_0$.

By defining a complete Green's function

$$G^{-1}(z) = z - h_0 - \tilde{u}\,, \tag{3.15}$$

one may rewrite Eq. (3.8) as

$$T_z = \tilde{u} + \tilde{u} G(z) \tilde{u}\,. \tag{3.16}$$

By using the completeness relationship (3.12) we have

$$T_z = \tilde{u} + \int \frac{d^3p}{(2\pi)^3} \frac{\tilde{u}|\psi_p^{(-)}\rangle\langle\overline{\psi}_p^{(-)}|}{z - W_p} \tilde{u} \tag{3.17}$$

or

$$T_z(q,k) = \tilde{u} + \int \frac{d^3p}{(2\pi)^3} \frac{\overline{T}^\dagger_{W_p}(p,q) T_{W_p}(p,k)}{z - W_p}\,. \tag{3.18}$$

In general there is no simple relationship between $\overline{T}$ and $T$ so that Eq. (3.18) cannot be further refined. However, for the potentials under consideration one may show that

$$\overline{T}_z = h_0 T_z \frac{1}{h_0}\,. \tag{3.19}$$

To prove Eq. (3.19) consider the relationship between $\tilde{u}$ and $\tilde{u}^\dagger$.

By taking the adjoint of Eq. (3.9) we have

$$\tilde{u}^\dagger = u \frac{1}{h_0} \left[ 1 \Big/ \left(1 - \frac{1}{h_0} u \frac{1}{h_0}\right)\right]\,. \tag{3.20}$$

By multiplying Eq. (3.9) by $h_0$ from the left and $1/h_0$ from the right, we have

$$h_0 \tilde{u} \frac{1}{h_0} = u \frac{1}{h_0} + u \frac{1}{h_0} \frac{1}{h_0} h_0 \tilde{u} \frac{1}{h_0} \tag{3.21}$$

which may be solved for

$$h_0 \tilde{u} \frac{1}{h_0} = u \frac{1}{h_0} \left[1 \Big/ \left(1 - \frac{1}{h_0} u \frac{1}{h_0}\right)\right] = \tilde{u}^\dagger\,. \tag{3.22}$$

From a similar procedure on Eq. (3.8)

$$\begin{aligned}h_0 T_z \frac{1}{h_0} &= h_0 \tilde{u} \frac{1}{h_0} \left(1 + \frac{1}{z-h_0}\right) h_0 T_z \frac{1}{h_0} \\ &= \tilde{u}^\dagger \left(1 + \frac{1}{z-h_0} h_0 T_z \frac{1}{h_0}\right)\,, \end{aligned} \tag{3.23}$$

so that $h_0 T(1/h_0)$ satisfies the same equation as $\overline{T}$ and the proof of (3.19) is complete.

The use of (3.19) in (3.18) gives

$$T_z(q,k) = \frac{\tilde{u}(q,k)}{z} + \int \frac{d^3p}{(2\pi)^3} \frac{W_p}{W_q} \frac{T^\dagger_{W_p}(p,q) T_{W_p}(p,k)}{z - W_p} \tag{3.24}$$

which becomes

$$T_z(q,k) = \frac{1}{h_0}\, u(q,k) + \Delta + \int \frac{d^3p}{(2\pi)^3}\, \frac{1}{W_q}\, z\, \frac{1}{z - W_p} \times T^{\dagger}_{W_p}(p,q)\, T_{W_p}(p,k) \tag{3.25}$$

when one uses

$$\frac{W_p}{z - W_p} = -1 + \frac{z}{W_p}\, \frac{1}{z - W_p}$$

and Eq. (3.9). The quantity $\Delta$ is given by

$$\Delta = \frac{1}{h_0}\, u\, \frac{1}{h_0} - \int \frac{d^3p}{W_q}\, T^{\dagger}_{W_p}(p,q)\, T_{W_p}(p,k) \tag{3.26}$$

or

$$\Delta = \frac{1}{h_0}\, u\, \frac{1}{h_0}\, \tilde{u} + \lim_{z\to 0}\, (T_z - \tilde{u})\,. \tag{3.27}$$

But

$$\lim_{z\to 0} T_z = \frac{1}{1 + \tilde{u}(1/h_0)}\, \tilde{u} = \tilde{u} - \tilde{u}\, \frac{1}{h_0}\, \frac{1}{1 + \tilde{u}(1/h_0)}\, \tilde{u}\,; \tag{3.28}$$

hence

$$\Delta = \left( \frac{1}{h_0}\, u\, \frac{1}{h_0} - \tilde{u}\, \frac{1}{h_0}\, \frac{1}{1 + \tilde{u}(1/h_0)} \right) \tilde{u}\,. \tag{3.29}$$

By solving Eq. (3.9) for $u(1/h_0)$

$$u\, \frac{1}{h_0} = h_0\, \tilde{u}\, \frac{1}{1 + \tilde{u}(1/h_0)}\,, \tag{3.30}$$

and $\Delta = 0$.

Upon using Eq. (3.6) in (3.25) we find

$$t_z(q,k) = \frac{u}{z}\,(q,k) + \int \frac{d^3p}{(2\pi)^3}\, \frac{t^{\dagger}_{W_p}(p,q)\, t_{W_p}(p,k)}{z - W_p} \tag{3.31}$$

which proves our assertion that the solutions of Eqs. (3.1) and (3.2) are equivalent.

Note that the only assumption about the form of the driving term is that it is given by a factor of $1/z$ times a Hermitian, energy-independent operator.

The physical content of Eq. (3.2) is examined by defining an equivalent diagrammatic expansion. The driving term is given and $v = u/z$ is defined in

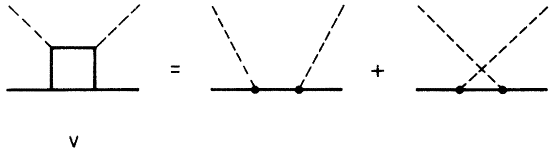


FIG. 1. The driving term $v$. The solid line represents nucleons and the dashed line represents pions.

