Normalization and finite-range effects in distorted-wave Born-approximation calculations for $(d, {}^3\mathrm{He})$ and (d, t) reactions*

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Exact finite-range distorted-wave Born-approximation calculations have been carried out for $(d, {}^3\mathrm{He})$ and (d, t) reactions at incident energies up to 80 MeV. It is found that the local energy approximation does not correctly reproduce the energy variation of exact calculations for $^{51}\mathrm{V}(d, {}^3\mathrm{He})^{50}\mathrm{Ti}$ (g.s.) between 30 and 80 MeV and, in addition, overestimates finite-range effects for $^{16}\mathrm{O}(d, {}^3\mathrm{He})^{15}\mathrm{N}$ (g.s.) at 80 MeV. The effects of various choices for the range function for (d, t) and $(d, {}^3\mathrm{He})$ reactions are discussed. Some new values for the zero range normalization constant are discussed.

[NUCLEAR REACTIONS Normalization and finite-range effects in DWBA. $^{51}\mathrm{V}(d, {}^3\mathrm{He})$, $^{51}\mathrm{V}(d,t)$, $^{16}\mathrm{O}(d, {}^3\mathrm{He})$, $T_d = 30$–80 MeV, calculate $\sigma(\theta)$.]

I. INTRODUCTION

Single-nucleon transfer reactions are widely studied in the hope of obtaining information about nuclear states. Typically, data are analyzed using the distorted-wave Born-approximation theory (DWBA).

For light ion projectiles ($A \leq 4$) most DWBA analyses utilize the so-called zero range (ZR) approximation[1] in which the interaction potential responsible for the transition is approximated by a δ function. This has the advantage of reducing the evaluation of the DWBA transition amplitude from a complicated six-dimensional integration to a much simpler three-dimensional integration. As a result there are considerable computational savings. The problem of evaluating the full integral has been faced for transfer reactions induced by heavy ions.[2] However, surprisingly few exact calculations have been reported for light ion transfer reactions.[3,4] Instead, it is usual to employ an approximate treatment of finite-range effects known as the local energy approximation[5-7] (LEA) which retains the simplicity of the ZR calculation. Tests of the validity of the LEA have been made by means of comparisons with exact finite-range (EFR) calculations for a few selected transitions.[6] In addition a higher order correction term (though not *all* such terms) has been shown to be small for certain cases.[7]

Clearly the LEA has acquired legitimacy largely by default. Thus, it should not be relied upon for transitions in which the kinematic conditions or optical potentials differ appreciably from the examples considered in Refs. 6 and 7. Such is the case for the $(d, {}^3\mathrm{He})$ reaction which has recently been studied at incident energies up to 80 MeV.[8,9]

In the present paper our purpose is threefold. Firstly, we wish to present a formulation of the EFR DWBA amplitude for light ion induced single-nucleon transfer reactions. Our expression is simpler to evaluate than that given in Ref. 3, is easily coded for the computer, and leads to quite rapid computations. Secondly, we present specific results for the $(d, {}^3\mathrm{He})$ reaction at or near energies corresponding to the data of Refs. 8 and 9. For selected transitions EFR calculations are compared with LEA results in order to investigate the validity of the latter approach. Detailed EFR analyses of the data have been presented elsewhere.[8,9]

Finally we investigate the effect of different choices for the deuteron and trinucleon internal wave functions and for the deuteron-nucleon interaction responsible for the transition in the EFR calculations for (d,t) and $(d, {}^3\mathrm{He})$ reactions. The effect of these choices upon the angular variation and the absolute magnitude of the predicted cross sections is discussed. Possible limitations of the treatment of the finite-range effects given by Bassel[10] are investigated.

II. FORMULATION

Exact formulas for finite-range DWBA calculations of transfer reactions have already been given by Austern and others.[3] We here present new expressions which are precisely equivalent to the expressions given in Ref. 3 but which employ a different coordinate system. The result is a somewhat simpler final expression which is convenient to program for the computer.

We consider a pickup reaction $A(a,b)B$. Then it may be shown[11] that the differential cross section is given by

$$\frac{d\sigma}{d\Omega} = \frac{\mu_a \mu_b}{(2\pi\hbar^2)^2} \frac{k_b}{k_a} b_{ST}^2 (T_B N_{B\frac{1}{2}T} | T_A N_A)^2 \sum_{LJ\Lambda} S_{LJ} | B_{L\Lambda} |^2 , \tag{1}$$

where, for simplicity, spin orbit coupling has been omitted. The quantities μ_i and k_i are, respectively, the reduced mass and wave number of particle i. The target has isospin T_A, projection N_A, and the residual nucleus isospin quantum numbers are defined similarly. The transferred nucleon has total angular momentum J, orbital angular momentum L (projection Λ), and isospin projection τ. The quantity S_{LJ} is the usual single-nucleon spectroscopic factor[12] and $b_{ST}{}^2$ is related to the overlap of the light particles a and b. In an isospin representation it is $\frac{3}{4}$ for (p,d) reactions and $\frac{1}{2}$ for $(d,{}^3\mathrm{He})$ and $({}^3\mathrm{He},\alpha)$ reactions.

The finite-range problem arises in evaluating $B_{L\Lambda}$. Specifically

$$B_{L\Lambda} = (2L+1)^{-1/2} \int\int \chi_b^{(-)*}(\vec{r}_b) V(s) Y_{00}(\hat{s}) i^L U_{NLJ}(R) \times Y_{L\Lambda}(\hat{R}) \chi_a^{(+)}(\vec{r}_{aA}) d\vec{s}\, d\vec{r}_b , \tag{2}$$

where the vectors $\vec{r}_a$, $\vec{r}_b$, $\vec{r}_A$, and $\vec{R}$ locate the particles a,b,A, and the transferred nucleon x, each with respect to the residual nucleus B. The vector $\vec{s} = \vec{r}_a - \vec{R}$ and $\vec{r}_{ij} = \vec{r}_i - \vec{r}_j$. The distorted waves $\chi_i^{(\pm)}$ describe the motion of the projectile and emitted light particle. The spatial part of the wave function of the nucleon bound in the target nucleus is $i^L U_{NLJ}(R) Y_{L\Lambda}(\hat{R})$ where $\hat{R}$ is a unit vector along $\vec{R}$. The quantity $V(s)$ we refer to as the "range function." It is defined by

