Developments in a new treatment of exchange effects in the theory of radioactive decay*

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We present some new developments in a treatment of exchange effects in radioactive decay published recently by us. By altering slightly the definition of the parent nucleus wave function in our expression for the radioactive decay width Γ, we find that the transformation of the volume integral expression for Γ to the surface integral expression becomes rigorous. We discuss also how the optical model wave functions that appear in our expressions must be defined in a new manner which will lead to much larger predictions for Γ.

[RADIOACTIVITY Width for radioactive decay by particle emission which includes effects of exchange symmetry is derived.]

In Ref. 1 an expression was derived for the width for radioactive decay. For the decay $P \to E + D$ this expression reads

$$\Gamma = \frac{\mu_{ED} k}{(2\pi\hbar)^2} \int d\hat{k} \langle \Phi_P | Q_{ED} V_{ED} A^\dagger_{ED} | \Phi_E \Phi_D \psi(\vec{k}, \vec{r}_{ED}) \rangle \times \frac{\langle \Phi_E \Phi_D \tilde{\psi}(\vec{k}, \vec{r}_{ED}) | V_{ED} A^\dagger_{ED} Q_{ED} | \Phi_P \rangle}{\langle \Phi_P | Q_{ED} | \Phi_P \rangle}, \tag{1}$$

where $\hbar^2 k^2 / 2\mu_{ED}$ is the kinetic energy of relative motion for the daughter nucleus D and the emitted particle E. V_{ED} is the interaction potential acting between E and D. A_{ED} is the antisymmetrizer

$$A_{ED} = N_{ED}^{-1} \sum_{n=1}^{N_{ED}} (-1)^{\sigma_{ED}(n)} P_{ED}(n), \tag{2}$$

made up of permutations that exchange like particles between E and D, which antisymmetrizes $\Phi_E \Phi_D \psi(\vec{k}, \vec{r}_{ED})$ given that Φ_E and Φ_D, the internal motion wave functions for E and D, are antisymmetric. Q_{ED} is the complement to P_{ED}, the projector onto the $E+D$ channel. Φ_P is the parent nucleus internal motion wave function. ψ and $\tilde{\psi}$ are solutions of

$$(E - H_{ED} - P_{ED} V_{ED} A^\dagger_{ED}) \Phi_E \Phi_D \psi(\vec{k}, \vec{r}_{ED}) = 0, \tag{3}$$

$$(E - H_{ED} - P_{ED} A_{ED} V^\dagger_{ED}) \Phi_E \Phi_D \tilde{\psi}(\vec{k}, \vec{r}_{ED}) = 0 \tag{4}$$

whose incoming parts are asymptotically equal to the incoming part of the plane wave $\Phi_E \Phi_D \times \exp(i\vec{k} \cdot \vec{r}_{ED})$. E is the total energy. The number N_{ED} is just

$$N_{ED} = \frac{(N_E + N_D)!}{N_E! N_D!}, \tag{5}$$

where N_X is the number of nucleons in nucleus X. $H = H_{ED} + V_{ED}$ is the total Hamiltonian.

Our expression for the decay width was the same as that which had previously been published.[2] However, some of the wave functions appearing in this expression are differently defined. The parent nucleus wave function Φ_P is the standard Schrödinger wave function. ψ and $\tilde{\psi}$ are related to the antisymmetrized coupled equations formalism (ACEF) wave functions.[3]

In this addendum we show that the decay width expression may be transformed rigorously from the volume integral form given in Eq. (1) to a simpler surface integral form if the parent nucleus wave function Φ_P is also taken to be an ACEF wave function. We also discuss here what sort of approximations appear to be suitable for our Φ_P and ψ in analyzing α decay. We conclude that Φ_P should be approximated by a shell model wave function in the usual way. For the wave function ψ, however, we are led to a new type of optical model wave function which will produce much greater values for the decay width than the conventional one.

In the scattering theory derivation of the decay width given in Ref. 1, the parent nucleus wave function Φ_P is introduced into the formalism to provide a one-state approximation for the operator g:

$$g = (E - H_{ED} - Q_{ED} \overline{Y})^{-1} Q_{ED}, \tag{6}$$

$$\overline{Y} = V_{ED} A^\dagger_{ED} + V_{ED} A^\dagger_{ED} (E - H_{ED} + i\epsilon)^{-1} P_{ED} \overline{Y}. \tag{7}$$

