Pole in $k\cot\delta$ for doublet, s-wave, n-d scattering

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The position of the pole in $k\cot\delta$, for doublet, s-wave, n-d scattering, and its residue are shown to be correlated with the doublet scattering length. An approximate, analytic solution of the N/D equations of Barton and Phillips indicates a linear dependence on the doublet scattering length for the pole position, and a quadratic dependence for the residue. These relationships are tested by means of exact numerical solutions of N/D equations and three-particle equations with separable two-particle interactions, and found to be qualitatively correct. The approximate, analytic solution of the N/D equations leads to a formula for $k\cot\delta$, which is of the same form as the phenomenological formula used previously by other authors. A formalism is presented which makes it possible to parametrize the effect of the omitted portion of the left hand cut in an N/D calculation.

[NUCLEAR REACTIONS Pole in n-d, doublet, s-wave $k\cot\delta$, N/D calculations; solutions of three-particle equations with separable interactions.]

I. INTRODUCTION

The earliest separable potential calculations on the three-nucleon system indicated that there are strong correlations among the low energy observables; correlations in the sense that a calculation which gives good agreement for one observable, gives good agreement for other observables. In the calculations of Aaron, Amado, and Yam,[1] and of Phillips,[2] a parameter was adjusted to give agreement with the experimental value of the doublet scattering length a, and it was found that a good value for the triton energy automatically resulted. Later calculations by Phillips[3] indicated that there is an approximately linear relationship between the triton energy and a; a result which has been confirmed in great detail by Brady *et al.*[4]

Using dispersion relations, Barton and Phillips[5] found that the low energy values of the doublet, s-wave, phase shift δ are determined mainly by a, and the single-nucleon exchange cut. The position of this branch cut and its discontinuity are determined by the binding energy of the deuteron, and the asymptotic normalization of the deuteron wave function. These parameters are known from the two-nucleon system, which implies that the low energy variation of δ is determined mainly by a. This conclusion is strengthened by the detailed N/D calculations of Avishai, Ebenhöh, and Rinat.[6]

Using his three-particle boundary condition model, Brayshaw[7] has also found strong correlations among three-nucleon observables. He has found that even the low energy breakup cross sections are closely correlated with a.

It has been known for some time that the doublet, s-wave effective range quantity $k\cot\delta$ has a pole in it, just below the elastic threshold. This pole has been incorporated in the phenomenological formula, which has been used to parametrize the low energy variation of $k\cot\delta$.[8,9] Here we shall show that the position of this pole and its residue are closely correlated with a. This will be demonstrated by means of an approximate N/D calculation, and by exact solutions of three-particle equations with separable interactions.

The N/D formalism that we shall use, makes it possible to include as much of the left hand cut as desired, and to parametrize, in a systematic way, the effect of the neglected portion of the cut. In its simplest version the formalism corresponds to the equations, which Barton and Phillips[5] solved by means of the approximate analytical method of Pagels.[10] Here we shall present another approximate analytical solution of the same equations, as well as exact numerical solutions. It is interesting that our approximate solution of the N/D equations gives a formula for $k\cot\delta$ which is identical in form to the phenomenological formula referred to above.[8,9] Furthermore, our formula predicts an approximately linear dependence of the position of the pole in $k\cot\delta$ on a, and a quadratic dependence on a, of the residue. Our exact numerical calculations give reasonable agreement with these predictions.

The N/D formalism we use is presented in Sec. II and our approximate analytic solution of the N/D equations is obtained. In Sec. III a comparison is made of the approximate solution of the N/D equations, the exact numerical solution of these equations, and the exact numerical solution of the three-particle equations with separable interactions. Section IV gives a summary and discussion of the results.

