Variational principles for the projected breakup amplitude

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Two alternate forms of variational principles for the breakup amplitude describing the two- to three-cluster transition are derived such that all the integrals involved in the intermediate stages are well defined. The first form contains a trial Green's function with which both the initial and final state trial wave functions are constructed. The earlier form of the Kohn-type variational principle derived by Lieber, Rosenberg, and Spruch is recovered, however, when this connection between the trial functions is removed. The second form of the variational principle is derived by projecting out from the trial functions all the open channel components which correspond to the two-cluster structures including the rearrangement channels. The remaining part of the wave functions describes the channels with three-cluster structures, and the integrals involving this part are then mathematically well defined.

[NUCLEAR REACTIONS Variational formulation of the breakup amplitude; projection of all the bound-pair channels from the trial functions.]

I. INTRODUCTION

Detailed theoretical studies of the low energy breakup reactions have not been possible until recently, mainly because of the lack of sufficient informations on the asymptotic behavior of scattering wave functions in the breakup channel. The works by Nuttall,[1] Gerjuoy,[2] and others have greatly clarified this question, and the first, fairly rigorous derivation of the Kohn-type variational principle has been given by Lieber, Rosenberg, and Spruch (LRS),[3] and also by Merkuriev.[4]

In the course of the derivation, LRS encountered mathematically ill-defined integrals, and these are treated by the radius-averaging procedure, which is justified in view of the more rigorous derivation of the same variational principle using the Faddeev equations.[3,4]

In the present paper, we derive two alternate forms of variational principles in which all the integrals which appear in the intermediate stages are well defined such that the radius averaging is not required. Such a study is desirable not only because it makes the derivation simple and straightforward, but may also be useful in the case with Coulomb interactions and may facilitate the application of the procedures to actual physical problems, perhaps in the form of a bound principle. We do not consider in this paper the problem of Coulomb potentials nor the possible derivation of bound principles themselves.

The first form derived in Sec. II is a slight modification of the LRS result in that the terms which appear in the variational expression in the intermediate stages are well defined through the use of the common trial Green's function. The same question has also been considered by Bryce and Mandl[5] from a somewhat different point of view. Unlike in LRS and in Ref. 5, the trial Green's function G_t plays a more crucial role in our derivation, as the trial functions for the initial and final states are not independent, but are connected by the *same* G_t. This restriction may, however, be removed by a further manipulation of the integrals involved. The resulting variational principle agrees with that of LRS, but we have to introduce the radius averaging in the course of this manipulation.

The second variational principle is derived in which all the open bound-pair channels (two-cluster channels) at a given scattering energy are projected out of the trial functions. It is not surprising that a separate treatment of the bound-pair channels in the variational formulation should remove the difficulty, since the ill defined integrals which appear in the LRS derivation are all of the types which connect the breakup and the bound-pair channels (continuum bound). It is also amusing to note the analogy between the projection procedure adopted here and the variational bound formulation.[6] Although the scattering energy here is too high to obtain bounds on scattering parameters, except perhaps at the breakup thresholds, the separation of all the open two-cluster channels from the rest of the channels turns out to be also useful for the breakup problem for entirely different reasons.

II. A MODIFIED FORM OF KOHN PRINCIPLE

For simplicity of discussion, we consider the reaction

$$1+(2+3)\underset{(i)}{\longrightarrow}1+2+3\underset{(f)}{} \tag{2.1}$$

of a three-particle system interacting by short-range forces. Using the notation of LRS,[3] we have for the breakup amplitude

$$\mathcal{T}_{fi}=(\Phi_f,[V^1+VG(Z)V^1]\Phi_i)\ , \tag{2.2}$$

where

$H=H_1+V^1$ for the i channel,

$H=H_0+V$ for the f channel

with

$$(H_i-E)\Phi_i=0\ ,$$

$$(H_0-E)\Phi_f=0\ .$$

In (2.2), $Z\equiv E+i\epsilon$, G is the full Green's function, and the limit $\epsilon\to 0^+$ is assumed *after* all the integrations implied here are performed. Using the identity

$$G=G_t+G+G(H-Z)G_t\ , \tag{2.3}$$

we have for (2.2)

$$\begin{aligned}\mathcal{T}_{fi}=&(\Phi_f,V^1\Phi_i)+(\Phi_f,VG_tV^1\Phi_i)+(\Phi_f,VGV^1\Phi_i)\\&+(\Phi_f,VG(H-Z)G_tV^1\Phi_i)\ .\end{aligned} \tag{2.4}$$

