New alternative to the resonating group method*

R. Raphael
Department of Physics, Oglethorpe University, Atlanta, Georgia 30316

P. C. Tandy
Department of Theoretical Physics, Australian National University, Canberra, ACT 2600 Australia

W. Tobocman
Department of Physics, Case Western Reserve University, Cleveland, Ohio 44106

(Received 7 July 1976)

When the cluster model approximation is used with the antisymmetrized coupled equations formalism for the scattering of complex nuclei there results an approximate method which appears to be an attractive alternative to the resonating group method. The antisymmetrized coupled equations formalism leads to a Schrödinger type of equation which has unique solutions in contrast to the resonating group method equation for which redundant solutions exist. The two methods are compared by applying them to the model problem of dineutron-dineutron scattering, and we find that the predicted cross sections are qualitatively similar. In addition, it is shown how the results of a cluster model calculation, which presumably provides a description of direct interaction effects, can be used as input into a Feshbach projection operator type calculation for compound nucleus formation effects.

[NUCLEAR REACTIONS Alternative to resonating group method for treating exchange effects in nuclear reactions. Calculation of dineutron-dineutron scattering.]

I. INTRODUCTION

The resonating group method[1] (RGM) has been widely used in recent years to analyze the scattering of complex nuclei. It is unique in its capacity to include the effects of exchange symmetry. Application of the RGM requires the solution of an integrodifferential equation having a kernel that is momentum dependent and energy dependent. The solution of this equation is not uniquely determined by the boundary conditions. This can lead to certain mathematical difficulties.[2] Formally exact integral equations for multiparticle scattering can provide a framework for applying the cluster model approximation to achieve an alternative to the RGM.[3]

In this article we introduce an alternative to the RGM which is based on the same physical assumptions as the RGM. The new method is the result of making the cluster model approximation in the antisymmetrized coupled equations formalism[4] (ACEF) for nuclear reactions. It results in an integrodifferential Schrödinger equation for the scattering function. The kernel of this equation is not energy dependent; it is derived directly from the interaction potential of the original Hamiltonian.

The new ACEF formalism appears to be somewhat easier to use than the RGM and should give equivalent results. This is demonstrated by applying both methods to the dineutron-dineutron (^2n $-$ ^2n) scattering problem. Since the ACEF is a formally exact multiparticle scattering theory which is still relatively new, it is instructive to compare it with other methods that have proven themselves to be trustworthy.

Besides comparing the RGM and the cluster model ACEF (CM-ACEF) in the single channel approximation, we also compare their multichannel forms. In addition, we indicate how these formalisms may be extended so as to include compound nucleus effects.

In Sec. II the RGM is presented. The cluster model approximation is applied to the ACEF in Sec. III. The application of the RGM and the CM-ACEF to the ^2n-^2n scattering problem is carried out in Secs. IV and V. The effectiveness of the Born approximation in the RGM and CM-ACEF is tested in Sec. V. In Sec. VII we describe how the results of a CM-ACEF can be used as input to a calculation that will account for compound nucleus formation effects on the scattering cross sections. Section VIII is devoted to a summary of our results.

II. RESONATING GROUP METHOD

Consider a system of N nucleons which we regard at the outset as being distinguishable. For each partition $\alpha, \beta, \ldots$ of the nucleons into two sets or clusters there is a decomposition of the Hamiltonian into two parts:

$$H = H_\alpha + V_\alpha = H_\beta + V_\beta = \cdots . \tag{1}$$

H_α contains the kinetic energy and the intracluster interactions for the partition α while V_α is the sum of the intercluster interactions for that partition. The channels a, a', a'', ... associated with partition α are identified with the various possible unit incident amplitude eigenstates Ψ_a of H_α having eigenvalue E:

$$(E - H_\alpha)\Psi_a = V_\alpha \Psi_a , \tag{2a}$$

$$\Psi_a \xrightarrow[r_\alpha \to \infty]{} \phi_a e^{i\vec{k}_a \cdot \vec{r}_\alpha} + \text{outgoing waves} , \tag{2b}$$

$$\phi_a = [\phi_{a1}^{J_1 M_1} \times \phi_{a2}^{J_2 M_2}]^{J_a M_a} , \tag{2c}$$

$$(E - H_\alpha)\phi_a e^{i\vec{k}_a \cdot \vec{r}_\alpha} = 0 , \tag{2d}$$

where $\vec{r}_\alpha$ is the relative displacement of the two clusters in the partition α. The internal motion wave functions like $\phi_{a1}^{J_1 M_1}$ for the individual clusters associated with the various channels will be taken to be antisymmetric with respect to nucleon exchange.