The approximation was

$$g \approx \frac{Q_{ED} | \Phi_P \rangle \langle \Phi_P | Q_{ED}}{\langle \Phi_P | Q_{ED} (E - H_{ED} - \overline{Y}) Q_{ED} | \Phi_P \rangle}, \tag{8}$$

$$(E_0 - H) \Phi_P = 0. \tag{9}$$

Then the imaginary part of the denominator of g was interpreted as one half of the decay width:

$$E - E'_0 - \Delta + i\tfrac{1}{2}\Gamma = \frac{\langle \Phi_P | Q_{ED} (E - H_{ED} - \overline{Y}) Q_{ED} | \Phi_P \rangle}{\langle \Phi_P | Q_{ED} | \Phi_P \rangle}, \tag{10}$$

or, by virtue of Eqs. (7) and (9),

$$\Delta - i\tfrac{1}{2}\,\Gamma = \langle \Phi_P \,|\, Q_{ED} V_{ED} A^\dagger_{ED} (E - H_{ED} + ie)^{-1} P_{ED}\, \overline{Y}\, Q_{ED} \,|\, \Phi_P \rangle / \langle \Phi_P \,|\, Q_{ED} \,|\, \Phi_P \rangle . \tag{11}$$

In making a one-state approximation to g we sought a function Φ_P which approximated an eigenfunction of $H_{ED} + \overline{Y}$. Now instead of using an eigenstate of $H = H_{ED} + V_{ED}$ for this purpose we might consider using an eigenstate of $H_{ED} + V_{ED} A^\dagger_{ED}$ inasmuch as $\overline{Y}$ contains the term $V_{ED} A^\dagger_{ED}$ rather than V_{ED}. This alternative leads to the following one-state approximation for g:

$$g \approx \frac{Q_{ED} \,|\, \Phi_P \rangle \langle \tilde{\Phi}_P \,|\, Q_{ED}}{\langle \tilde{\Phi}_P \,|\, Q_{ED} (E - H_{ED} - \overline{Y}) Q_{ED} \,|\, \Phi_P \rangle} , \tag{12}$$

$$(E_0 - H_{ED} - V_{ED} A^\dagger_{ED}) \Phi_P = 0 , \tag{13}$$

$$(E_0 - H_{ED} - A_{ED} V^\dagger_{ED}) \tilde{\Phi}_P = 0 . \tag{14}$$

The adjoint function $\tilde{\Phi}_P$ must be introduced because the operator $V_{ED} A^\dagger_{ED}$ is not Hermitian. This leads to an expression for the decay width which is identical to Eq. (1) except that the Φ_P's in the bras must be replaced by $\tilde{\Phi}_P$'s and the parent state wave functions are solutions of Eqs. (13) and (14) instead of Eq. (9):

$$\Gamma = \frac{\mu_{ED} k}{(2\pi\hbar^2)^2} \int d\hat{k} \langle \tilde{\Phi}_P \,|\, Q_{ED} V_{ED} A^\dagger_{ED} \,|\, \Phi_E \Phi_D \psi(\vec{k}, \vec{r}_{ED}) \rangle \times \frac{\langle \Phi_E \Phi_D \tilde{\psi}(\vec{k}, \vec{r}_{ED}) \,|\, V_{ED} A^\dagger_{ED} Q_{ED} \,|\, \Phi_P \rangle}{\langle \tilde{\Phi}_P \,|\, Q_{ED} \,|\, \Phi_P \rangle} . \tag{15}$$

The difference in definition of the parent state has an interesting consequence when the attempt is made to convert the matrix elements that appear in the decay width expression to surface integrals. The attempt in Ref. 1 was only partly successful in that we found a volume integral remainder persisting after making the transformation to surface integrals. With the new definition of Φ_P the transformation is complete:

$$\begin{aligned} N(\vec{k}) &= \langle \tilde{\Phi}_P \,|\, Q_{ED} V_{ED} A^\dagger_{ED} \,|\, \Phi_E \Phi_D \psi(\vec{k}, \vec{r}_{ED}) \rangle \\ &= \langle \tilde{\Phi}_P \,|\, H_{ED} - H^\dagger_{ED} \,|\, \Phi_E \Phi_D \psi(\vec{k}, \vec{r}_{ED}) \rangle \\ &= \frac{\hbar^2}{2\mu_{ED}} \langle \tilde{\Phi}_P \,|\, \overrightarrow{\nabla}_{ED}{}^2 - \overleftarrow{\nabla}_{ED}{}^2 \,|\, \Phi_E \Phi_D \psi(\vec{k}, \vec{r}_{ED}) \rangle , \end{aligned} \tag{16}$$

where we have used Eqs. (3), (14), and the fact that $Q_{ED} = 1 - P_{ED}$. Similarly,

$$\begin{aligned} M(\vec{k})^* &= \langle \Phi_E \Phi_D \tilde{\psi}(\vec{k}, \vec{r}_{ED}) \,|\, V_{ED} A^\dagger_{ED} Q_{ED} \,|\, \Phi_P \rangle \\ &= \langle \Phi_E \Phi_D \tilde{\psi}(\vec{k}, \vec{r}_{ED}) \,|\, H^\dagger_{ED} - H_{ED} \,|\, \Phi_P \rangle \\ &= \frac{\hbar^2}{2\mu_{ED}} \langle \Phi_E \Phi_D \tilde{\psi}(\vec{k}, \vec{r}_{ED}) \,|\, \overrightarrow{\nabla}_{ED}{}^2 - \overleftarrow{\nabla}_{ED}{}^2 \,|\, \Phi_P \rangle , \end{aligned} \tag{17}$$

where we have used Eqs. (4), (13), and the fact that $Q_{ED} = 1 - P_{ED}$.