Obviously, the ambiguity arises both in (2.3) and (2.4) if $\lim\epsilon\to 0^+$ is taken before the integrations are carried out, specially those integrals connected with G_t. Therefore, at least for the G_t and G integrations are concerned, ϵ should be held at some finite value.

We can now introduce the scattering functions

$$\begin{aligned}\Psi_i&=\Phi_i+GV^1\Phi_i\equiv\Phi_i+\tilde{\Psi}_i\ ,\\ \Psi_f&=\Phi_f+GV\Phi_f\equiv\Phi_f+\tilde{\Psi}_f\ ,\end{aligned} \tag{2.5}$$

and similarly for the trial functions as

$$\begin{aligned}\Psi_{it}&=\Phi_i+G_tV^1\Phi_i\equiv\Phi_i+\tilde{\Psi}_{it}\ ,\\ \Psi_{ft}&=\Phi_f+G_tV\Phi_f\equiv\Phi_f+\tilde{\Psi}_{ft}\ ,\end{aligned} \tag{2.6}$$

where Ψ_i and Ψ_f, and Ψ_{it} and Ψ_{ft}, are related by the *same* G and G_t, respectively. $\mathcal{T}_{fi}$ may now be rewritten in terms of $\tilde{\Psi}_i$, $\tilde{\Psi}_{it}$, etc., in many different ways, depending on which sides G or G_t is to be attached to inside the integrals in (2.4). Thus, for example,

$$\begin{aligned}\mathcal{T}_{fi}=&(\Phi_f,V^1\Phi_i)+(\Phi_f,V\tilde{\Psi}_{it})+(\tilde{\Psi}_f,V^1\Phi_i)\\&+(\tilde{\Psi}_f,[H-Z]\tilde{\Psi}_{it})\end{aligned} \tag{2.7}$$

or

$$\begin{aligned}\mathcal{T}_{fi}=&(\Phi_f,V^1\Phi_i)+(\tilde{\Psi}_{ft},V^1\Phi_i)+(\tilde{\Psi}_f,V^1\Phi_i)\\&+(\tilde{\Psi}_f,[H-Z]\tilde{\Psi}_{it})\ .\end{aligned} \tag{2.8}$$

The expression (2.7) was used by LRS in their derivation of the breakup variational principle, with a careful discussion of the second term in the right-hand side using the radius-averaging procedure. Bryce and Mandl[5] also considered the same expression, but with ϵ kept finite until all the algebraic manipulations are completed. They found then that the same variational expression may be obtained without introducing the radius averaging if certain asymptotic conditions on the trial functions are met.

Instead of (2.7), we consider in this section the expression (2.8) which, for given $\tilde{\Psi}_f$, $\tilde{\Psi}_{ft}$, and $\tilde{\Psi}_{it}$, contains only well-defined integrals because of the $V^1\Phi_i$ factor and $(H-Z)$. However, unlike in (2.7), *both* $\tilde{\Psi}_{ft}$ and $\tilde{\Psi}_{it}$ are now involved. They are related through G_t and thus are not independent trial functions. (This restriction will be removed later on.) With $\epsilon\to 0^+$ now, we rewrite (2.8) in the form

$$\begin{aligned}\mathcal{T}_{fi}=&(\Phi_f,[H-E]\Phi_i)+(\tilde{\Psi}_{ft},[H-E]\Phi_i)\\&+(\tilde{\Psi}_f,[H-E]\Phi_i)+(\tilde{\Psi}_f,[H-E]\tilde{\Psi}_{it})\\=&(\Phi_f,[H-E]\Phi_i)+(\tilde{\Psi}_{ft},[H-E]\Phi_i)\\&+(\tilde{\Psi}_f,[H-E]\Psi_{it})\\=&(\Psi_{ft},[H-E]\Phi_i)+(\tilde{\Psi}_f,[H-E]\Psi_{it})\ .\end{aligned} \tag{2.9}$$