II. N/D FORMALISM

Here we shall only be concerned with the doublet, s-wave, elastic amplitude for n-d scattering. This amplitude is given by

$$\tilde{f}=e^{i\delta(k)}\sin\delta(k)/k\ , \tag{1}$$

where δ is the phase shift and k is the wave number in the c.m. frame. The total three-body energy E, the deuteron binding energy α^2, and k are related by the on-shell condition

$$E=-\alpha^2+\tfrac{3}{4}k^2\ . \tag{2}$$

We are working in units in which $\hbar^2$ divided by the nucleon mass is one. Following Barton and Phillips,[5] we introduce a dimensionless energy variable and amplitude by means of the relations

$$z=3k^2/(4\alpha^2)\ , \tag{3}$$

$$f(z)=e^{i\delta}\sin\delta/z^{1/2}\ . \tag{4}$$

In general,[5,6] this amplitude has a right hand branch cut due to two- and three-particle unitarity, and a left hand branch cut associated with the exchange of nucleons and mesons. The cut structure is indicated in Fig. 1. Here we shall ignore the three-particle unitarity cut; i.e., we shall assume that the phase shift is real for all positive energies. The justification for this is that we are only interested in studying the amplitude very close to the elastic threshold. The inelasticity could be included by means of the Frye-Warnock equations.[6,11] Taking δ to be real, we have from Eq. (4),

$$\mathrm{Im}f^{-1}(z)=-z^{1/2}\quad(z\geq 0)\ . \tag{5}$$

The discontinuity across the single-nucleon exchange cut can be calculated from the Born term that arises in the three-particle formalism of Alt, Grassberger, and Sandhas.[12] The on-shell Born term is given by[13]

$$\tilde{f}_B=\frac{\pi}{6}\int_{-1}^{1}dx\,\frac{g(|\tfrac{1}{2}\vec{k}'+\vec{k}|)g(|\tfrac{1}{2}\vec{k}+\vec{k}'|)}{\alpha^2+(\tfrac{5}{4})k^2+\vec{k}'\cdot\vec{k}}, \tag{6}$$

$$x=\vec{k}'\cdot\vec{k}/k^2\ ,$$

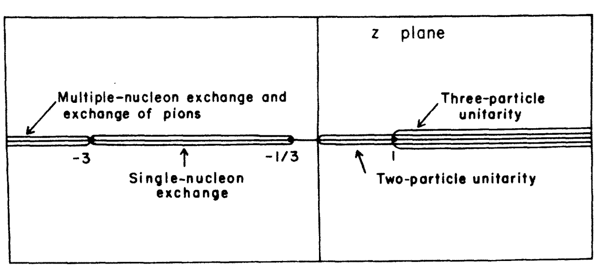

FIG. 1. Cut structure for elastic, n-d, partial wave, scattering amplitudes.

where g is related to the spatial part of the deuteron wave function $|B\rangle$ by[13]

$$g(p)=-(\alpha^2+p^2)\langle\vec{p}\,|B\rangle\sqrt{4\pi}\ , \tag{7}$$

and has the normalization

$$g^2(i\alpha)=4\alpha/[\pi(1-\alpha\rho)]\ . \tag{8}$$

Here ρ is the effective range for the deuteron. In general, of course, the Born amplitude depends on the deuteron wave function; however, its discontinuity across the single-nucleon exchange cut, which comes from the vanishing of the denominator in Eq. (6), depends only on α and ρ and is given by

$$[f_B(z+i\epsilon)-f_B(z-i\epsilon)]/(2i)$$
$$=\mathrm{Im}f_B(z+i\epsilon)=\pi/[\sqrt{3}\,(1-\alpha\rho)z],\quad -3\leq z\leq-\tfrac{1}{3}\ . \tag{9}$$

As is well known,[5,6] an amplitude that satisfies Eq. (5), and has the left hand cut structure of Fig. 1 can be written in the form

$$f(z)=N(z)/D(z)\ , \tag{10}$$

where N and D satisfy the equations

$$N(z)=\frac{1}{\pi}\int_{-\infty}^{-1/3}dy\,\frac{D(y)\mathrm{Im}f(y)}{y-z}\ , \tag{11}$$

$$D(z)=1-\frac{z}{\pi}\int_{0}^{\infty}dy\frac{N(y)}{y^{1/2}(y-z)}\ . \tag{12}$$

Upon substituting Eq. (11) into Eq. (12), we find that D can be obtained from the equation,

$$D(z)=1-\frac{iz^{1/2}}{\pi}\int_{-\infty}^{-1/3}\frac{dy\,D(y)\mathrm{Im}f(y)}{y^{1/2}(z^{1/2}+y^{1/2})}\ . \tag{13}$$

Equation (12) implies that N can be obtained from D by means of the relation

$$N(z)=-z^{-1/2}\mathrm{Im}D(z)\quad(z\geq 0)\ . \tag{14}$$

From Eq. (13) it follows that $D(z)$ is an analytic function in the $z^{1/2}$ plane, except for a cut along the negative imaginary axis, beginning at $-i/\sqrt{3}$.