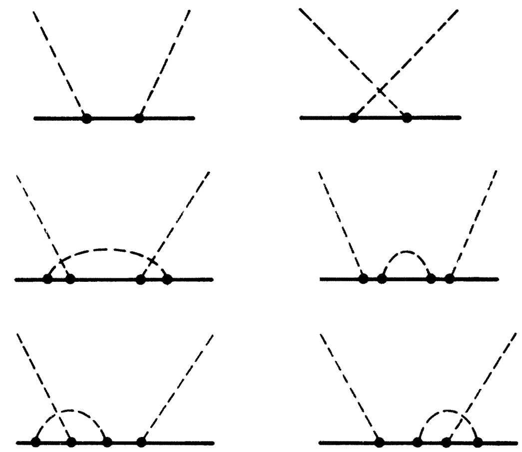

FIG. 2. The diagrammatic expansion of Eq. (3.2) up to fourth order in $g$.

Fig. 1. Some higher-order terms are given in Fig. 2. The intermediate Green's function contains the factor $(z/h_0)^2$. Some comments are necessary to explain the seeming occurrence in Fig. 2 of nucleon mass renormalization and meson nuclear vertex corrections. Each vertex is renormalized and the renormalized physical coupling constant, form factors, and physical nucleons are used. The "unusual" graphs of Fig. 2 are simply finite pieces of those Feynman graphs which remain after the infinite divergences have been removed.

As an example consider the Feynman diagram of Fig. 3(a) which is designed as the divergent self-energy $\Sigma(p)$ which may be given as expansion in powers $\not{p} - m$. Then as explained in Lurie,[18] for example

$$\Sigma(p) = \Sigma_0(p) + (\not{p} - m)\Sigma_1(p) + \Sigma_2(p)\,.$$

The terms $\Sigma_0$ and $\Sigma_1$ are quadratically and linearly

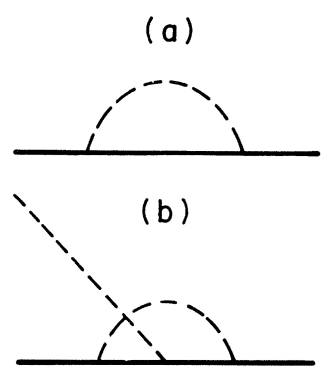


FIG. 3. Divergent Feynman diagrams: (a) Nucleon self-energy; (b) meson-nucleon vertex.

divergent and produce shifts in the effective fermion mass and coupling constant which are absorbed by renormalization. The term $\Sigma_2(p)$ vanishes when $\not p = m$ and is finite. It is an approximation to this finite and physical term which is included in the graphs of Fig. 2 while the divergent pieces are absorbed by the renormalization inherent in Low's treatment.

A similar argument applies for the graph of Fig. 3(b). The terms included in Fig. 2 which have this term include only the finite contributions which remain after renormalization.

It is useful to introduce the concept of reducible and irreducible graphs. A graph is reducible if it is obtained by an iteration of lower order diagrams. Thus in the series of Fig. 2 only the Born term $v$ is irreducible.

Contact may be made with conventional graphical notation by recalling that $v$ is defined in Fig. 1 and giving Eq. (3.2) as in Fig. 4. Thus a field-theoretic definition of a $\pi$-nucleon potential, as the driving term of the Low equation, has been provided.

## IV. INCLUSION OF THE CROSSED TERM

In this section we find a potential which, when inserted into a linear equation, gives the same $T$ matrix as the solution to the field-theoretic, crossing-symmetric Low equation. This potential and linear equation depend on the solution and it is necessary to set up an iterative procedure to obtain these quantities.

We start with the field-theoretic Low equation, Eq. (2.28), which is repeated for the sake of clarity

$$t_z(qk) = \frac{u}{z}(q,k) + \int \frac{d^3p}{(2\pi)^3} \frac{t^\dagger_{W_p}(p,q)t_{W_p}(p,k)}{z - W_p} + \int \frac{d^3p}{(2\pi)^3} \frac{t^\dagger_{W_p}(p,-k)t_{W_p}(p,-q)}{-z - W_p}, \quad (4.1)$$

where the notation $-k$ designates $(-\vec{k}, i)$ for

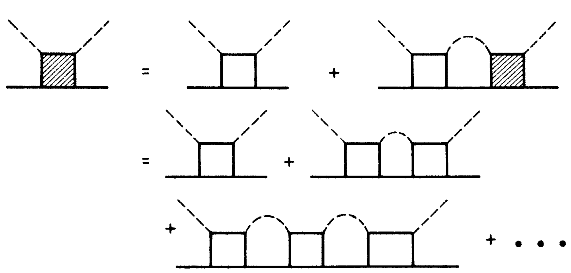

FIG. 4. The uncrossed $T$ matrix is given by the cross-hatched object.

Hermitian isospin operators. A schematic version of (4.1) is

$$t_z = v_z + t^\dagger D_z t + C[t^\dagger D_z t]. \quad (4.2)$$

By redefining the driving term as $w_z$

$$w_z = v_z + C[t^\dagger D_z t] \quad (4.3)$$

we have

$$t_z = w_z + t^\dagger D_z t. \quad (4.4)$$

Note that Eq. (4.4) has the same general form as Eq. (3.1). However, the energy and momentum dependence of $w$ is unknown. The simplified form of the interaction current, Eq. (2.24), allows us to make further statements about the form of the operators appearing in Eqs. (4.1)–(4.4). Our interaction current differs from that of Chew and Low ony by the evaluation of the pion momenta in the center-of-mass frame. Hence, just as in the static model, the driving term $u/z$ and the $T$ matrix are separable. That is, the driving term has the form of Eq. (2.19) with momenta and energy given by the center-of-mass values. We have

$$u(q,k) = \Sigma\lambda_\alpha 4\pi \frac{f(q)f(k)}{\sqrt{2\omega_q}\sqrt{2\omega_k}} P_\alpha(q,k) = \Sigma\lambda_\alpha u_\alpha(q,k), \quad (4.5)$$

where $P_\alpha$ is the projection operator for the given spin-isospin (angular momentum = 1) channel $\alpha$. Just as in the static model the $T$ matrix has the form

$$t_z(q,k) = 4\pi \sum_\alpha h_\alpha(z) \frac{f(q)f(k)}{\sqrt{2\omega_q}\sqrt{2\omega_k}} P_\alpha(q,k) \equiv \sum_\alpha t_\alpha(z). \quad (4.6)$$

The use of (4.6) allows us to write the third term on the right-hand side of Eq. (4.1) as

$$C[t^\dagger D_z t] = -\sum_{\alpha,\beta} u_\alpha(q,k) \int \frac{p^2 dp\, |t_\beta(W_p)|^2}{4\pi^2(z+W_p)} \times \frac{f^2(p)}{2\omega_p} A_{\beta\alpha}, \quad (4.7)$$

where $A_{\alpha\beta}$ is the familiar[13] crossing matrix. The function $b_\alpha(z)$ is defined by

$$C[t^\dagger D_z t] = \sum_\alpha b_\alpha(z) u_\alpha(q,k). \quad (4.8)$$

An examination of Eq. (4.7) tells us that $b_\alpha(z)$ has a branch cut along the negative real $z$ axis for $z < -\mu$. We further assume, as do Chew and Low, that $b_\alpha$ has no other singularities off the real axis.