A more symmetric form is

$$\begin{aligned}\mathcal{T}_{fi}=&[(\Psi_{ft},[H-E]\Phi_i)-(\Phi_f,[H-E]\Psi_{it})]\\&+(\Psi_f,[H-E]\Psi_{it})\ ,\end{aligned} \tag{2.10}$$

where $[H-E]$ always operates to its right. Note that *all* the integrals in (2.9) and (2.10) are well defined. In particular the integrals in the square bracket of (2.10) give simply a c number for given Ψ_{it} and Ψ_{ft}. We immediately identify them as $\mathcal{T}_{fi,t}$, i.e.,

$$\begin{aligned}\mathcal{T}_{fi,t}\equiv&(\Psi_{ft},[H-E]\Phi_i)-(\Phi_f,[H-E]\Psi_{it})\\=&(\tilde{\Psi}_{ft},[H-E]\Phi_i)-(\Phi_f,[H-E]\tilde{\Psi}_{it})\ ,\end{aligned} \tag{2.11}$$

which is justified because, as $\Psi_{it}\to\Psi_i$ and simultaneously $\Psi_{ft}\to\Psi_f$, we have the first term in (2.11) approach $\mathcal{T}_{fi}$, while the second term $\to 0$. In fact, $\mathcal{T}_{fi,t}$ of (2.11) should be identical to that defined by LRS, where the identification was made only after the radius averaging and some partial integrations. Obviously, such steps are not needed in the present case. We will come back to this question later on.

In order to obtain a variational principle, we simply set

$$\begin{aligned}\Psi_i&=\Psi_{it}+\delta\Psi_i\ ,\\ \Psi_f&=\Psi_{ft}+\delta\Psi_f\ ,\end{aligned} \tag{2.12}$$

where again $\delta\Psi_i$ and $\delta\Psi_f$ are related by (2.6) through G_t, and substitute them into the identity

(2.11). Then,

$$\mathcal{T}_{fi} = \mathcal{T}_{fi,t} + (\Psi_{ft}, [H-E]\Psi_{it}) - \tau_{fi}\,, \tag{2.13}$$

where

$$\tau_{fi} = (\delta\Psi_f, [H-E]\,\delta\Psi_i)$$

= second order in the error functions.

Thus, by neglecting the term τ_{fi} in (2.13), we finally obtain

$$[\mathcal{T}_{fi}] = \mathcal{T}_{fi,t} + (\Psi_{ft}, [H-E]\Psi_{it})\,, \tag{2.14}$$

with the important constraint on the form of the trial functions that Ψ_{it} and Ψ_{ft} be related through G_t as defined in (2.6). Thus, although the form of $[\mathcal{T}_{fi}]$ is identical to that given by LRS, (2.14) is little more restrictive in the variations of its parameters. On the other hand, this restriction is physically reasonable in view of the fact that the only unknown quantity common to both Ψ_{it} and Ψ_{ft} is G_t, and the approximation is made naturally on G. From a practical point of view, the construction of Ψ_{it} and Ψ_{ft} from the same G_t involves one additional set of integrations, and this may not be such a trivial task as it seems. But, for a simple approximation to G of the form, e.g.,

$$G_t \approx X_t)\,(X_t, [Z-H]X_t)^{-1}(X_t\,, \tag{2.15}$$

we obtain immediately

$$\begin{aligned}\Psi_{it} &= \Phi_i + C_i X_t\,,\\ \Psi_{ft} &= \Phi_f + C_f X_t\,,\end{aligned} \tag{2.16}$$

where

$$\begin{aligned}C_i &= (X_t, V^1\Phi_i)/(X_t, [Z-H]X_t)\,,\\ C_f &= (X_t, V\Phi_f)/(X_t, [Z-H]X_t)\,.\end{aligned} \tag{2.17}$$

Thus, the amount of work involved in (2.15) is not any more than the earlier variational principle, because the integrals which occur in C_i and C_f will also appear in the variational expressions. C_i and C_f are now complicated functions of nonlinear parameters introduced in X_t. Extensions of (2.15) to forms involving higher ranks are straightforward.