Since, in general, it is not practical to include all of the left hand cut, it is desirable to have a systematic way of parametrizing the effect of the omitted portion of the cut. This can be done by breaking the integral in Eq. (13) into an integral on the range $-\infty$ to $-b$, say, and one on the range $-b$ to $-\frac{1}{3}$. The integral on the range $-\infty$ to $-b$ can be expanded as a power series in $z^{1/2}$, which will converge when $|z|<b$. Thus we can replace Eq. (13) by

$$D(z)=1+\sum_{n=1}^{\infty}c_n(-z)^{n/2}$$
$$-\frac{iz^{1/2}}{\pi}\int_{-b}^{-1/3}\frac{dy\,D(y)\mathrm{Im}f(y)}{y^{1/2}(z^{1/2}+y^{1/2})}\ , \tag{15}$$

where the c_n's are to be taken as adjustable parameters. It is easy to verify that they are real.

The equations solved by Barton and Phillips[5] are recovered if b is taken to be 3, and only the $n=1$ term in the series is retained. We shall now derive an approximate analytic solution of this truncated equation. We write the equation in the form

$$D(z)=1-ic_1z^{1/2}+\frac{z^{1/2}}{2\pi}\oint_C\frac{dy\,D(y)f_B(y)}{y^{1/2}(z^{1/2}+y^{1/2})},\qquad(16)$$

where the closed contour C goes around the single-nucleon exchange cut in the counterclockwise sense. If we take g in Eq. (6) to be the constant value given by Eq. (8) we obtain

$$f_B(z)=\frac{2}{\sqrt{3}\,(1-\alpha\rho)z}\,Q_0\left(\frac{5z+3}{4z}\right),\qquad(17)$$

where Q_0 is an associated Legendre function of the second kind.[14] It possesses an expansion of the form[14]

$$Q_0(\zeta)=\sum_{n=0}^{\infty}\frac{1}{2n+1}\,\frac{1}{\zeta^{2n+1}}\quad(|\zeta|>1).\qquad(18)$$

Since we are only interested in small z (large ζ), we shall approximate Eq. (17) by

$$f_B(z)=\frac{d}{z+\frac{3}{5}},\quad d=\frac{8}{5\sqrt{3}(1-\alpha\rho)}.\qquad(19)$$

Substituting this approximation into Eq. (16), we find

$$D(z)=1-ic_1z^{1/2}+\frac{dz^{1/2}D(-\frac{3}{5})}{\sqrt{\frac{3}{5}}\,(z^{1/2}+i\sqrt{\frac{3}{5}})},\qquad(20)$$

where $D(-\frac{3}{5})$ can be found in terms of c_1 by simply setting $z=-\frac{3}{5}$ in the equation. By combining Eqs. (4), (10), and (14) it is easy to show that

$$z^{1/2}\cot\delta=\frac{\mathrm{Re}D(z)}{N(z)}=-z^{1/2}\,\frac{\mathrm{Re}D(z)}{\mathrm{Im}D(z)},\quad(z>0).\qquad(21)$$

Using this relation, the unknown parameter c_1 can be eliminated in favor of the doublet scattering length a. When this is done, we obtain the result

$$z^{1/2}\cot\delta=\left(1-\frac{z}{z_0}\right)^{-1}\times\left\{-\frac{1}{\beta}-\frac{5z}{3\beta}\left[1-\sqrt{\tfrac{3}{5}}\,\frac{\beta}{z_0}\,(z_0+\tfrac{3}{5})\right]\right\},\qquad(22)$$

where

$$\beta=2\alpha a/\sqrt{3},\qquad(23)$$

and

$$z_0=-\frac{[1+(\sqrt{\frac{5}{3}}\,d/2)](3\beta/5)}{(5d/3)+[1-(\sqrt{\frac{5}{3}}\,d/2)]\beta}.\qquad(24)$$

This expression for $z^{1/2}\cot\delta$ is of the same form as the phenomenological formula referred to above.[8,9] In particular there is a simple pole at $z=z_0$, and in our approximation, the pole position is simply related to the doublet scattering length by means of Eqs. (23) and (24).