The fact that the N nucleons comprising the system are physically indistinguishable causes it to be possible to arrange the partitions of the system into sets or families containing physically indistinguishable configurations. Suppose that n_α and $N - n_\alpha$ are the numbers of nucleons in the two clusters of partition α. Then there are

$$N_\alpha = \frac{N!}{n_\alpha ! (N - n_\alpha)!} \tag{3}$$

partitions in the family of physically indistinguishable partitions associated with partition α. Let us label the partitions in this family by $\alpha(1)$, $\alpha(2)$, ..., $\alpha(N_\alpha)$. The associated channels must be similarly distinguished. So we have families of physically indistinguishable channels which we identify by $a(1), a(2), \ldots, a(N_\alpha)$. Let $P_\alpha(n)$ be the nucleon exchange operator which transforms $\Psi_{a(1)}$ into $\Psi_{a(n)}$. Then the transition amplitude for scattering from the physical channel a (associated with the family $[a(i)]$ of unphysical channels) to physical channel b is[5]

$$\mathcal{T}_{ba} = \langle \Phi_{b(1)} | V_{\beta(1)} A_{\alpha(1)} | \Psi_{a(1)} \rangle N_\beta^{1/2} N_\alpha^{-1/2} , \tag{4a}$$

$$\Phi_{b(1)} = \phi_{b(1)} e^{i\vec{k}_{b(1)} \cdot \vec{r}_{\beta(1)}} , \tag{4b}$$

$$A_{\alpha(1)} = \sum_{n=1}^{N_\alpha} (-1)^{\sigma_\alpha(n)} P_\alpha(n) , \tag{4c}$$

where $\sigma_\alpha(n)$ is the parity of permutation $P_\alpha(n)$.

Equations (2) and (4) summarize the theory of scattering reactions. To calculate the transition amplitude one must solve the Schrödinger equation, Eq. (2a), subject to the boundary condition Eq. (2b). Then the solution is substituted into Eq. (4a). The execution of this task is difficult because of the multidimensional nature of the differential equation and the boundary condition.

We restrict ourselves at first to situations where only a single channel need be included. The more general case will be discussed later.

The RGM simplifies the problem by assuming that the cluster model approximation to the scattering wave function,

$$A_{\alpha(1)} \Psi_{a(1)} \approx A_{\alpha(1)} \phi_{a(1)} \psi_a(\vec{r}_\alpha) , \tag{5}$$

is adequate. Inasmuch as

$$[A_{\alpha(1)}, H] = 0 , \tag{6}$$

Eq. (2a) can be transformed into

$$\langle \phi_{a(1)} \delta(\vec{r} - \vec{r}_\alpha) | (E - H) A_{\alpha(1)} | \Psi_{A(1)} \rangle = 0 . \tag{7}$$

Combining Eqs. (5) and (7) gives

$$\frac{\hbar^2}{2m_\alpha}(k_a^2 + \nabla^2)\psi_a(\vec{r}) = \int d^3r' W_a(\vec{r}, \vec{r}') \psi_a(\vec{r}') , \tag{8a}$$

$$W_a(\vec{r}, \vec{r}') = \langle \phi_{a(1)} \delta(\vec{r} - \vec{r}_\alpha) | V_{\alpha(1)} A_{\alpha(1)} + (E - H_{\alpha(1)})(1 - A_{\alpha(1)}) | \phi_{a(1)} \delta(\vec{r}' - \vec{r}_\alpha) \rangle . \tag{8b}$$

Equation (8) is the RGM equation. The interaction kernel $W_a(\vec{r}, \vec{r}')$ is seen to be both energy and momentum dependent.