Now we return to (2.10) and (2.11), and consider the possibility of eliminating the restriction on the trial functions discussed above. Since both integrals in (2.11) are well defined, the radius averaging on these quantities is merely a matter of technical convenience and would not change their character. (Similar procedure was also adopted in LRS-II.) In contrast, we recall that the radius averaging was introduced in LRS-I to an integral which was not well defined in the limit $\epsilon \to 0^+$. This is the essential difference between the present derivation of (2.14) and the LRS result. By introducing the radius averaging in (2.11), followed by the limit $\epsilon \to 0$, we obtain, with partial integrations,

$$\begin{aligned}(\tilde{\Psi}_{ft}, [H-E]\Phi_i) &\to \lim_{\epsilon\to 0^+} \langle((H-E)\Phi_f, \vec{G}_t(Z)[H-E]\Phi_i)\rangle\\ &= \lim_{\epsilon\to 0^+} \langle((H-E)\Phi_f, \vec{G}_t(Z)[H-E]\Phi_i)\rangle\\ &= \langle((H-E)\Phi_f, \tilde{\Psi}_{it})\rangle\end{aligned} \tag{2.18}$$

and

$$\begin{aligned}(\Phi_f, [H-E]\tilde{\Psi}_{it}) &\to \langle(\Phi_f, [H-E]\tilde{\Psi}_{it})\rangle\\ &= -\,\mathcal{T}_{fi,t} + \langle((H-E)\Phi_f, \tilde{\Psi}_{it})\rangle\,,\end{aligned} \tag{2.19}$$

where $\langle\ \rangle$ denotes the radius averaging.[3] Thus, we recover (2.11) after (2.18), cancelling the same term in (2.19). In (2.18), $\tilde{\Psi}_{ft}$ is converted into $\tilde{\Psi}_{it}$ by shifting the G_t, which acts toward left, toward right, holding ϵ finite. During the course of this operation, we encounter an ill-defined integral if the radius averaging were not introduced. The $\tilde{\Psi}_{ft}$ no longer appears in (2.10), and the dependence of $\tilde{\Psi}_{ft}$ on $\tilde{\Psi}_{it}$ is eliminated. As a result, we obtain the Kohn-type principle of LRS obtained earlier; that is, the form (2.14) with no restrictions on the trial functions other than the asymptotic boundary conditions[1-3]

$$\tilde{\Psi}_{it} \sim a_{ci}\,\phi_c(\vec{r}_1)\,e^{iK\rho}/\rho^{5/2} + a_{ci}\,\psi_0(\vec{r}_1)\,e^{ik_0\rho}/\rho \tag{2.20}$$

and similarly for $\tilde{\Psi}_{ft}$. In (2.20), we used $K = (2mE)^{1/2}/\hbar$ with m taken here to be the reduced mass of the projectile 1, and $\rho \equiv (\vec{r}_1^{\,2} + \vec{R}_1^{\,2})^{1/2}$, where $\vec{r}_1$ is the relative coordinate of the pair $2+3$ and $\vec{R}_1$ is for their center of mass and the particle 1. The functions $\phi_c(\vec{r}_1)$ and $\psi_0(\vec{r}_1)$ are the zero energy and the negative energy wave functions respectively for the $2+3$ pair.