$$V(s) Y_{00}(\hat{s}) = \langle \phi_b | V_{ax} | \phi_a \rangle , \tag{3}$$

where ϕ_a and ϕ_b are internal spatial wave functions for the respective particles. Each is assumed to be symmetric and to carry zero orbital angular momentum. The quantity V_{ax} is the effective interaction between the projectile and transferred nucleon. It will be noted that an additional term $V_{aB} - U_{aA}$ involving the interaction between a and B, V_{aB}, and the entrance channel optical potential U_{aA} has been omitted. This has been shown to be a reasonable procedure for (d,p) reactions.[4] In the present paper we shall assume that this is also the case for all light ion induced single-nucleon transfer reactions. We shall return to the details of the evaluation of Eq. (3) later. For the present we assume that $V(s)$ can be found and proceed to the evaluation of $B_{L\Lambda}$.

Inspection of Eq. (3) reveals that it is identical in form to an expression given by Chant in an exact finite-range calculation for two-nucleon transfer.[13] Thus Eq. (11) of Ref. 13 reduces to the present Eq. (2) provided we restrict the summation to $l=0$ $L'=L$ and replace the quantity $F_{0L}(s,R)$ by $V(s)U_{NLJ}(R)$. Proceeding precisely as in Ref. 13 we thus obtain

$$B_{L\Lambda} = \frac{4\pi}{k_a k_b} \sum_{l_a l_b} i^{L+l_a-l_b} \frac{\hat{l}_a \hat{l}_b}{\hat{L}} (L\Lambda l_a 0 | l_b \Lambda) d^{l_b}_{\Lambda 0}(\theta) I_{l_a l_b} , \tag{4}$$

where $\hat{x} = (2x+1)^{1/2}$ and $d^l_{mn}(\theta)$ is a reduced rotation matrix element. The quantity $I_{l_a l_b}$ is given by

$$I_{l_a l_b} = \int_0^\infty U_{l_b}(r_b) G_{l_a l_b}(r_b) r_b dr_b , \tag{5}$$

where

$$G_{l_a l_b} = \int_0^\infty g^{l_b}_{0L l_a l_b}(s, r_b) s^2 ds \tag{6}$$

and where

$$g^{l_b}_{0L l_a l_b}(s, r_b) = \frac{\hat{L}\hat{l}_a}{2\hat{l}_b} \sum_M (LMl_a - M | l_b 0) \times \int_{-1}^{+1} V(s) U_{NLJ}(R) \frac{U_{l_a}(r_{aA})}{r_{aA}} \times d^L_{M0}(\theta_R) d^{l_a}_{M0}(\theta_{aA}) dx . \tag{7}$$

In Eqs. (5) and (7) the quantities U_{l_a} and U_{l_b} are defined by

$$\chi_a^{(+)}(\vec{r}_{aA}) = \frac{4\pi}{k_a r_{aA}} \sum_{l_a \lambda_a} i^{l_a} U_{l_a}(r_{aA}) Y^*_{l_a \lambda_a}(\hat{k}_a) Y_{l_a \lambda_a}(\hat{r}_{aA}) \tag{8}$$

and

$$\chi_b^{(-)*}(\vec{r}_b) = \frac{4\pi}{k_b r_b} \sum_{l_b \lambda_b} i^{-l_b} U_{l_b}(r_b) Y^*_{l_b \lambda_b}(\hat{k}_b) Y_{l_b \lambda_b}(\hat{r}_b) . \tag{9}$$

In evaluating the integral in Eq. (7) the needed geometrical relationships are

$$R = (r_b{}^2 + \alpha^2 s^2 + 2\alpha r_b s x)^{1/2} , \tag{10}$$

$$r_{aA} = (\gamma^2 r_b{}^2 + \mu^2 s^2 + 2\gamma\mu r_b s x)^{1/2} , \tag{11}$$

$$\cos\theta_R = \frac{r_b + \alpha s x}{R} , \tag{12}$$

$$\cos\theta_{aA} = \frac{\gamma r_b + \mu s x}{r_{aA}} , \tag{13}$$

where $\alpha = -a/b$, $\gamma = B/A$, and $\mu = (b-a)(A+a)/Ab$.

On inspection Eqs. (6) and (7) are found to be identical in form with the expressions given by Bayman and Kallio[14] in calculating zero range form factors for two-nucleon transfer reactions. The calculation of each $G_{l_a l_b}$ is thus equivalent in difficulty to the evaluation of the Bayman-Kallio form factor for two-nucleon transfer for a transition in which the nucleons originate from orbits carrying angular momenta L and l_a, respectively, and transfer total orbital angular momentum l_b. Such a calculation is quite feasible and it is convenient to employ Gauss-Legendre integration in

evaluating the angular integration in Eq. (7) and Gauss-Hermite integration in the radial integration of Eq. (6). Unlike Ref. 14 we are unable to give a prescription for choosing the number of integration points. For the $(d,{}^3\mathrm{He})$ calculation at 80 MeV which follows we find empirically that eight points for the angular integration and four points for the radial integration are sufficient to insure better than 1% accuracy in the final cross sections. As a result, the calculation proceeds quite rapidly. For example, an EFR calculation for transferred angular momentum $L=3$ using 50 partial waves takes only ~60% longer than the corresponding ZR calculation using the March 1974 version of the code DWUCK.