The quantity $w(z)$ is given in terms of $b_\alpha(z)$ as

$$w(z)=\sum\left(\frac{\lambda_\alpha}{z}+b_\alpha(z)\right)u_\alpha=\sum\frac{a_\alpha(z)}{z}u_\alpha$$
$$=\sum_\alpha w_\alpha(z)\,. \tag{4.9}$$

It is tempting to use Eq. (4.9) as a potential, or driving term, in a linear equation. However, it is necessary to define the auxiliary functions $f'_\alpha$, $a'_\alpha$, and $u'_\alpha$ where

$$u'_\alpha(q,p)=\frac{f'_\alpha(q)f'_\alpha(p)}{(2\omega_q)^{1/2}(2\omega_p)^{1/2}}P_\alpha(q,p) \tag{4.10}$$

with $a'_\alpha$ and $f'_\alpha$ to be determined below except that $a'_\alpha$ has the same analytic properties as $a_\alpha$. That is, $a'_\alpha$ has a pole at $z=0$ and a left-hand cut. The quantity $w'_\alpha(z)$ is given by

$$w'_\alpha(z)=\frac{a'_\alpha(z)}{z}u'_\alpha\,. \tag{4.11}$$

In the Appendix it is shown that the linear equation

$$t'_\alpha(z)=w'_\alpha(z)+w'_\alpha(z)\frac{z}{h_0}\frac{1}{z-h_0}\frac{z}{h_0}\frac{a'_\alpha(h_0)}{a'_\alpha(z)}t'_\alpha(z) \tag{4.12}$$

is equivalent to the nonlinear equation

$$t'_\alpha(z)=w'_\alpha(z)+\int\frac{d^3p}{(2\pi)^3}\frac{t'^\dagger_\alpha(W_p)t'_\alpha(W_p)}{z-W_p}\frac{a'_\alpha(z)}{a'_\alpha(W_p)}, \tag{4.13}$$

where for notational simplicity we ignore the momentum dependence in the integral.

We again stress that Eq. (4.12) is a potential equation and not a field-theoretic equation. We work in a Hilbert space consisting of point nucleons and point pions and hope to obtain a linear equation which has the same $T$-matrix elements as the field-theoretic solution.

The presence of the factor $a'_\alpha(z)/a'_\alpha(W_p)$ in the integrand means that the left- and right-hand cuts are not separated as they are in the low equation. Let us define another $T$ matrix $t''_z$:

$$t''(z)=\sum_\alpha t''_\alpha(z)$$
$$=\sum_\alpha\frac{a'_\alpha(z)}{[a'_\alpha(h_0)]^{1/2}}t'_\alpha(z)u_\alpha\frac{1}{[a'_\alpha(h_0)]^{1/2}} \tag{4.14}$$

where we intend to show that $t_z$ of Eq. (4.1) and $t''_z$ are equivalent. It is necessary to comment on the presence of square root operators in Eq. (4.14). The quantity $[a'_\alpha(h_0)]^{1/2}$ is always evaluated between our plane wave states where $h_0$ takes on positive values only. Hence we use $[a'_\alpha(h_0)]^{1/2}$ in regions where $a'_\alpha(h_0)$ is real and there is no ambiguity about which root to use.

By multiplying Eq. (4.12) by $[a'_\alpha(h_0)]^{-1/2}$ from the left and from the right and multiplying Eq. (4.12) by an overall factor of $a'_\alpha(z)$ and using the definition (4.14) we have

$$t''_\alpha(z)=\frac{a'_\alpha(z)}{[a'_\alpha(h_0)]^{1/2}}w'_\alpha(z)\frac{1}{[a'_\alpha(h_0)]^{1/2}}$$
$$+\int\frac{d^3p}{(2\pi)^3}\frac{t''^\dagger_\alpha(W_p)t''_\alpha(W_p)}{z-W_p}\,. \tag{4.15}$$

If one makes the identification

$$a'_\alpha(z)\frac{1}{[a'_\alpha(h_0)]^{1/2}}w'_\alpha(z)\frac{1}{[a'_\alpha(h_0)]^{1/2}}=w_\alpha(z) \tag{4.16}$$

or

$$\frac{a'^2_\alpha(z)}{z}\frac{f'_\alpha(k)f'_\alpha(p)}{[a'_\alpha(\omega_k)]^{1/2}[a'_\alpha(\omega_p)]^{1/2}}=\frac{a_\alpha(z)}{z}f(k)f(p) \tag{4.17}$$

which implies

$$a'^2_\alpha(z)=a_\alpha(z)\,,$$
$$f'_\alpha(k)=[a_\alpha(\omega_k)]^{1/2}f(k) \tag{4.18}$$

then one has

$$t''_\alpha(z)=w_\alpha(z)+\int\frac{d^3p}{(2\pi)^3}\frac{t''^\dagger_\alpha(W_p)t''_\alpha(W_p)}{z-W_p}\,. \tag{4.19}$$

Because of our choice of $w(z)$, Eq. (4.19) is of the same form as Eq. (4.1) and we make the identification $t_z=t''_z$. One may obtain a linear equation for $t_z$ by multiplying Eq. (4.13) from the left and right by and by a factor of $a_\alpha(z)$. Then

$$t_\alpha(z)=w_\alpha(z)+w_\alpha(z)\frac{z}{h_0}\frac{1}{z-h_0}\frac{z}{h_0}\left(\frac{a_\alpha(h_0)}{a_\alpha(z)}\right)^2t_\alpha(z) \tag{4.20}$$

and the problem is formally solved. Equation (4.20) represents a linear equation, the solution of which is equivalent to that of Eq. (4.1).