III. PROJECTED CONTINUUM VARIATIONAL PRINCIPLE

We return to the expression (2.7) used earlier by LRS-I and reconsider the difficulty. According to the asymptotic behavior of the scattered functions derived by Nuttall,[1] the divergent part of the term

$$(\Phi_f, V\tilde{\Psi}_{it}) \tag{3.1}$$

comes from the bound-pair components in Ψ_{it} combined with the disconnected part in $V\Phi_f$. This required the radius averaging. In Sec. II, we managed to manipulate the terms slightly differently such that the integrals which appear are all well defined. However, it is also obvious that if the bound-pair components can be projected out from the trial functions Ψ_{ft} and Ψ_{it} from the start, the difficulty mentioned above would not arise. As will be shown below, this, in fact, can be done. In this connection, Carew and Rosenberg[7] considered

independently an approach similar to ours. But the projection method used in their work is quite different from the one used below; they modify the binding potentials in each bound-pair channel and put them into the Faddeev equations, along the line of the quasiparticle approach,[8] while we utilize the channel projections of the Feshbach[9] types. Their interconnection was clarified by Tobocman.[10]

For simplicity, we assume that the bound-pair channel is allowed only in the $1+(2+3)$ configuration with only one bound state of the $2+3$ system. Cases in which a finite number of bound states can occur and also other bound-pair rearrangement channels are present can be treated by straightforward extensions.[11] The present formalism is not applicable when an infinite number of bound state channels are present, as in the case with Coulomb interactions.

We define the bound-pair channel projection $P=|\psi_0(\vec{r}_{23})\rangle\langle\psi_0^*(\vec{r}'_{23})|$ and the corresponding "static" Green's function in that channel space as

$$G^P=[P(Z-H)P]^{-1}. \tag{3.2}$$

Then, we have the identity

$$G(Z)=G^P(Z)+F^T\mathcal{G}^Q F\,, \tag{3.3}$$

where

$$\begin{aligned} &G(Z)=(Z-H)^{-1},\\ &F=1+DG^P,\quad D\equiv H-Z\,,\\ &F^T=1+G^PD\,,\\ &\mathcal{G}^Q=[Q(Z-\mathcal{K})Q]^{-1} \end{aligned} \tag{3.4}$$

with

$$\begin{aligned} &\mathcal{K}=H+(H-Z)G^P(H-Z)\,,\\ &Q=1-P,\quad QP=0\,. \end{aligned}$$

The important properties of F and $\mathcal{K}$ are[11]

$$PF=0,\quad \text{and thus}\quad F=QF \tag{3.5}$$

and

$$\begin{aligned} &P(\mathcal{K}-Z)=(\mathcal{K}-Z)P=0\,,\\ &Q(\mathcal{K}-Z)Q=(\mathcal{K}-Z)\,. \end{aligned} \tag{3.6}$$

Therefore, (2.2) may be rewritten in the form

$$\begin{aligned} \mathcal{T}_{fi}&=[(\Phi_f,V^1\Phi_i)+(\Phi_f,VG^PV^1\Phi_i)]\\ &\quad+(\Phi_f,VF^T\mathcal{G}^QFV^1\Phi_i)\\ &=(\Phi_f,\chi_i^{(+)})+(\chi_f^{(-)},\mathcal{G}^Q\chi_i^{(+)})\,, \end{aligned} \tag{3.7}$$

where

$$\begin{aligned} &\chi_i^{(+)}=FV^1\Phi_i=Q\chi_i\,,\\ &\chi_f^{(-)}=F^*V\Phi_f=Q\chi_f\,. \end{aligned} \tag{3.8}$$

Note that, using the definition for F,

$$\begin{aligned} \chi_i^{(+)}&=V^1\Phi_i+DG^{P(+)}V^1\Phi_i\\ &=DF\Phi_i=DP\Psi_i^P\,, \end{aligned}$$

where

$$P\Psi_i^P=\Phi_i+G^{P(+)}V^1\Phi_i\,.$$

Also,

$$\begin{aligned} \chi_f^{(-)}&=D[\Phi_f+G^{P(-)}V\Phi_f]\\ &=DF^*\Phi_f\,. \end{aligned}$$

The integration involving $G^{P(-)}$ in $\chi_f^{(-)}$ is precisely the form (3.1) and should be done carefully. The important difference here, however, is the fact that $G^{P(\pm)}$ are assumed to be known exactly rather than through a trial function such as $\tilde{\Psi}_{it}$. Thus, we can either hold $\epsilon\neq 0$ and do the integration first, or, introduce radius averaging. In any case, the evaluation of χ_i, χ_f, and $\mathcal{K}$ in terms of G^P should not present any difficulty, and we *assume* in the following that these quantities are available.