III. COMPARISONS OF ZR, LEA, AND EFR CALCULATIONS FOR $(d,{}^3\mathrm{He})$ REACTIONS

A. Calculations

For $(d,{}^3\mathrm{He})$ reactions the problems of normalization and finite-range effects have been discussed previously by Bassel.[10] He chose to approximate the range function by a Gaussian

$$V(s)Y_{00}(\hat{s}) = D_0\left(\frac{1}{\pi R^2}\right)^{3/2}\exp(-s^2/R^2)\,, \tag{14}$$

where $D_0 = -172.8$ MeV fm$^{3/2}$ and $R = 1.54$ fm. Integrating we find

$$\int V(s)Y_{00}(\hat{s})\,d\vec{s} = D_0 \tag{15}$$

so that the equivalent ZR approximation is

$$V(s)Y_{00}(\hat{s}) = D_0\delta(\vec{s})\,. \tag{16}$$

The LEA approximation is discussed extensively elsewhere.[5-7] The procedure involves a Taylor's series expansion of all functions of $\vec{R}$ and $\vec{r}_{aA}$ about $\vec{r}_b$. For the range function defined in Eq. (14) the result is the ZR expression Eq. (16) together with an additional operator

$$\mathcal{O} = \exp[\tfrac{1}{4}R^2(C_a\nabla_a{}^2 + C_x\nabla_x{}^2 + C_b\nabla_b{}^2)] \tag{17}$$

acting upon the wave functions. The quantities C_i are known functions of the particle masses and $\nabla_i{}^2$ acts upon the wave function of particle i. The operator is evaluated by using the series expansion of the exponential together with the Schrödinger equation for each particle. The result is

$$\mathcal{O}(r_b) \approx \exp\left\{\frac{2}{\hbar^2}\frac{m_x m_a}{m_b}\frac{R^2}{4}[U_a(\gamma r_b) - E_a + U_x(r_b) - E_x - U_b(r_b) + E_b]\right\}, \tag{18}$$

where m_i is the mass, U_i is the optical potential, and E_i is the energy of particle i. Using this expression our LEA approximation is

$$V(s)Y_{00}(\hat{s}) = D_0\mathcal{O}(r_b)\delta(\vec{s})\,. \tag{19}$$

Notice that, while a calculation using the expression for the operator given in Eq. (17) would be exact, the expression for $\mathcal{O}(r_b)$ given in Eq. (18) is an approximation in which terms involving $\nabla_i{}^2$ operating on U_i have been neglected in comparison with the derivatives of the wave functions themselves.

In summary our ZR approximation is Eq. (16), our LEA approximation is Eq. (19), and our EFR calculation is defined by Eq. (14). If finite-range effects are indeed small we would expect to obtain identical results in all three cases.

B. Results

The EFR calculation described above has been coded for the computer[15] and calculations have been carried out for $(d,{}^3\mathrm{He})$ transitions at various incident energies. We here consider representative results which are compared with ZR and LEA calculations.

The transition considered is $^{51}\mathrm{V}(d,{}^3\mathrm{He})^{50}\mathrm{Ti}$(g.s.) in which a proton is removed from the $1f_{7/2}$ shell. Optical potentials used in the calculation are listed in Table I and are consistent with elastic scattering data in this region of the Periodic Table.[8] For the proton bound state the potential was adjusted to reproduce the empirical separation energy. Nonlocality ranges[16] used were 0.54 fm for deuterons, 0.2 fm for ^{3}He ions, and 0.85 fm for the bound proton.

Results are shown in Figs. 1–3 for incident energies of 30, 52, and 80 MeV, respectively. For each differential cross section shown we have assumed that C^2S_{LJ}, the product of the isospin coupling Clebsch-Gordan coefficient and the spectroscopic factor appearing in Eq. (1), is unity. Three results emerge. Firstly, the shapes of the calculated cross sections are quite insensitive to finite-range effects, even at 80 MeV. Secondly, the ZR calculation generally underestimates the absolute cross section. Thirdly, the LEA somewhat overestimates finite-range effects and leads to cross sections somewhat larger than the EFR results.

It is clear from the figures that these effects, though small, are not energy independent. In Table II we list differential cross sections at the first peak of the calculated angular distributions each divided into the EFR result at the same energy. The resultant ratios are thus the spectroscopic factors which would result from analyses of hypothetical experimental data identical to the three EFR calculations. We see that, while the EFR necessarily yields an energy independent spectroscopic factor (of unity), the LEA introduces

TABLE I. Optical potentials.

System	E [a]	V	r_0	a	W	W_D	r'_0	a'	r_c	Ref.
$^{51}V+d$	30	96.64	1.061	0.81	···	17.14	1.28	0.751	1.3	24
$^{51}V+d$	52	89.00	1.05	0.852	···	12.40	1.29	0.761	1.3	25
$^{51}V+d$	80	73.00	1.2	0.74	···	12.20	1.14	0.930	1.3	26
$^{16}O+d$	80	78.90	1.05	0.78	···	8.14	1.32	0.86	1.3	27
$^{50}Ti+^{3}He$	30	174.5	1.14	0.723	15.5	···	1.62	0.85	1.3	28
$^{50}Ti+^{3}He$	52	173.5	1.14	0.723	16.5	···	1.62	0.85	1.3	28
$^{50}Ti+^{3}He$	80	158.32	1.2	0.671	···	18.51	1.095	0.942	1.3	29
$^{15}N+^{3}He$	80	145.0	1.34	0.580	15.0	···	1.56	1.15	1.4	27
Bound state [b,c]			1.20	0.65					1.3	
Bound state [b,d]			1.25	0.65					1.25	

[a] Energies are in MeV and lengths in fm. The optical potential is

$$U(r) = -\frac{V}{1+e^{x}} - \frac{iW}{1+e^{x'}} - \frac{4iW_D e^{x'}}{(1+e^{x'})^2} + U_C(r_c),$$

where

$$x = \frac{r - r_0 A^{1/3}}{a}, \quad x' = \frac{r - r'_0 A^{1/3}}{a'},$$

and $U_C(r_c)$ is the Coulomb potential due to a sphere of charge of radius $r_c A^{1/3}$. E is the incident deuteron laboratory energy for the transition in which the listed optical potential is used.

[b] Also includes a spin-orbit potential of 25 times the Thomas term.

[c] Used in ^{51}V calculations.

[d] Used in ^{16}O calculations.

a spurious energy variation of about 7.5%. Roughly twice this energy variation would result from a ZR analysis which, it is seen, would also overestimate the spectroscopic factor by about 20% at the lower energies.