The function $a_\alpha(z)$, hence the potential $w(z)$ and Green's function, depend on the solution $t_z$. In order to demonstrate the validity of Eq. (4.20), it is necessary to show how to determine $w$ from the known driving potential $v$ and to explain the physics contained in that equation.

One method of solution is to use a self-consistent procedure. First one neglects the crossed term, calculates a $T$ matrix, and uses it to obtain an approximation to $a_\alpha(z)$. Then one solves the linear equation with the new potential and Green's function to obtain a better estimate of the

$T$ matrix and an improved function $a_\alpha(z)$. The improved $a_\alpha(z)$ is then used to generate a new $T$ matrix. This procedure is continued until convergence is achieved. The remainder of this section is devoted to a detailed discussion of this procedure.

The procedure is begun by defining a first approximation to the potential $w$:

$$w_0=\sum_\alpha \frac{u_\alpha}{z}\equiv v=\sum_\alpha v_\alpha\,. \tag{4.21}$$

The notation $v$ given by (4.21) is used in the remainder of this section. The first approximation to $t$ is simply obtained by summing the diagrams of Fig. 4 via an integral equation as in Eq. (3.2), i.e.,

$$\begin{aligned} t_0&=v+vg_0t_0\,,\\ g_0&=\left(\frac{z}{h_0}\right)^2\frac{1}{z-h_0}\,. \end{aligned} \tag{4.22}$$

Whereas $v$ is crossing-symmetric, $t_0$ is not, as is clearly seen in Fig. 4. In order to remedy the lack of crossing symmetry one might add the crossed version of the iterated diagrams $C[t_0-v]$ to $t_0$. The resulting $T$ matrix is crossing-symmetric, but it is not complete because the new term, as shown in Fig. 5, is itself irreducible and should be used to generate even more higher order diagrams.

One therefore defines a potential

$$\begin{aligned} w_1&=v+C[t_0-v]=v+C[vg_0t_0]\\ &=v+C[t_0^\dagger Dt_0]\equiv\sum\frac{a_\alpha^{(0)}(z)}{z}u_\alpha\\ &=\sum_\alpha w_{1,\alpha}(z)\,, \end{aligned} \tag{4.23}$$

and immediately notices that the third line of Eq. (4.23) looks more like Eq. (4.3) than does Eq. (4.21). Thus the estimate of $w$ has been improved. However, Eq. (4.23) is not the final solution because $t_0$ is not the same as $t$. The function $a_\alpha^{(0)}(z)$ is assumed to have the same analytic properties as $a_\alpha(z)$.

The next step is to calculate the $T$ matrix corresponding to the potential of Eq. (4.23). One has

$$t_{1,\alpha}(t)=w_{1,\alpha}[1+g_{1,\alpha}(z)t_{1,\alpha}(z)]\,, \tag{4.24}$$

where the notation $t_{n,\alpha}(z)$ designates the $n$th iterate in the channel $\alpha$. The Green's function is given by

$$g_{1,\alpha}=\frac{1}{z-h_0}\left(\frac{z}{h_0}\right)^2\frac{a_\alpha^{(0)}(h_0)}{a_\alpha^{(0)}(z)}\,. \tag{4.25}$$

Some of the terms of $t_1$ not included in $t_0+C[t_0-v]$ are shown in Fig. 6. Note that each of the graphs contains at least one state in which another virtual meson has been produced. Although a new, infinite class of diagrams is included, $t_1$ is not crossing-symmetric because the crossed version of the diagrams of Fig. 6 are not included. The addition of the crossed terms, $C[t_1]-C[t_0-v]-t_0$, some of which are shown in Fig. 7, to $t_1$ results in a crossing-symmetric $T$ matrix. However, these new terms are irreducible and should also be used to generate more diagrams. This step is

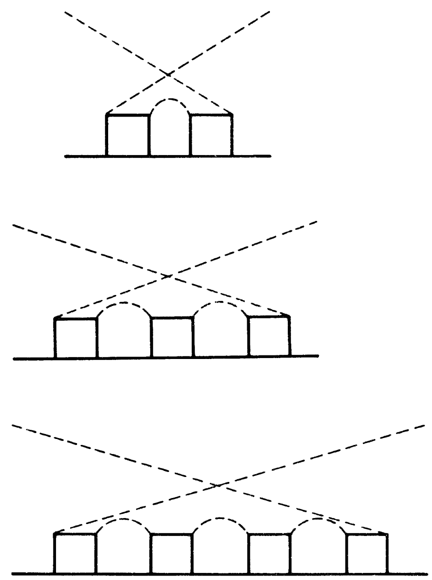

FIG. 5. Some terms introduced by crossing the reducible term of Fig. 5.

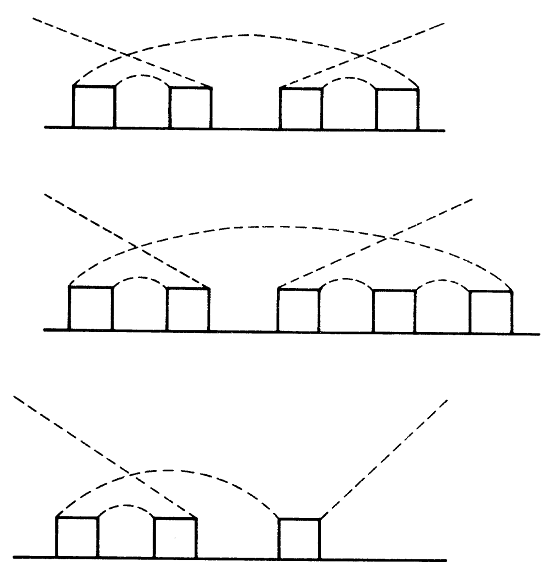

FIG. 6. Some terms caused by using terms of Fig. 5 as driving terms added to $v$.

facilitated by defining

$$\begin{aligned} t_1 &= t_0 + \Delta t \ , \\ w_1 &= \frac{u}{z} + \Delta v \equiv v + \Delta v \ , \\ g_1 &= g_0 + \Delta g \ . \end{aligned} \tag{4.26}$$

Then

$$\begin{aligned} t_1 &= \left(\frac{u}{z} + \Delta v\right)\left[1 + (g_0 + \Delta g)(t_0 + \Delta t)\right] \\ &= t_0 + \Delta v + \frac{u}{z} g_0 \Delta t + \sum_\alpha \frac{u_\alpha}{z} \Delta g_\alpha t_{1,\alpha} \\ &\quad + \sum_\alpha \Delta v_\alpha g_{1,\alpha} t_{1,\alpha} . \end{aligned} \tag{4.27}$$