The main problem of calculating $\mathcal{T}_{fi}$ lies therefore in the estimation of the second term in (3.7), that is, the Green's function $\mathcal{G}^Q$. The situation with (3.7) is then identical to (2.2), where previously G assumed the role of $\mathcal{G}^Q$. Consequently, the derivation of the variational principle follows exactly the same way.

In order to derive the variational principle, we use the identity, with $\epsilon\neq 0$,

$$\mathcal{G}^Q=\mathcal{G}_t^Q+\mathcal{G}^Q+\mathcal{G}^Q(\mathcal{K}-Z)\mathcal{G}_t^Q\,, \tag{3.9}$$

where, as in (3.7), we omit the superscripts $(\pm)$ for the asymptotic boundary conditions unless ambiguity arises. Substitution of (3.9) into (3.7) gives

$$\begin{aligned} \mathcal{T}_{fi}&=(\Phi_f,\chi_i)+(\chi_f,\tilde{\Psi}_{it}^Q)+(\tilde{\Psi}_f^Q,\chi_i)\\ &\quad+(\tilde{\Psi}_f^Q,[\mathcal{K}-E]\tilde{\Psi}_{it}^Q)\,, \end{aligned} \tag{3.10}$$

where we set

$$\begin{aligned} &\tilde{\Psi}_i^Q=\mathcal{G}^Q\chi_i\,,\\ &\tilde{\Psi}_f^Q=\mathcal{G}^Q\chi_f\,, \end{aligned} \tag{3.11}$$

and

$$\tilde{\Psi}_{it}^Q=\mathcal{G}_t^Q\chi_i,\quad \tilde{\Psi}_{ft}^Q=\mathcal{G}_t^Q\chi_f\,.$$

Again, $\tilde{\Psi}_i^Q$ and $\tilde{\Psi}_f^Q$ are related by the common $\mathcal{G}^Q$, but we do not need $\tilde{\Psi}_{ft}^Q$ in (3.10), so do not have to introduce it at this stage. (3.10) is the identity analogous to (2.4). Now, let

$$\begin{aligned} &\tilde{\Psi}_i^Q\equiv\tilde{\Psi}_{it}^Q+\delta\Psi_i^Q\,,\\ &\tilde{\Psi}_f^Q\equiv\tilde{\Psi}_{ft}^Q+\delta\Psi_f^Q\,, \end{aligned} \tag{3.12}$$

where $\tilde{\Psi}_{it}^Q$ and $\tilde{\Psi}_{ft}^Q$ are *independent* trial functions.

Using the relation

$$(\mathcal{K}-E)\tilde{\Psi}_i^Q=-FV^1\Phi_i=-\chi_i\,, \tag{3.13}$$

and similarly for $\tilde{\Psi}_f^Q$, we finally have, with $\epsilon \to 0$,

$$\mathcal{T}_{fi} = [\mathcal{T}_{fi}^Q] - \tau_{fi}^Q \ , \tag{3.14}$$

where

$$[\mathcal{T}_{fi}^Q] = (\Phi_f, \chi_i) + (\chi_f, \tilde{\Psi}_{it}^Q) + (\tilde{\Psi}_{ft}^Q, \chi_i) + (\tilde{\Psi}_{ft}^Q, [\mathcal{H} - E]\tilde{\Psi}_{it}^Q) \tag{3.15}$$

and

$$\tau_{fi}^Q = (\delta\Psi_f^Q, [\mathcal{H} - E]\,\delta\Psi_i^Q) \ . \tag{3.16}$$