We conclude this section with a warning that, while the above results may apply qualitatively to other $(d, {}^3He)$ analyses, such small finite-range

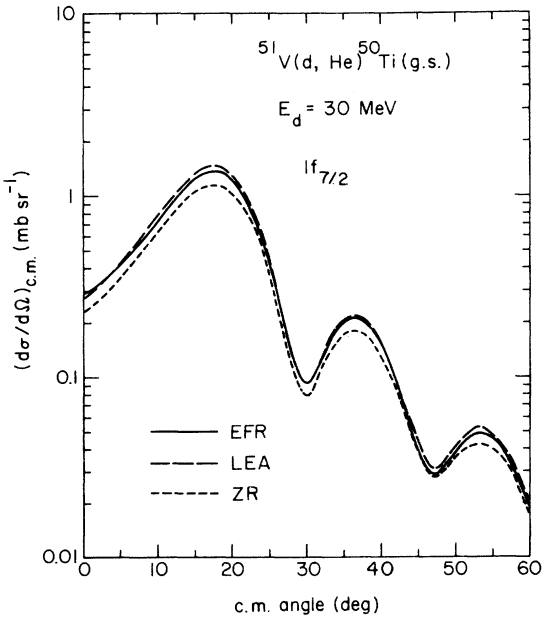

FIG. 1. Comparison of EFR, LEA, and ZR calculations for the $^{51}V(d, {}^3He)^{50}Ti$(g.s.) reaction at 30 MeV.

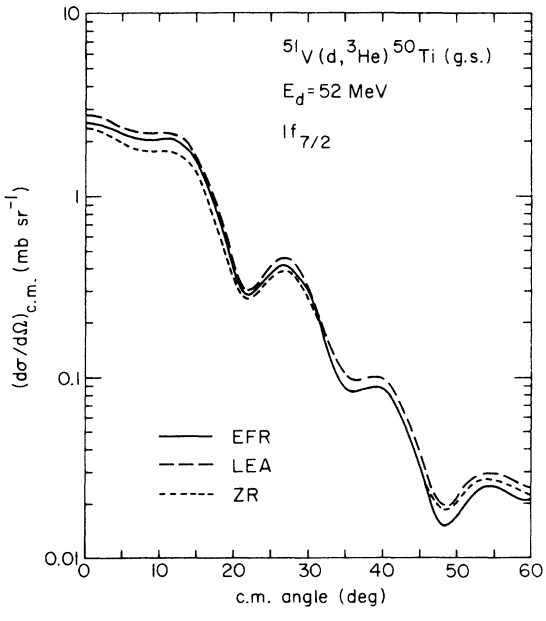

FIG. 2. Comparison of EFR, LEA, and ZR calculations for the $^{51}V(d, {}^3He)^{50}Ti$(g.s.) reaction at 52 MeV.

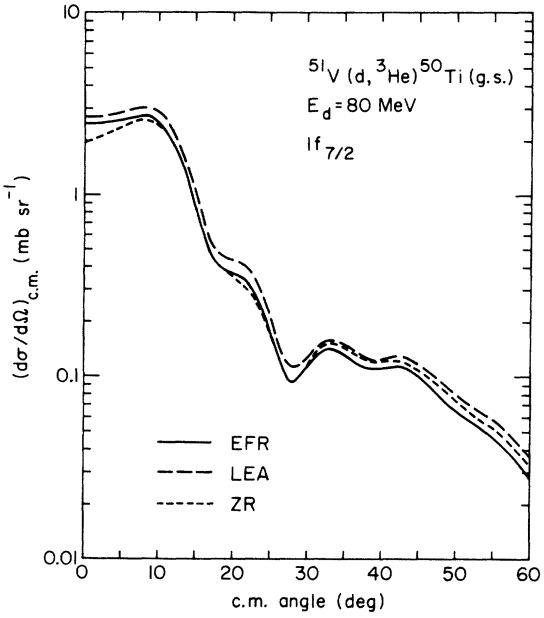

FIG. 3. Comparison of EFR, LEA, and ZR calculations for the ^{51}V(d, ^{3}He)^{50}Ti(g.s.) reaction at 80 MeV.

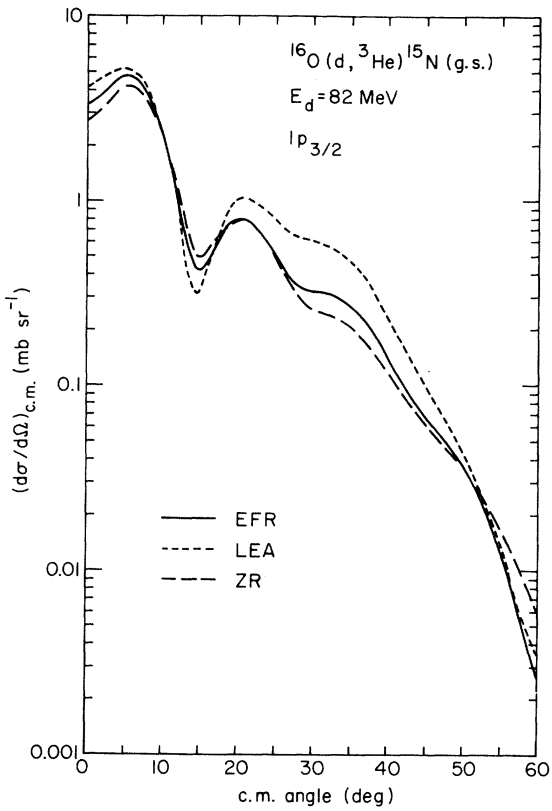

FIG. 4. Comparison of EFR, LEA, and ZR calculations for the ^{16}O(d, ^{3}He)^{15}N(g.s.) reaction at 82 MeV.

effects cannot be guaranteed. Thus, in Fig. 4 we show ZR, LEA, and EFR calculations for the ^{16}O(d, ^{3}He)^{15}N(g.s.) reaction at 82 MeV. In this transition a proton is removed from the $1p_{3/2}$ shell. The optical potentials used are listed in Table I. The same nonlocality ranges were used as before. It is seen that not only do the three calculations differ somewhat in absolute magnitude but also there are large differences in shape between all three calculations. Indeed the LEA overestimates finite-range effects so seriously in this case that the ZR calculation bears a greater resemblance to the EFR results. While this difference in behavior is not understood in detail it is worth noting that the calculations shown in Figs. 1–3 are each matched for $l = 3$ transfer at some nonzero scattering angle. On the other hand, for the calculation shown in Fig. 4 the preferred angular momentum transfer even at zero degrees is slightly greater than $1\hbar$. Thus the transition is mismatched at all angles and owing to the resultant destructive interference effects, can be expected to show an enhanced sensitivity to the treatment of the nuclear interior.