By crossing the last three terms and using them as an additional potential, one finds a new total potential $w_2$:

$$\begin{aligned} w_2 &= v + C[v g_0 t_0] + C[v g_0 \Delta t] + \sum_\alpha \Delta v_\alpha g_{1,\alpha} t_{1,\alpha} \\ &\quad + \sum_\alpha v_\alpha \Delta g t_{1,\alpha} \\ &= v + C[w_1 g_1 t_1] \\ &= v + C[t_1^\dagger D t_1] = \sum_\alpha \frac{a_\alpha^{(1)}(z)}{z} u_\alpha = \sum_\alpha w_{2,\alpha} . \end{aligned} \tag{4.28}$$

Equation (4.28) shows that the addition given by crossing the terms of $t_1$ not in $C[t_0 - v]$ results in an equation for a potential which has the same form as Eq. (4.4).

The new terms involve the production of another (virtual) meson. If one recalls Chew's idea[19] that (multi)meson production is associated with large energy denominators, there is then some hope that $t_1$ is not very different from $t_0$. This means that there is a good chance that this iterative scheme will converge.

Note that each of the successive graphs is computed with a different Green's function. The successive potentials $w_n$ have no cuts for positive $z$ and are real even at energies high enough so that physical multimeson production could occur. The graphs of Figs. 4–9 are not Feynman diagrams.

The next step is to use $w_2$ to generate still another $T$ matrix:

$$t_{2,\alpha} = w_{2,\alpha}(1 + g_{2,\alpha} t_{2,\alpha}) , \tag{4.29}$$

where $g_2$ is obtained from the energy dependence of $w_2$. A typical term of $t_2$ not included in $t_1$ $+ C[t_1 - w_1]$ is shown in Fig. 8. Clearly $t_2$ is not crossing-symmetric, because the crossed version of the new terms is not included. One such term is shown in Fig. 9. These crossed terms are irreducible. By the reasoning of the paragraphs preceding Eq. (4.28), one has

$$w_3 = v + C[t_2^\dagger D t_2] . \tag{4.30}$$

The procedure of this section leads to an infinite sequence of potentials which may be summarized by

$$\begin{aligned} w_0 &= v = \sum \frac{\lambda_\alpha u_\alpha}{z} , \\ t_0 &= v(1 + g_0 t_0) , \\ w_n &= v + C[v_{n-1} g_{n-1} t_{n-1}] \\ &= v + C[t_{n-1}^\dagger D t_{n-1}] = \sum_\alpha \frac{a_\alpha^{(n-1)}(z)}{z} u_\alpha \\ &= \sum_\alpha w_{n,\alpha} , \\ g_{n,\alpha} &= \left(\frac{a_\alpha^{(n-1)}(h_0)}{a_\alpha^{(n-1)}(z)}\right)^2 g_0 , \\ t_{n-1,\alpha} &= w_{n-1,\alpha}(1 + g_{n-1,\alpha} t_{n-1,\alpha}) . \end{aligned} \tag{4.31}$$

Furthermore, if the sequence converges (inclusion of terms in which $m$ additional mesons are pro-

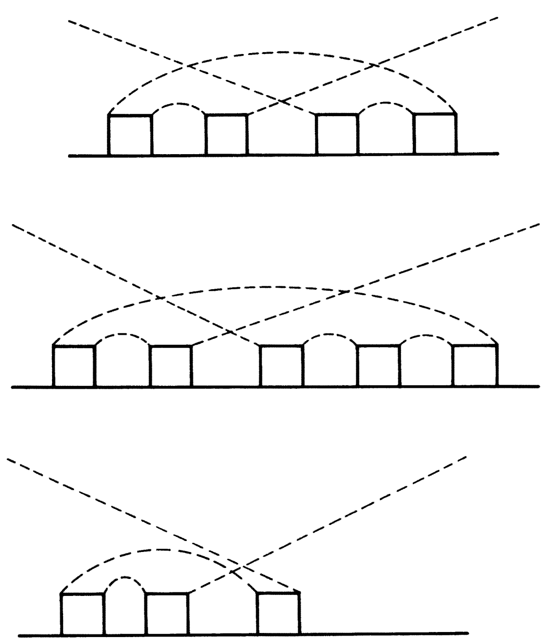

FIG. 7. New terms introduced by crossing terms of Fig. 6.

duced is negligible for a given energy), then there is an integer $m$ such that

$$t_{m-1} = t_m . \tag{4.32}$$

This means that

$$\begin{aligned} w_m &= v + C[\, t_m^\dagger D t_m] , \\ t_m &= w_m + t_m^\dagger D t_m \\ &= v + t_m^\dagger D t_m + C[t_m^\dagger D t_m] \end{aligned} \tag{4.33}$$

which implies that $w_m$ and $t_m$ are solutions of Eq. (4.2). Now $t_m$ is a physically meaningful solution as long as none of the intermediate potentials $w_m$ introduce an additional $\pi$-nucleon bound state. Thus $w_m = w$ and $t_m = t$. Since $w$ is known, its energy dependence is known so that one may write the linear equation (4.20).

The requirement that the iterative procedure converge is not more restrictive than the requirements of the original Chew-Low solution. Indeed, in order for Chew and Low to obtain their function $h_\alpha(z)$ from $g_\alpha(z)$ it is necessary to assume that the value of the $\pi$-nucleon coupling constant be small enough so a power series expansion for their "crossed term" converges.

It is useful to describe in words, the procedures of this section. A potential $v$ generates a $T$ matrix. Only $v$ is crossing-symmetric and the iterated terms are not crossing-symmetric. The crossed version of these terms may be used to generate an additional potential which is added to $v$. This potential, so formed, generates another $T$ matrix which contains an infinite set of non-crossing-symmetric diagrams (as well as an infinite set of terms which are crossing-symmetric). The crossed version of any remaining nonsymmetric diagrams is added to the cumulative potential. This potential generates another $T$ matrix and an iterative procedure is established. Each new step results in the addition of terms with an intermediate state containing an additional meson. Thus even under the one-meson truncation, an infinite number of (multimeson) irreducible terms is contained in the solution of the Low equation.