Therefore, the variational functional (3.15) is an estimate of $\mathcal{T}_{fi}$ to second order in the error functions. It resembles the variational bound formulation[6] for the low energy scatterings below the breakup threshold. Since, by construction, both χ_i and χ_f are already in the Q space, the trial functions $\tilde{\Psi}_{ft}^Q$ and $\tilde{\Psi}_{it}^Q$ are automatically orthogonal to the bound-pair channels and thus necessarily in the breakup continuum channels. In practice, the explicit form of the operator Q is *not* required as both F and $\mathcal{H} - E$ are also in the Q space.[11] This point is especially relevant in connection with the rearrangement channels involving other bound pairs. Except for the definition of G^P and $P\Psi_i^P$ in χ's and $\mathcal{H}$, the formalism is unchanged for the more complicated open channels. We do not consider these questions here, but simply refer to Ref. 11 for the detail.

Finally, we note that the complications of (3.15) over the Kohn-type form (2.14) or the LRS result are that both G^P and $P\Psi^P$ have to be calculated explicitly, exactly. A partial elimination of these requirements may be possible by formulating an iterative procedure. As was discussed earlier[12] in connection with the variational bound formulation and its applications, consider, for example, the function $\chi_i^{(+)}$ defined by (3.8). The G^P-dependent part

$$Y_i \equiv G^P V^1 \Phi_i$$

can be rewritten to satisfy an inhomogeneous equation

$$P(E - H)PY_i = PV^1\Phi_i \ , \tag{3.17}$$

and this can be solved rather trivially numerically. Similarly, for the G^P-dependent part in $\mathcal{G}^Q$, we may define

$$\mathcal{W}_i \equiv \mathcal{G}^Q \chi_i^{(+)} \ ,$$

which can be rewritten as a set of coupled equations

$$\begin{aligned} Q(E - H)\,Q\mathcal{W}_i &= \chi_i^{(+)} - Q(H - E)\,\mathcal{Y}_i \ , \\ P(E - H)P\mathcal{Y}_i &= P(H - E)\,Q\mathcal{W}_i \ . \end{aligned} \tag{3.18}$$

Thus, we have again eliminated the explicit dependence on G^P, and (3.18) now requires an iterative procedure as both the functions $\mathcal{W}_i$ and $\mathcal{Y}_i$ are not known. For cases in which the P space contains rearrangement channels, we refer to the result in Ref. 11.

As remarked earlier, the formalism of the present section cannot be applied when the pairs interact by an unscreened Coulomb potential; we then have to project out an infinite number of bound-continuum channels in order to construct the operators F and $\mathcal{H}$. We also have not considered the case when the exact pair-bound state wave functions are not available[13-15]; this problem further complicates the application of our result.

It is conjectured that the form (3.15) may provide a variational bound on the proper sum of the open channel amplitudes including the breakup at the breakup threshold energy.

IV. DISCUSSION

We have derived two alternate expressions of the variational principle for the breakup process without the explicit use of the radius averaging procedure.

The first form given in (2.14) differs from the earlier versions[3,4] in that the initial and final channel trial functions are not independent but connected by G_t. However, this may not seem to be a major drawback when G_t is approximated by a sum of separable forms, as discussed in Sec. II; the additional integrals needed to construct Ψ_{it} and Ψ_{ft} in this way are essentially of the same types as those involved in the variational functionals themselves. We have shown also that the above restriction on the trial functions *can* be removed and thus the LRS form recovered.

The projected form of the variational principle given in Sec. III is similar to the form used in the variational bounds formulation[6] at low energy scatterings and also in the unitary variational principle of Carew and Rosenberg.[7] The trial functions in (3.15) are all of the *continuum* type, while the bound-pair component of the scattering function is dealt with separately. Such a procedure is especially useful when the bound-pair channels dominate the scattering.

In view of lack of any serious computations which are available for comparison, it is difficult to determine at this time whether any one of the variational principles derived thus far is to be preferred over the others.

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