TABLE II. Energy variation of the spectroscopic factor for ^{51}V(d, ^{3}He)^{50}Ti(g.s.). The quantity tabulated is $[d\sigma/d\Omega_{\mathrm{EFR}}(\theta)]/[d\sigma/d\Omega_i(\theta)]$ where i = EFR, LEA, or ZR. The angle θ is 18° at 30 MeV, 11° at 52 MeV, and 8° at 80 MeV.

Calculation	C^2S_{LJ} 30 MeV	52 MeV	80 MeV
EFR	1.00	1.00	1.00
LEA	0.95	0.95	0.88
ZR	1.20	1.17	1.04

Clearly a proper treatment of finite-range effects is an essential ingredient in any analysis of data[9,27] for the latter transition. Thus, it is no surprise that Doubre *et al.*,[27] in a DWBA analysis of their experimental data for ^{16}O(d, ^{3}He)^{15}N at 82 MeV, obtain closest agreement with experiment using a ZR calculation and find that inclusion of finite-range effects in LEA leads to some deterioration in the fit to experiment. In addition, their extracted spectroscopic factor C^2S is 2.8 which is unexpectedly large. In contrast, our EFR calculations, being very similar to the ZR results would reproduce the experimental data quite well and, in addition, would yield a spectroscopic factor of ~2.0 which is the expected $(1p_{1/2})^{-1}$ shell model value. It should be noted that our LEA calculations yield a spectroscopic factor of ~1.83 while our ZR calculation leads to a value of $C^2S \sim 2.23$. This behavior is consistent with an enhancement of the ZR form factor at large radii which, as we shall see in Sec. IV, is to be ex-

pected in LEA. In contrast, Doubre's LEA result of $C^2S = 2.8$ is quite similar to his ZR value ($C^2S = 2.5$). Such a result is consistent with an LEA form factor incorrectly normalized to reproduce the ZR form factor at large radii.

IV. EFFECTS OF DIFFERENT RANGE FUNCTIONS

The effects of different choices for the light particle wave functions and deuteron-nucleon interaction which determine the range functions for $(d, {}^3\mathrm{He})$ and (d,t) reactions have already been investigated in part.[10,17-20] Most calculations merely evaluate the zero range normalization constant D_0 although Goldfarb and co-workers[17-20] use an LEA approach which permits the evaluation of an equivalent finite-range constant D which is argued to be a useful measure of the overall normalization of the cross section at low energies where finite-range effects are negligible.

In comparing their results for the constants D and D_0, Goldfarb *et al.* find that a proper treatment of the long range behavior of the range function is a crucial ingredient of the calculation and, in addition, that quite large differences in absolute magnitude between ZR and EFR calculations are to be expected. As a result of this sensitivity to the long range part of $V(s)$, Goldfarb and Parry[17] argue that Bassel's decision to approximate $V(s)$ obtained from a Hulthén deuteron wave function and an Irving-Gunn trinucleon wave function by a Gaussian may lead to error. This possibility is investigated in Sec. IV A in which, in order to avoid explicit consideration of a possible Coulomb term[10] in the interaction for $(d, {}^3\mathrm{He})$, we have considered only the (d,t) reaction.

In Sec. IV B we turn to the more general problem of the extent to which calculations of D_0^2 for different light particle wave functions and interactions accurately reflect the relative magnitudes of the corresponding EFR calculations. Goldfarb and Parry's arguments that the results should be sensitive to the long range behavior of the corresponding range functions imply that relative values of D_0^2 may be misleading. On the other hand values for D^2/D_0^2 reported in Ref. 20 do not show much variation. Also presented are some new values for D_0^2 together with a discussion of existing results.

A. Accuracy of the Gaussian approximation

We first compare the Gaussian range function of Eq. (14) using Bassel's values for (d,t) of $D_0 = -183.6$ MeV fm$^{3/2}$ and $R = 1.69$ fm with the "exact" result which Bassel hopes to simulate with the Gaussian expression, i.e.:

$$V(s)Y_{00}(\hat{s}) = \int \phi_d(r)(B_t - T_s)\phi_t(r,s)d\vec{\mathbf{r}}\,, \qquad (20)$$

where B_t is the empirical $t \to d+n$ separation energy and T_s is the kinetic energy operator acting on the coordinate s. The expression $(B_t - T_s)$ is the result of using the Schrödinger equation to obtain the interaction V_{dn}. The functions ϕ_d and ϕ_t are, respectively, the Hulthén and Irving-Gunn spatial wave functions used by Bassel. In Fig. 5(a) the two range functions are shown. Although

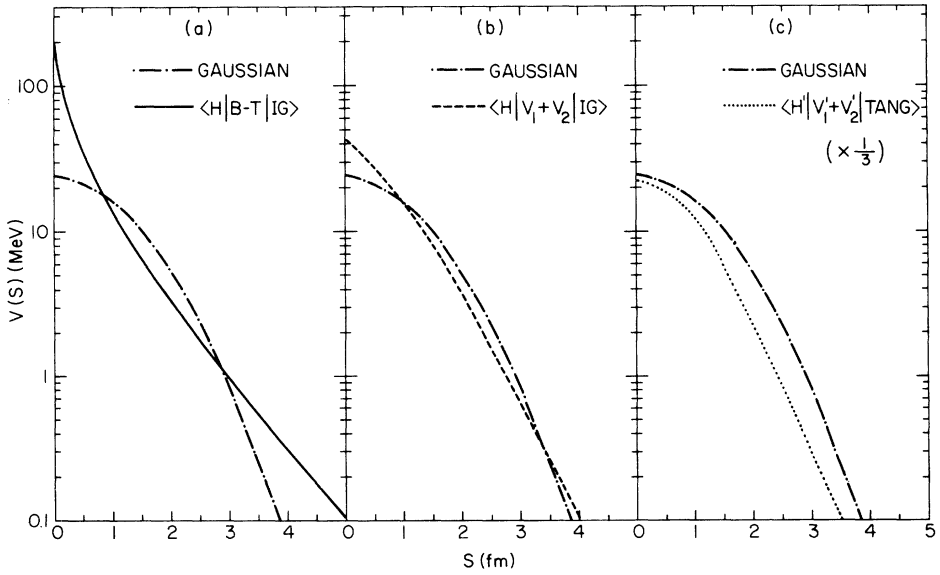

FIG. 5. Range functions for the (d,t) reaction. The notation is identical to that of Table III.

the Gaussian function has been adjusted to reproduce the momentum dependence of Eq. (20) to order K^2, it is clear that the two range functions differ appreciably in detail. Most notable are the differences at radii less than 1 fm and the inevitable difference in asymptotic behavior (radii greater than ~3 fm).