Any solution of the Low equation, whether by our iterative procedure or by other means, necessarily contains the full set of terms discussed here.

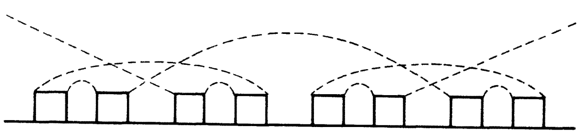

FIG. 8. An iterate of Fig. 7.

## V. EXPLICIT COMPARISON WITH THE CHEW-LOW SOLUTION

In this section we compare the solution inherent in Eq. (4.20) with the solution of Ref. 13. To do this we use the static approximation which involves the neglect of all $1/m$ terms. We first state the Chew-Low solution. The driving term is

$$v(q,k) = \sum_\alpha \frac{\lambda_\alpha}{z}\, u_\alpha(q,k) . \tag{5.1}$$

One has

$$\lambda = \frac{f_r^{\,2}}{\mu^2}\,\frac{2}{3}\begin{pmatrix} 4 \\ 1 \\ 1 \\ -2 \end{pmatrix}, \tag{5.2}$$

where $f_r$ is the renormalized coupling constant ($f_r^{\,2} = 0.088$) and $\mu$ is the meson mass. The $P_\alpha$ are the channel projection operators of Ref. 13. Chew and Low find the following expression for the $T$ matrix:

$$t_z(p,k) = \frac{4\pi f(p) f(k)}{(4\omega_p \omega_k)^{1/2}} \sum_\alpha h_\alpha(z)\, P_\alpha(p,k) \tag{5.3}$$

with

$$\begin{aligned} h_\alpha(z) &= \frac{\lambda_\alpha}{z}\, D_\alpha^{-1}(z) = -e^{i\delta_\alpha} \sin\delta_\alpha / p^3 v^2(p) , \\ D_\alpha(z) &= 1 - \frac{\lambda_\alpha}{z}\,\frac{1}{\pi} \int \frac{z^2}{\omega_q^{\,3}}\, \frac{q^4 dq\, f^2(q)}{z - \omega_q} \\ &\quad + \frac{\lambda_\alpha}{z\pi} \int \frac{z^2 q^4 dq}{\omega_q^{\,3}(z + \omega_q)}\; B_\alpha(\omega_q) , \end{aligned} \tag{5.4}$$

where $B_\alpha(\omega_q)$ is a function which insures that the solution is crossing-symmetric[20]:

$$B_\alpha(\omega) = |D_\alpha(-\omega)|^2 \sum_\beta \frac{A_{\alpha\beta}\lambda_\beta^{\,2}}{\lambda_\alpha |D_\beta(\omega)|^2} \tag{5.5}$$

and $A_{\alpha\beta}$ is the crossing matrix.

Before comparing the solution of Eq. (4.20) to Eq. (5.4) it is worthwhile to examine the Chew-Low solution in the absence of the left-hand cut. In the limit $B_\alpha = 0$, one sees that Eq. (5.4) differs from the solution to the Lippmann-Schwinger equation with an energy-dependent potential by

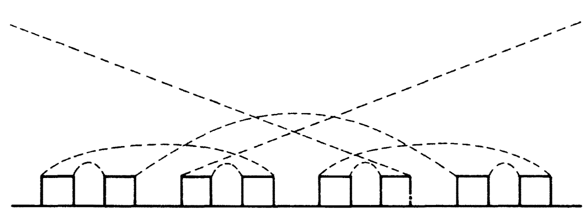

FIG. 9. The crossed version of Fig. 8.

the presence of the factor $z^2/\omega_p^2$ in the integral. This factor is the result of a subtraction made by Chew and Low to insure the amplitude has a pole when $z=0$. Consider now Eqs. (3.8) and (3.9). The solution of Eq. (3.9) gives

$$\tilde{u}=\sum_\alpha \frac{\gamma_\alpha}{h_0} u_\alpha , \tag{5.6}$$

where

$$\gamma_\alpha=\lambda_\alpha\Big/\left[1-\frac{\lambda_\alpha}{\pi}\int\frac{p^4dp}{\omega_p^3}f^2(p)\right] . \tag{5.7}$$

The separability of $\tilde{u}$ leads to the solution

$$\begin{aligned}T_z(q,k)&=\frac{z}{h_0}t_z(q,k)\\&=\frac{4\pi}{(4\omega_q\omega_k)^{1/2}}\sum \bar{h}_\alpha(z)\frac{f(q)}{\omega_q}f(k)P_\alpha(q,k),\end{aligned} \tag{5.8}$$

where

$$\bar{h}_\alpha(z)=\gamma_\alpha\Big/\left[1-\frac{\gamma_\alpha}{\pi}\int\frac{p^4dp}{(z-\omega_p)}\frac{f^2(p)}{\omega_p^2}\right] . \tag{5.9}$$

We see that (5.9) is the solution of a Lippman-Schwinger equation with a potential with a different strength than $v$. The next step is to write $\bar{h}_\alpha(z)$ in terms of $\lambda_\alpha$. Using (5.7) in (5.9) one has

$$\begin{aligned}\bar{h}_\alpha(z)&=\lambda_\alpha\Big/\left[1-\frac{\lambda_\alpha}{\pi}\int\frac{p^4dp}{\omega_p^3}f^2(p)\right.\\&\qquad\left.-\frac{\lambda_\alpha}{\pi}\int\frac{p^4f^2(p)}{(z-\omega_p)\omega_p^2}\right] \qquad (5.10)\\&=\lambda_\alpha\Big/\left[1-\frac{\lambda_\alpha}{\pi}z\int\frac{p^4dpf^2(p)}{\omega_p^3(z-\omega_p)}\right] .\end{aligned}$$

Thus the use of $\tilde{u}$ in a Lippman-Schwinger equation instead of $v$ insures that the amplitude has a pole at $z=0$.