If we take $\alpha=0.2316\ \mathrm{fm}^{-1}$ and $\rho=1.701$ fm [see Sec. III], then from Eq. (19) we find $d=1.524$, which is approximately $2\sqrt{\frac{3}{5}}=1.549$. Thus for reasonable values of β, the second term in the denominator of Eq. (24) is negligible compared with the first. If we approximate d everywhere by $2\sqrt{\frac{3}{5}}$ and use Eq. (3), we find that

$$k\cot\delta\approx\left(1-\frac{k^2}{k_0^{\,2}}\right)^{-1}\left[-\frac{1}{a}+\frac{5k^2}{2a\alpha^2}\left(\frac{\alpha a}{\sqrt{5}}-1\right)\right],\qquad(25)$$

$$k_0^{\,2}\approx-\frac{8}{5\sqrt{5}}\,\alpha^3a,\qquad(26)$$

$$\text{residue}=\lim_{k^2\to k_0^2}(k^2-k_0^{\,2})k\cot\delta\approx-\frac{8\alpha^3}{5\sqrt{5}}\left(1-\frac{2\alpha a}{\sqrt{5}}\right)^2.\qquad(27)$$

Thus our approximate solution of the N/D equations suggests that the position of the pole in $k\cot\delta$ varies linearly with a, and the residue varies quadratically. In the next section we shall present results of exact numerical calculations, which were carried out to test these implications of our simple formula.

III. NUMERICAL RESULTS

In this section we present the results of two sets of numerical calculations; one set is based on the N/D formalism of Sec. II, and the other set consists of exact solutions of the three-particle equations that arise when separable potentials are used to describe the two-nucleon interaction. In all of the calculations we took $\alpha=0.2316\ \mathrm{fm}^{-1}$ and $\rho=1.701$ fm. This insured that all of our exact numerical calculations produced amplitudes with the same discontinuity across the single-nucleon exchange cut.

The exact numerical solutions of the N/D equations were carried out with only the $n=1$ term retained in Eq. (15), with b set equal to 3 and with $\mathrm{Im}f(z)$ taken from Eq. (9). The integral in Eq. (15) was replaced with a Gauss-Legendre quadrature rule, and the resulting equation was solved by matrix inversion. The parameter c_1 was adjusted to a set of values for the doublet scattering length a, and for each value of a the position of the pole and its residue were determined.

The three-particle equations that were solved, and the method used to solve them are given in

Ref. 13. Two rank one, spin-dependent, s-wave, separable interactions were used; one employed the Gaussian form factor, as in Ref. 13, and the other the standard Yamaguchi form factor.[15] The triplet Gaussian interaction was fitted to a deuteron binding energy of 2.2246 MeV, and a triplet effective range of 1.747 fm. This leads to the values for α and ρ stated above. The Yamaguchi triplet interaction was fitted to the same values of α and ρ. Both singlet interactions were fitted to a singlet scattering length of -23.715 fm. The singlet effective range was allowed to vary from 2.2 to 3.6 fm, so as to sweep out a range of values for the doublet scattering length. Clearly some of these effective ranges are unphysical, but this is irrelevant to the point we are trying to make; namely, that the position of the pole in k cotδ and its residue are determined mainly by the doublet scattering length, and the single-nucleon exchange cut. For each value of the doublet scattering length the position of the pole and its residue were calculated.

The results of the calculations are shown in Fig. 2. The simple formula results were obtained from Eqs. (25)–(27), and all of the residues were calculated as in the first line of Eq. (27). It is seen that the results of the two sets of separable potential calculations almost lie on the same curves. The exact solutions of the N/D equations lie closer to these curves than the simple formula results. Even though there are some discrepancies among the various calculations, it is clear that the qualitative implications of the simple formula appear to be correct; namely, the pole position and residue are closely correlated with the doublet scattering length, and this correlation comes about mainly through the single-nucleon exchange cut.