Comparisons between EFR calculations using the two range functions were carried out for the ^{51}V$(d,t)^{50}$V(g.s.) transition at 30, 52, and 80 MeV incident deuteron energy. The transferred neutron was assumed to be picked up from a $1f_{7/2}$ orbit and all optical potentials and nonlocality ranges were identical to those used in the earlier calculations for $(d,{}^3\mathrm{He})$ on the same target. At all three incident energies differences in the angular variation of the two calculations were quite small. Thus, at 30 and 52 MeV the shapes of the predicted cross sections were indistinguishable, while at 80 MeV, as shown in Fig. 6, very minor differences were apparent. Clearly, even at 80 MeV, differences between the two calculations are very small though very precise data might possibly differentiate between them. Differences in absolute magnitude are, however, more significant and are not energy independent. Thus, at 80 MeV, the Gaussian range function underestimates the result obtained using Eq. (20) by about 3% whereas at 30 MeV the Gaussian result is roughly 8% too low.

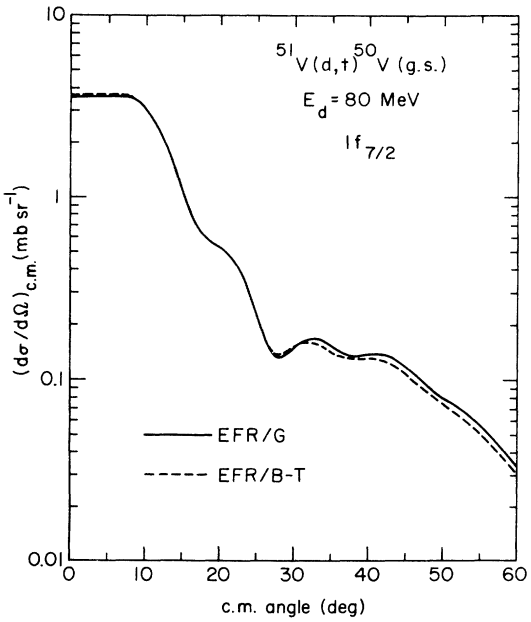

FIG. 6. Comparison of EFR calculations for the ^{51}V-$(d,t)^{50}$V(g.s.) reaction at 80 MeV. The curve denoted by EFR/G results from the use of Bassel's Gaussian range function while EFR/$B-T$ is the result of an "exact" calculation using the range function given in Eq. (20).

Provided the Coulomb part of the d-p interaction may be neglected [Bassel's approximation (b)], quite similar results are to be expected for $(d,{}^3\mathrm{He})$ reactions.

Clearly, for most purposes, the Gaussian approximation is quite adequate. On the other hand the above results indicate that greater care may be necessary in the analysis of very precise data.[8]

B. Alternative wave functions and interaction potentials

In Table III we compare the results of calculations for ^{51}V$(d,t)^{50}$V(g.s.) at 30 and 80 MeV using various simple choices for $V(s)$ including the two range functions discussed in Sec. IV A. In no case were significant changes in shape observed. Consequently, in Table III we present values for $N=\frac{3}{2}D_0^2$ and the ratio $[d\sigma/d\Omega(\mathrm{EFR})]/[d\sigma/d\Omega(\mathrm{ZR})]$ evaluated at the first maximum in the angular distribution. If Goldfarb's LEA approximation is satisfactory the latter ratio should approach D^2/D_0^2 at low energies; where available this ratio is also listed.

On comparing the Gaussian range function with its "exact counterpart" [Eq. (20)], which is denoted by $\langle H|B-T|IG\rangle$, the differing energy dependence of the two calculations discussed in Sec. IV A is quite evident. Significantly, for the second range function, Goldfarb and Parry find $D^2/D_0^2=0$, a consequence of the fact that the Irving-Gunn triton wave function does not correctly reproduce the empirical $t\to d+n$ separation energy at large distances. On reducing the incident deuteron energy to ≈6 MeV so that Coulomb distortions dominate we find $[d\sigma/d\Omega(\mathrm{EFR})]/[d\sigma/d\Omega(\mathrm{ZR})]$ of ≈1.5 at 130°. Thus, at least for range functions which do not have proper asymptotic behavior, it appears to be necessary to modify Goldfarb and Parry's low energy LEA procedure by including at least some of the omitted higher order terms.[17]

A third choice for $V(s)$ is to replace the operator B_t-T_s appearing in Eq. (20) by

$$V_{dn}=V(r_{pn})+V(r_{nn})\,, \tag{21}$$

where $V(r_{ij})$ reproduces low energy nucleon-nucleon scattering. Specifically, we use[21]

$$V^{ST}(r_{ij})=-U^{ST}\exp(\beta_{ST}{}^2 r_{ij}{}^2)\,, \tag{22}$$

where, in the singlet even state $U^{01}=29.05$ MeV and $\beta_{01}{}^2=0.292$ fm^{-2}, while in the triplet even state $U^{10}=66.92$ MeV and $\beta_{10}{}^2=0.415$ fm^{-2}. Since ϕ_t is space symmetric in our calculations it is unnecessary to specify the interaction in the odd states. Properly[19] taking into account the spin integrations, Eq. (21) becomes

$$V_{dn}=\sum_{j=n,p}\left[-\tfrac{3}{4}V^{01}(r_{nj})-\tfrac{1}{4}V^{10}(r_{nj})\right]. \tag{23}$$

TABLE III. Effects of different range functions for (d,t) reactions.