Let us turn to Eq. (4.20); recall

$$w(z)=\sum_\alpha\frac{a_\alpha(z)}{z}u_\alpha(q,p) ,$$
$$a_\alpha(z)=\lambda_\alpha+zb_\alpha(z) . \tag{5.11}$$

Because the potential is separable, the result for $\bar{h}_\alpha(z)$ may be given immediately by

$$\begin{aligned}\bar{h}_\alpha(z)&=\frac{a_\alpha(z)}{z}\bar{D}_\alpha^{-1}(z) ,\\ \bar{D}_\alpha(z)&=1-\frac{a_\alpha(z)}{z\pi}\int\frac{z^2}{\omega_q^3}\frac{q^4dq}{z-\omega_q}u^2(a)\frac{a_\alpha^2(\omega_q)}{a_\alpha^2(z)} .\end{aligned} \tag{5.12}$$

To compare (5.12) with (5.4) multiply the numerator and denominator of (5.12) by $\lambda_\alpha/z[\lambda_\alpha/z+b_\alpha(z)]^{-1}$. Then

$$\bar{h}_\alpha(z)=\frac{\lambda_\alpha}{z}E_\alpha^{-1}(z) \tag{5.13}$$

with

$$E_\alpha(z)=1-\frac{\lambda_\alpha}{2\pi}\int\frac{z^2}{\omega_q^3}\frac{q^4dqf^2(q)}{z-\omega_q}+\Delta_\alpha(z) , \tag{5.14}$$

where

$$\begin{aligned}\Delta_\alpha(z)=&-\frac{zb_\alpha(z)}{\lambda_\alpha+z\lambda_\alpha b_\alpha(z)}\\&-\frac{\lambda_\alpha z}{\pi}\int\frac{q^4dqf^2(q)}{\omega_q^3(z-\omega_q)}\frac{a_\alpha^2(\omega_q)-a_\alpha^2(z)}{a_\alpha^2(z)} .\end{aligned} \tag{5.15}$$

The function $\Delta_\alpha(z)$ has the following properties:

(1) $\lim_{z\to 0}\Delta_\alpha(z)=0$;
(2) $\Delta_\alpha(z)$ has a branch cut along the real negative $z<-\mu$;
(3) $\Delta_\alpha(z)$ has no cut for $z>\mu$.

The third property obtains from the presence of the $a_\alpha^2(\omega_q)-a_\alpha^2(z)$ term in the integral. Thus we may write

$$\Delta_\alpha(z)=-\frac{\lambda_\alpha z}{\pi}\int_0^\infty\frac{dqC'_\alpha(\omega_q)}{z+\omega_q} , \tag{5.16}$$

where $C'_\alpha(z)$ is an unknown function which is assumed to fall off so that the integral converges. To make contact with Eq. (5.4) one may define another unknown function $C_\alpha(\omega_q)$ such that

$$C'_\alpha(\omega_q)=\frac{q^4}{\omega_q^3}f^2(q)C_\alpha(\omega_q) . \tag{5.17}$$

The use of Eqs. (5.16) and (5.17) in (5.14) shows that our solution (5.12) has the same form as the Chew-Low solution. We have

$$\begin{aligned}E_\alpha(z)=1&-\frac{\lambda_\alpha}{2\pi}\int\frac{z^2}{\omega_q^3}\frac{q^4dqf^2(q)}{z-\omega_q}\\&+\frac{\lambda_\alpha z}{\pi}\int\frac{q^4dqf^2(q)}{\omega_q^3}\frac{C_\alpha(\omega_q)}{(z+\omega_q)} .\end{aligned} \tag{5.18}$$

Our solution is the same as the Chew-Low solution if $C_\alpha(x)=B_\alpha(x)$. If the crossing relation uniquely[20] specifies $B_\alpha(x)$, this condition will obtain because $\bar{h}_\alpha(z)$ is constructed to be crossing-symmetric.

It is worthwhile to comment on the convergence properties of the expansion of Sec. IV. From Eqs. (5.15) and (5.18) one see that, even with the first estimate of $a_\alpha(z), a_\alpha^{(0)}(z)$, the form of $\bar{h}_\alpha(z)$ is the same as that for $h_\alpha(z)$.

## VI. SUMMARY OF RESULTS

(1) A derivation of a linear equation (3.3) equivalent to the Low equation (neglecting crossing) with nucleon recoil included [to order $(1/m)$] is given. The renormalizability properties of the Low equa-

tion are maintained so that a physically meaningful potential is defined. This linear equation embodies in a potential scattering problem the scattering solution of the field-theoretic Hamiltonian Eq. (2.11).

(2) A similar linear equation (4.20) is derived for the case when the crossed term of the Low equation is included. In this case, the potential and Green's function of the linear equation depend on the solution. An iterative procedure to compute these quantities is derived. A diagrammatic interpretation of this procedure is made and the specific procedure for calculating the potential and Green's function is given in Eq. (4.31).

(3) Our solution is equivalent to the solution of Chew and Low in the static limit. Furthermore, it seems that the iterative procedure of Sec. IV converges.

These results show that the concept of a $\pi$-nucleon potential, even when crossing is included, is a useful way to characterize $\pi$-nucleon scattering. This makes $\pi$-nucleon input easier to insert into the many-body problem.

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## APPENDIX

In this Appendix we prove that the solution of the equation

$$t_\alpha(z) = w_\alpha(z) + w_\alpha(z)\frac{z}{h_0}\frac{1}{z-h_0}\frac{z}{h_0}\frac{a_\alpha(h_0)}{a_\alpha(z)}t_\alpha(z)\,, \tag{A1}$$

where

$$\frac{a_\alpha(z)}{z} = \frac{\lambda_\alpha}{z} + b_\alpha(z) \tag{A2}$$

with $b_\alpha(z)$ regular except for a branch cut along the negative real $z$ axis for $z < -\mu$; and

$$w_\alpha(z) = \frac{a_\alpha(z)}{z}u_\alpha \tag{A3}$$

is equivalent to the solution of

$$t_\alpha(z) = w_\alpha(z) + \int\frac{d^3p}{(2\pi)^3}\frac{t_\alpha^\dagger(W_p)t_\alpha(W_p)}{z-W_p}\frac{a_\alpha(z)}{a_\alpha(W_p)}\,, \tag{A4}$$

where the momentum labels of Eq. (A4) have been suppressed.