Next we use Green's theorem and find

$$N(\vec{k}) = \frac{\hbar^2 R^2}{2\mu_{ED}} \int d\hat{r}_{ED} \left(\psi \frac{\partial}{\partial r_{ED}} \tilde{\xi}^* - \tilde{\xi}^* \frac{\partial}{\partial r_{ED}} \psi \right)_{r_{ED} = R} , \tag{18}$$

$$\tilde{\xi}(\vec{r}_{ED}) = (\Phi_E \Phi_D \,|\, \tilde{\Phi}_P \rangle , \tag{19}$$

$$M(\vec{k})^* = \frac{\hbar^2 R^2}{2\mu_{ED}} \int d\hat{r}_{ED} \left(\tilde{\psi}^* \frac{\partial}{\partial r_{ED}} \xi - \xi \frac{\partial}{\partial r_{ED}} \tilde{\psi}^* \right)_{r_{ED} = R} , \tag{20}$$

$$\xi(\vec{r}_{ED}) = (\Phi_E \Phi_D \,|\, \Phi_P \rangle . \tag{21}$$

The decay width is thus just

$$\Gamma = \frac{\mu_{ED} k}{(2\pi\hbar^2)^2} \int \frac{d\hat{k} N(\vec{k}) M(\vec{k})^*}{\langle \tilde{\Phi}_P \,|\, Q_{ED} \,|\, \Phi_P \rangle} , \tag{22}$$

where N and M^* require only the values and slopes of ψ, $\tilde{\psi}$, ξ, and $\tilde{\xi}$ at the radius R, where R must be greater than the range of $(\Phi_E \Phi_D \,|\, V_{ED} \,|\, \Phi_E \Phi_D)$. This expression would be identical with the one given by Mang[2] if we let $\Phi_P = \tilde{\Phi}_P$ be an eigenstate of $H = H_{ED} + V_{ED}$ and if we let $\psi = \tilde{\psi}$ be the resonating group method wave function for the $E + D$ channel.

For light nucleus decay we might seek to calculate the required wave functions from their definitions. This is not feasible for heavy nucleus decay. In the latter case one would like to be able to relate ξ and $\tilde{\xi}$ to a shell model wave function for the parent and to relate ψ and $\tilde{\psi}$ to an optical model wave function for $D + E$ scattering.

The Schrödinger type equations we encounter in this analysis for Φ_P and $\tilde{\Phi}_P$ are single-partition cases of equations of the coupled-equations nuclear reaction formalism. These equations are discussed in Ref. 3. The transition amplitude for the $E + D \to E' + D'$ elastic or inelastic scattering reaction is

$$T_{c'c} = \langle \Phi_{E'} \Phi_{D'} e^{i\vec{k}' \cdot \vec{r}_{ED}} \,|\, V_{ED} A^\dagger_{ED} \,|\, \Psi_{ED} \rangle N_{ED} , \tag{23}$$

where

$$(E - H_{ED} - V_{ED}) \Psi_{ED} = 0 , \tag{24}$$

$$(E - H_{ED}) \Phi_{E'} \Phi_{D'} e^{i\vec{k}' \cdot \vec{r}_{ED}} = 0 , \tag{25}$$

$$\Psi_{ED} \to \Phi_E \Phi_D e^{i\vec{k} \cdot \vec{r}_{ED}} + \text{outgoing waves.} \tag{26}$$

In Ref. 3 it is shown that $T_{c'c}$ is related to the solutions of Eqs. (13) and (14) by

$$T_{c'c} = \langle \Phi_E, \Phi_D, e^{i\vec{k}'\cdot\vec{r}_{ED}} \,|\, V_{ED} A^{\dagger}_{ED} \,|\, \Phi_P \rangle N_{ED}$$
$$= \langle \Phi_E, \Phi_D, e^{i\vec{k}'\cdot\vec{r}_{ED}} \,|\, V_{ED} A^{\dagger}_{ED} \,|\, \tilde{\Phi}^*_P \rangle N_{ED} \qquad (27)$$

where Φ_P and $\tilde{\Phi}^*_P$ fulfill the same asymptotic boundary conditions as Ψ_{ED}. It would appear that the antisymmetric parts of Ψ_{ED}, Φ_P, and $\tilde{\Phi}^*_P$ are very similar inside the range of V_{ED}.