	Range function [a] $\langle\phi_d \mid V_{dn} \mid \phi_t\rangle$	$N=\frac{3}{2}D_0^2$ (units of 10^4 MeV^{2}fm^3)	$\frac{d\sigma'}{d\Omega}$(EFR)/$\frac{d\sigma}{d\Omega}$(ZR) 80 MeV [b]	30 MeV [c]	D^2/D_0^2	Notes
1	Gaussian	5.06	1.10	1.27	1.34 [d]	Bassel. $D_0=-183.6$ MeVfm$^{3/2}$, $R=1.69$ fm
2	$\langle H \mid B-T \mid IG\rangle$	5.06	1.13	1.37	0.00 [e]	$\delta=0.768$ fm^{-1}
3	$\langle H \mid V_1+V_2 \mid IG\rangle$	4.09	1.11	1.27	0.00 [e,f]	$\delta=0.768$ fm^{-1}
4	$\langle H \mid V_1+V_2 \mid G\rangle$	3.79	1.10	1.27	0.00 [e,f]	$\langle H \mid B-T \mid G\rangle$ gives $N=1.43\times10^4$ MeV^{2}fm^3
5	$\langle H' \mid V_1'+V_2' \mid$ TANG$\rangle$	14.11	1.01	1.14	···	In error due to omission of surface terms
6	$\langle H' \mid B \mid$ TANG$\rangle$	3.51	···	···	···	Assuming that T operator cancels surface terms; $r_c=0.45$ fm
7	$\langle H'' \mid B-T \mid$ TANG$\rangle$	5.00	···	···	1.92 [e]	Goldfarb's result; $r_c=0.4$ fm

[a] H: Hulthen deuteron, Ref. 10; H': modified deuteron wave function Eq. (27); H'': unspecified deuteron wave function used by Goldfarb and Parry; $B-T$: see Eq. (20); V_1+V_2: Gaussian interactions Eqs. (22) and (23); $V_1'+V_2'$: hard core interactions Eqs. (26) and (23); IG: Irving-Gunn triton; G: Gaussian triton; TANG: hard core triton.

[b] Evaluated at 7° c.m.

[c] Evaluated at 17° c.m.

[d] Using Eq. (18), $D^2/D_0^2=\lim_{r_b\to\infty}|\mathcal{O}(r_b)|^2$.

[e] Reference 17.

[f] Provided V_1+V_2 is replaced by $B-T$.

This choice for the range function is motivated by the notion that, although Eq. (20) has the merit of self-consistency, the generation of V_{dn} by operating with (B_t-T_s) on ϕ_t may lack precision. The resultant range function $\langle H \mid V_1+V_2 \mid IG\rangle$ is plotted in Fig. 5(b) where it is seen to resemble most closely the Gaussian range function, particularly at large radii. In Table III, it is seen that these two range functions lead to nearly identical values for the ratio of EFR and ZR cross sections and differ only by about 20% in absolute normalization.

In our fourth choice for the range function, $\langle H \mid V_1+V_2 \mid G\rangle$, the Irving-Gunn triton wave function is replaced by the Gaussian function

$$\Phi_t(r,s)=N_G\exp[-\eta^2(s^2+\tfrac{3}{2}r^2)]\,, \tag{24}$$

where $\eta=0.242$ fm^{-1} and N_G is a normalization constant, and the $d-n$ interaction is given by Eq. (23). The only effect is a slight (~8%) reduction in overall normalization with respect to the previous calculation using the Irving-Gunn wave function. This result is contrary to the result obtained by Bassel who, using (B_t-T_s) to generate V_{dn}, obtained $N=1.43\times10^4$ MeV^{2}fm^3. Evidently, in the present context, the Gaussian wave function is deficient mostly in its ability to specify the d-n interaction. Provided this interaction is specified independently, the use of the Gaussian wave function does not lead to appreciable error.

The fifth entry in Table III, $\langle H' \mid V_1'+V_2' \mid$ TANG$\rangle$ uses the triton wave function and nucleon-nucleon potential of Tang and Herndon.[22] Specifically

$$\phi_t=f(r_{12})f(r_{23})f(r_{31})\,, \tag{25}$$

where $f(r_{ij})$ is the trial function discussed in Ref. 22 and

$$V^{ST}(r_{ij})=-U^{ST}\exp[-K_{ST}(r-r_c)]\,,\quad r>r_c$$
$$=+\infty\,,\quad r<r_c\,, \tag{26}$$

where the hard core radius $r_c=0.45$ fm. In the singlet even state $U^{01}=277.07$ MeV, $K_{01}=2.211$ fm^{-1} and in the triplet even state $U^{10}=549.26$ MeV, $K_{10}=2.735$ fm^{-1}.

The deuteron wave function used in conjunction with Eqs. (25) and (26) is denoted by H' in the Table III. It is given by

$$\phi(r)=N_d\frac{\exp[-\alpha(r-r_c)]-\exp[-\beta(r-r_c)]}{r}\,, \tag{27}$$

where $\alpha=0.231$ fm^{-1}, $\beta=1.438$ fm^{-1}, $r_c=0.45$ fm, and N_d is a normalization constant. This wave function is correct asymptotically but does slightly overestimate the deuteron rms radius. The results are, however, quite insensitive to the details of the deuteron wave function and no serious error results.

As shown in Fig. 5(c) the function $\langle H' \mid V_1'+V_2' \mid$ TANG$\rangle$ has a significantly shorter

range than the other range functions considered. Thus, it is no surprise that the ratios of EFR and ZR cross sections which result are ~10% to 15% smaller than for the other range functions. Presumably this is a consequence of high momentum components which arise due to the introduction of a hard core in the nucleon-nucleon interaction.