The proof proceeds by using the techniques of Sec. III. In analogy with Eq. (3.6) define

$$T_\alpha(z) = \frac{z}{h_0}\frac{a_\alpha(h_0)}{a_\alpha(z)}t_\alpha(z)\,. \tag{A5}$$

By multiplying (A4) by

$$\frac{z}{h_0}\frac{a_\alpha(h_0)}{a_\alpha(z)}$$

from the left, we have

$$T_\alpha(z) = \frac{1}{h_0}a_\alpha(h_0)u_\alpha + \frac{1}{h_0}a_\alpha(h_0)u_\alpha\frac{z}{h_0}\frac{1}{z-h_0}T_\alpha(z) \tag{A6}$$

which is expressed as an equivalent Lippmann-Schwinger equation:

$$T_\alpha(z) = \tilde{u}_\alpha + \tilde{u}_\alpha\frac{1}{z-h_0}T_\alpha(z)\,, \tag{A7}$$

with

$$\tilde{u}_\alpha = \frac{1}{h_0}a_\alpha(h_0)u_\alpha\left(1+\frac{1}{h_0}\tilde{u}_\alpha\right)\cdot \tag{A8}$$

The procedure of Eqs. (3.11)–(3.18) as augmented by the definitions

$$\begin{aligned}T_\alpha(W_p) &= \langle\psi_{W_p}^{(-)}|\tilde{u}_\alpha|\phi_k\rangle\,,\\ \overline{T}_\alpha(W_p) &= \langle\psi_{W_p}^{(-)}|\tilde{u}_\alpha^\dagger|\phi_k\rangle\end{aligned} \tag{A9}$$

gives

$$T_\alpha(z) = \tilde{u}_\alpha + \int\frac{d^3p}{(2\pi)^3}\frac{\overline{T}_\alpha^\dagger(W_p)T_\alpha(W_p)}{z-W_p}\,. \tag{A10}$$

The next step is to determine the relationship between $\overline{T}$ and $T$ which is obtained from the relationship between $\tilde{u}_\alpha$ and $\tilde{u}_\alpha^\dagger$. At this point we use the fact that in order to calculate $T$ or $\overline{T}$ one uses matrix elements of $\tilde{u}_\alpha$ and $\tilde{u}_\alpha^\dagger$ in a plane wave basis. For such matrix elements, $h_0$ is always greater than or equal to $\mu$ and $a_\alpha^\dagger(h_0) = a_\alpha(h_0)$. It is then straightforward to use (A8) to show that

$$\tilde{u}_\alpha^\dagger = \frac{h_0}{a_\alpha(h_0)}\tilde{u}_\alpha\frac{a_\alpha(h_0)}{h_0}\,. \tag{A11}$$

By multiplying (A7) by $h_0/a_\alpha(h_0)$ from the left and by $a_\alpha(h_0)/h_0$ from the right and using (A11), we obtain

$$\frac{h_0}{a_\alpha(h_0)}T_\alpha(z)\frac{a_\alpha(h_0)}{h_0} = \tilde{u}_\alpha^\dagger + \tilde{u}_\alpha^\dagger\frac{1}{z-h_0}\frac{h_0}{a_\alpha(h_0)}T_\alpha(z)\frac{a_\alpha(h_0)}{h_0}\,; \tag{A12}$$

hence,

$$\overline{T}_\alpha(z) = \frac{h_0}{a_\alpha(h_0)}T_\alpha(z)\frac{a_\alpha(h_0)}{h_0}\,. \tag{A13}$$

The use of (A8) and (A13) in (A10) gives

$$T_\alpha(z)=\frac{a_\alpha(h_0)}{h_0}u_\alpha+\frac{a_\alpha(h_0)}{h_0}u_\alpha\frac{1}{h_0}\tilde{u}_\alpha$$
$$+\int\frac{d^3p}{(2\pi)^3}\frac{W_p}{a_\alpha(W_p)}\frac{a_\alpha(h_0)}{z-W_p}T_\alpha^\dagger(W_p)T_\alpha(W_p) \tag{A14}$$

$$=\frac{a_\alpha(h_0)}{h_0}u_\alpha+\frac{a_\alpha(h_0)}{h_0}u_\alpha\frac{1}{h_0}\tilde{u}_\alpha$$
$$-\int\frac{d^3p}{(2\pi)^3}\frac{a_\alpha(h_0)}{a_\alpha(W_p)}T_\alpha^\dagger(W_p)T_\alpha(W_p)$$
$$+\int\frac{d^3p}{(2\pi)^3}\frac{z a_\alpha(h_0)}{a_\alpha(W_p)}\frac{T_\alpha^\dagger(W_p)T_\alpha(W_p)}{z-W_p}. \tag{A15}$$

The next step is to show that the sum of the second and third quantities on the right-hand side of (A15) is zero. From (A14)

$$-\int\frac{d^3p}{(2\pi)^3}\frac{a_\alpha(h_0)}{a_\alpha(W_p)}T_\alpha^\dagger(W_p)=\lim_{z\to 0}T_\alpha(z)-\tilde{u}_\alpha \tag{A16}$$

and the solution of (A7) at $z=0$ gives

$$\lim_{z\to 0}T_\alpha(z)-\tilde{u}_\alpha=-\tilde{u}_\alpha\frac{1}{h_0}\frac{1}{1+\tilde{u}_\alpha(1/h_0)}\tilde{u}_\alpha. \tag{A17}$$

The relation

$$\frac{a_\alpha(h_0)}{h_0}u_\alpha\frac{1}{h_0}=\tilde{u}_\alpha\frac{1}{h_0}\frac{1}{1+\tilde{u}_\alpha(1/h_0)} \tag{A18}$$

which is obtained from (A8) gives

$$\lim_{z\to 0}T_\alpha(z)-\tilde{u}_\alpha=-\frac{a_\alpha(h_0)}{h_0}u_\alpha\frac{1}{h_0}\tilde{u}_\alpha. \tag{A19}$$

The use of (A16) and (A19) in (A15) gives

$$T_\alpha(z)=\frac{a_\alpha(h_0)}{h_0}u_\alpha+\int\frac{d^3p}{(2\pi)^3}\frac{z}{z-W_p}\frac{a_\alpha(h_0)}{a_\alpha(W_p)}\times T_\alpha^\dagger(W_p)T_\alpha(W_p) \tag{A20}$$

which, when expressed in terms of $T_\alpha(z)$ via (A5), gives

$$t_\alpha(z)=\frac{a_\alpha(z)}{z}u_\alpha+\int\frac{d^3p}{(2\pi)^3}\frac{t_\alpha^\dagger(W_p)t_\alpha(W_p)}{z-W_p}\frac{a_\alpha(z)}{a_\alpha(W_p)} \tag{A21}$$

which is the desired relationship.

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