The equations for $\Phi_{ED} = \Phi_E \Phi_D \psi$ and $\tilde{\Phi}_{ED} = \Phi_E \Phi_D \tilde{\psi}$, Eqs. (3) and (4), are those you get by operating on the equations for Φ_P and $\tilde{\Phi}_P$ by the projection operator P_{ED}.

At the energy $E \approx E_0'$ there is a sharp resonance in the transition amplitude $T_{c'c}$. This results from Ψ_{ED} being dominated by a large compound nucleus configuration at this energy. Thus if we write

$$\Psi_{ED} = \Psi_{ED}(\mathrm{CN}) + \Psi_{ED}(\mathrm{DI}) , \qquad (28)$$

separating the exact scattering wave function into a compound nucleus part and a direct interaction part, then at $E \approx E_0$ the $\tilde{\Psi}_{ED}(\mathrm{CN})$ term will be dominant. This part might well be approximated by a shell model wave function $\Phi_P(\mathrm{SM})$. Thus at $E \approx E_0$ we might set

$$\Psi_{ED} \approx \Phi_P \approx \tilde{\Phi}^*_P \approx \Psi_{ED}(\mathrm{CN}) \approx \Phi_P(\mathrm{SM}) . \qquad (29)$$

There will be a nonresonant background term in $T_{c'c}$ due to $\Psi_{ED}(\mathrm{DI})$. This term must be such that the shape elastic scattering amplitude is

$$T^{(\mathrm{shape})}_{cc} = \langle \Phi_E \Phi_D e^{i\vec{k}'\cdot\vec{r}_{ED}} \,|\, V_{ED} A^{\dagger}_{ED} \,|\, \Psi_{ED}(\mathrm{DI}) \rangle N_{ED} . \qquad (30)$$

The optical model wave function $\Psi_{ED}(\mathrm{OM}) = \Phi_E \Phi_D \chi(\vec{k}, \vec{r}_{ED})$ which serves to represent the shape elastic scattering is such that

$$T^{(\mathrm{shape})}_{cc} \approx \langle e^{i\vec{k}'\cdot\vec{r}_{ED}} \,|\, U(\vec{r}_{ED}) \,|\, \chi(\vec{k}, \vec{r}_{ED}) \rangle , \qquad (31)$$

$$\left[k^2 + \nabla_{ED}{}^2 - \frac{2\mu_{ED}}{\hbar^2} U(\vec{r}_{ED}) \right] \chi(\vec{k}, \vec{r}_{ED}) = 0 . \qquad (32)$$

We would like to identify the optical model relative motion wave function χ with the functions ψ and $\tilde{\psi}$ which appear in our expression for the decay width. From Eqs. (3) and (23) we can see that this is not possible because the corresponding relations for ψ are

$$T^{(\mathrm{shape})}_{cc} \approx \langle e^{i\vec{k}'\cdot\vec{r}_{ED}} \,|\, \tilde{U}(\vec{r}_{ED}) \,|\, \psi(\vec{k}, \vec{r}_{ED}) \rangle N_{ED} , \qquad (33)$$

$$\left[k^2 + \nabla_{ED}{}^2 - \frac{2\mu_{ED}}{\hbar^2} \tilde{U}(\vec{r}_{ED}) \right] \psi(\vec{k}, \vec{r}_{ED}) = 0 \qquad (34)$$

where $\tilde{U}(\vec{r}_{ED})$ is the nonlocal operator

$$\tilde{U}(\vec{r}) \psi(\vec{k}, \vec{r}) = \langle \Phi_E \Phi_D \delta(\vec{r} - \vec{r}_{ED}) | V_{ED} A^{\dagger}_{ED} | \Phi_E \Phi_D \psi(\vec{r}_{ED}) \rangle . \qquad (35)$$

Presumably, the operator $\tilde{U}$ can be approximated by a local potential operator. Because of the factor N_{ED} in Eq. (33), the optical potential $\tilde{U}$ (we use the same symbol to represent the operator and its local optical potential approximation) will be much weaker than U, and the approximate ψ will be much less distorted than χ. Thus the optical model approximation for ψ will be something rather different from the usual optical model wave function.

We have noted that our expression for the decay width is identical to that given by Mang[1] except that our wave functions Φ_P, $\tilde{\Phi}_P$, ψ, and $\tilde{\psi}$ have special definitions. Like Mang we conclude that Φ_P and $\tilde{\Phi}_P$ should be approximated by a shell model wave function for the parent nucleus. However, for ψ and $\tilde{\psi}$ we are led to a choice different from that of Mang. Essentially, we require a much shallower optical potential than the customary one. Thus our predicted width will be much larger than that predicted by Mang.

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[2]H. J. Mang, Phys. Rev. 119, 1069 (1960).
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