The normalization constant $N = \frac{3}{2}D_0^2$ for this range function is $N = 14.11 \times 10^4$ MeV^2 fm^3. This is in moderate agreement with the value of $N = 11.93 \times 10^4$ MeV^2 fm^3 obtained by Rook[19] in a similar calculation using a 0.4 fm hard core nucleon-nucleon interaction in conjunction with the Ohmura triton wave function. However, both results are considerably larger than the other values listed in Table III and are in serious disagreement with experiment. In Ref. 19 Rook interpreted this result to be a consequence of the rather simple form of the Ohmura triton wave function. It is our contention, however, that the difficulty lies in the treatment of the hard core potentials. Thus, Dobes[23] in a discussion of (d,p) calculations, points out that the use of a hard core nucleon-nucleon potential necessitates the inclusion of an additional surface term in the interaction, evaluated at the hard core radius. Dobes also points out that similar terms must arise in $(d,{}^3\mathrm{He})$ and (d,t) calculations and that, for (d,p) neglecting the surface term leads to a value of D_0^2 which is 2 to 3 times too large. Thus Rook's calculation and our own calculation for (d,t) using the range function $\langle H' \,|\, V_1' + V_2' \,|\, \mathrm{TANG}\rangle$ are probably in error. While we have not yet carried out EFR calculations including the surface terms, we are able to obtain a revised estimate of N. Following Dobes result for (d,p) that, using the operator $(B-T)$ to replace the interaction potential, the integrals of the kinetic energy operator T and the surface term exactly cancel, we can obtain N simply by evaluating the volume integral of $\langle H' \,|\, B \,|\, \mathrm{TANG}\rangle$. The result is $N = 3.5 \times 10^4$ MeV^2 fm^3 which is similar to other values in Table III. This is to be compared with $N = 5.0 \times 10^4$ MeV^2 fm^3 obtained by Goldfarb and Parry[17] using a slightly different hard core radius and an unspecified deuteron wave function. These authors also find $D^2/D_0^2 = 1.92$, a surprisingly large value. However, this ratio could be in error if proper account has not been taken of the surface terms.

Finally, values of $N = \frac{3}{2}D_0^2$ for the $(d,{}^3\mathrm{He})$ reaction are listed in Table IV. Provided the Coulomb part of the interaction should be excluded, values of $[d\sigma/d\Omega(\mathrm{EFR})]/[d\sigma/d\Omega(\mathrm{ZR})]$ will be quite similar to the values quoted for (d,t) in Table III. In general, the results for N are also quite similar to the corresponding entries in Table III. It should be noted that the Irving-Gunn and Tang-Herndon wave functions are identical to those used for the (d,t) calculations.

V. SUMMARY AND CONCLUSIONS

An expression for the exact finite-range DWBA amplitude has been obtained which is convenient for light ion transfer reactions. The method has been used to investigate the accuracy of the LEA approach for $(d,{}^3\mathrm{He})$ and (d,t) reactions.

For the $^{51}\mathrm{V}(d,{}^3\mathrm{He})^{50}\mathrm{Ti}$(g.s.) reaction it has been found that the shape of the differential cross section is well reproduced in LEA. However, the LEA calculations lead to an erroneous energy dependence of the absolute cross section between 30 and 80 MeV incident deuteron energy. The energy dependence is changed by roughly 7.5% as a result of specifically LEA effects while an additional change of about 5% is expected if Bassel's Gaussian range function is replaced by the "exact" result $\langle H \,|\, B - T - V_C \,|\, IG\rangle$. For the $^{16}\mathrm{O}(d,{}^3\mathrm{He})^{15}\mathrm{N}$(g.s.) reaction at 82 MeV it is found that the LEA seriously overestimates finite-range effects. The EFR result is, in fact, closer to ZR.

An investigation of the effects of different choices for the range function $V(s)$ largely con-

TABLE IV. Different range functions for $(d,{}^3\mathrm{He})$ reactions.

	Range function [a] $\langle \phi_d \,|\, V_{dp} \,|\, \phi_{^3\mathrm{He}} \rangle$	$N = \frac{3}{2} D_0^2$ (units of 10^4 MeV^2 fm^3)	Notes
1	$\langle H \,|\, B - T \,|\, IG \rangle$	3.84	Coulomb term included $\delta = 0.768$ fm^{-1}
2	$\langle H \,|\, B - T - V_C \,|\, IG \rangle$	4.42	Coulomb term excluded $\delta = 0.768$ fm^{-1}
3	$\langle H \,|\, V_1 + V_2 \,|\, IG \rangle$	4.09	Coulomb term excluded $\delta = 0.768$ fm^{-1}
4	$\langle H \,|\, V_1 + V_2 \,|\, G \rangle$	4.05	Coulomb term excluded $\eta = 0.206$ fm^{-1}
5	$\langle H' \,|\, B \,|\, \mathrm{TANG} \rangle$	2.69	Coulomb term included

[a] V_C: deuteron-proton Coulomb interaction. The remaining notation is identical to Table III.

forms the validity of the simple Gaussian function chosen by Bassel. A comparison with the exact range function $\langle H|B-T|IG\rangle$ for (d,t) reactions revealed slight differences in energy dependence but no serious discrepancies. The use of a Gaussian, spin dependent, nucleon-nucleon interaction to obtain the deuteron-nucleon interaction resulted in a 20% reduction in the value of N for (d,t) and some change in energy dependence. However, the subsequent introduction of a Gaussian triton wave function had little effect. A calculation using the Tang-Herndon variational triton wave function was attempted and difficulties associated with the use of a hard core nucleon-nucleon interaction were discussed.

Normalization constants were also calculated for the $(d,{}^3\mathrm{He})$ reaction. Generally results were rather similar to the (d,t) results. It is interesting to note that the normalization constants for (d,t) and $(d,{}^3\mathrm{He})$ obtained by Bassel using the Irving-Gunn trinucleon wave function are the largest values in Tables III and IV. Despite the many other uncertainties in DWBA analyses, experiment does seem to indicate that Bassel's normalization is too large, and a reduction in N to a value close to our result obtained with the Tang-Herndon wave function is to be preferred.[30] It is significant that Goldfarb, Gonzalez, and Phillips,[20] using a variety of methods find

$$N(d,t) = (3.45 \pm 0.9) \times 10^4 \text{ MeV}^2 \text{ fm}^3 ,$$
$$N(d,{}^3\mathrm{He}) = (3.0 \pm 0.9) \times 10^4 \text{ MeV}^2 \text{ fm}^3 \qquad (28)$$

which values are in reasonable agreement with experiment.

In conclusion we point out that the LEA cannot always be relied upon in analyses of $(d,{}^3\mathrm{He})$ and (d,t) reactions, particularly at energies of 80 MeV or more. In general, EFR calculations are desirable, and realistic treatments of the light particle wave functions are essential if we hope to extract meaningful spectroscopic factors or to investigate refinements of the DWBA approach. Indeed precise work may necessitate more sophisticated three body wave functions than those considered here which, among their other deficiencies, are limited to being totally space symmetric.[18]

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