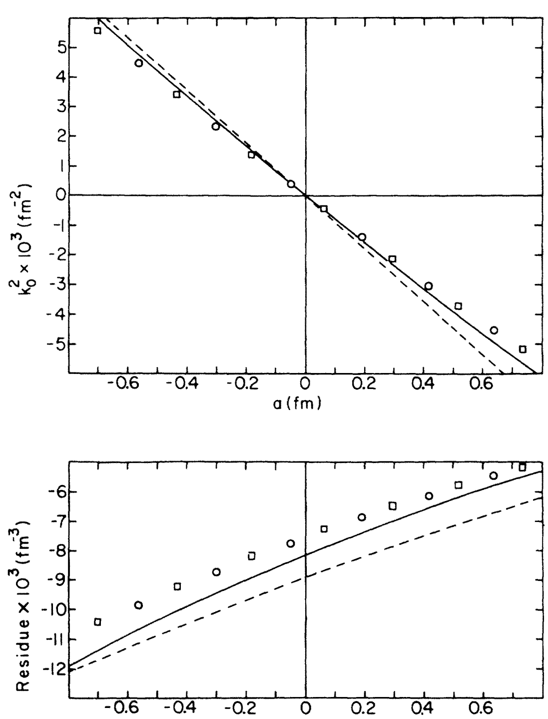

FIG. 2. Pole position and residue as a function of the doublet scattering length. Solid lines are from the exact N/D calculations. Dashed lines are from Eqs. (26) and (27). Circles and squares are from the Yamaguchi and Gaussian separable potential calculations, respectively.

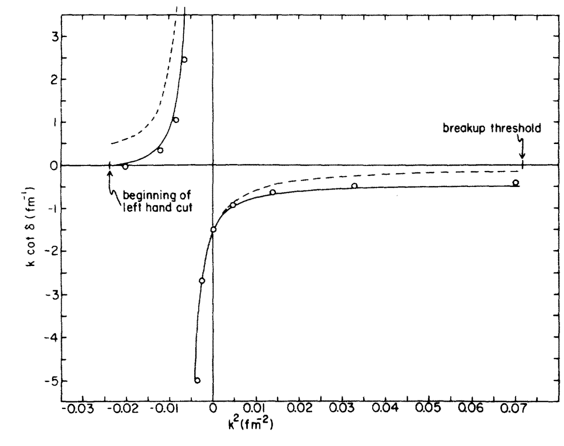

FIG. 3. k cotδ for the doublet, s state as a function of k^2, where k is the relative wave number. Solid lines are from the exact N/D calculations. Dashed lines are from Eq. (25). Circles are for either separable potential. In all cases the doublet scattering length is 0.652 fm.

In Fig. 3 we show the results of four different calculations for k cotδ, all of which were adjusted to give a doublet scattering length of 0.652 fm. There are two separable potential calculations; one with Gaussian form factors (singlet effective range = 2.8625 fm) and one with Yamaguchi form factors (singlet effective range = 3.5190 fm). The two sets of points are indistinguishable, and appear as one in the figure. The N/D calculation is of the type described above, and gives almost the same results as the separable potential calculations. The results of the simple formula are also shown. These calculations strongly support the idea that the low energy values of k cotδ are determined mainly by the doublet scattering length and the single-nucleon exchange cut.

IV. SUMMARY AND DISCUSSION

We have developed a systematic way of parametrizing the effect of the neglected portion of the left hand cut in an N/D calculation of the elastic amplitude for n-d scattering. From the truncated form of this scheme, which corresponds to the equations solved by Barton and Phillips,[5] we have obtained a simple, approximate formula for $k\cot\delta$ in the doublet, s-wave channel. This formula predicts a simple pole in $k\cot\delta$, whose position varies linearly with the doublet scattering length a, and whose residue has a quadratic dependence on a. Exact numerical solutions of the N/D equations, and of the separable potential, three-particle equations give reasonable agreement with these predictions.

It is rather remarkable that the two separable potential calculations give almost identical results for $k\cot\delta$ [see Fig. 3], since their singlet effective ranges are quite different, and moreover the one for the Yamaguchi potential is quite unphysical. It is also surprising that the numerical N/D calculation does so well, since there is a complete neglect of three-particle unitarity, and only one parameter is used to treat the neglected part of the left hand cut. This was already seen by Barton and Phillips.[5]

At the present time, we are studying the two-nucleon exchange cut in order to see how model dependent it is. The triton energy falls just above the junction of the one- and two-nucleon exchange cuts, and it is known[5] that neglect of the two-nucleon exchange cut in an N/D calculation leads to a poor result for the triton energy. We also plan to take into account the inelasticity by means of the Frye-Warnock equations, in order to study its relationship to the other three-nucleon parameters.

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