Let us transform to momentum space:

$$\psi_a(\vec{r}) = \int d^3k \chi_a(\vec{k}) e^{i\vec{k} \cdot \vec{r}} , \tag{9}$$

$$\frac{\hbar^2}{2m_\alpha}(k_a^2 - k^2)\chi_a(\vec{k}) = \left(\frac{1}{2\pi}\right)^3 \int d^3k' M_a(\vec{k}, \vec{k}') \chi_a(\vec{k}') , \tag{10a}$$

$$M_a(\vec{k}, \vec{k}') = \langle \phi_{a(1)} e^{i\vec{k} \cdot \vec{r}_\alpha} | V_{\alpha(1)} A_{\alpha(1)} + (E - H_{\alpha(1)})(1 - A_{\alpha(1)}) | \phi_{a(1)} e^{i\vec{k}' \cdot \vec{r}_\alpha} \rangle . \tag{10b}$$

The integral equation form of the RGM equation is thus

$$\chi_a(\vec{k}) = \delta(\vec{k} - \vec{k}_a) + \left(\frac{1}{2\pi}\right)^3 \int d^3k' \frac{2m_\alpha M_a(\vec{k}, \vec{k}')\chi_a(\vec{k}')}{\hbar^2(k_a^2 - k'^2 + i\epsilon)}. \tag{11}$$

Symbolically, this may be written

$$\chi_a = 1 + g_a M_a \chi_a \tag{12}$$

and assigned the formal solution

$$\chi_a = (1 - g_a M_a)^{-1}. \tag{13}$$

Combining Eqs. (9), (5), and (4a) gives

$$\mathcal{T}_{aa} = \int d^3k\, U_a(\vec{k}'_a, \vec{k})\chi_a(\vec{k}) = \mathcal{T}_{aa}(\vec{k}'_a, \vec{k}_a), \tag{14a}$$

$$U_a(k'_a, k) = \langle \phi_{a(1)} e^{i\vec{k}'_a \cdot \vec{r}_\alpha} \,|\, V_{\alpha(1)} A_{\alpha(1)} \,|\, \phi_{a(1)} e^{i\vec{k}\cdot\vec{r}} \rangle. \tag{14b}$$

Now make use of Eq. (13):

$$\begin{aligned} \mathcal{T}_{aa} &= U_a \chi_a \\ &= U_a (1 - g_a M_a)^{-1} \\ &= U_a + \mathcal{T}_{aa} g_a M_a. \end{aligned} \tag{15}$$

In explicit notation this reads

$$\mathcal{T}_{aa}(\vec{k}'\vec{q}) = U_a(\vec{k}', \vec{q}) + \left(\frac{1}{2\pi}\right)^3 \int d^3k \frac{\mathcal{T}_{aa}(\vec{k}', \vec{k})2m_\alpha M_\alpha(\vec{k}, \vec{q})}{\hbar^2(k_a^2 - k^2 + i\epsilon)}. \tag{16}$$

Equation (16) is the RGM integral equation for the transition amplitude.

III. CLUSTER MODEL ANTISYMMETRIZED COUPLED EQUATIONS FORMALISM

For the case where there is only one open channel, the equations of the ACEF[4] reduce to

$$\mathcal{T}_{aa}(\vec{k}'_a, \vec{k}_a) = \langle \phi_{a(1)} e^{i\vec{k}'_a \cdot \vec{r}_\alpha} \,|\, T_\alpha \,|\, \phi_{a(1)} e^{i\vec{k}_a \cdot \vec{r}_\alpha} \rangle, \tag{17}$$

$$T_\alpha = V_{\alpha(1)} A_{\alpha(1)} + T_\alpha (E - H_{\alpha(1)} + i\epsilon)^{-1} N_\alpha^{-1} A_{\alpha(1)} V_{\alpha(1)}. \tag{18}$$

The cluster model approximation applied to the ACEF integral equation, Eq. (18), results from setting

$$(E - H_{\alpha(1)} + i\epsilon)^{-1} = \left(\frac{1}{2\pi}\right)^3 \int d^3k \frac{|\phi_{a(1)} e^{i\vec{k}\cdot\vec{r}_\alpha}\rangle 2m_\alpha \langle \phi_{a(1)} e^{i\vec{k}\cdot\vec{r}_\alpha}|}{\hbar^2(k_a^2 - k^2 + i\epsilon)}. \tag{19}$$

Combining Eqs. (17)–(19) leads to the following integral equation for the transition amplitude:

$$\mathcal{T}_{aa}(\vec{k}', \vec{q}) = U_a(\vec{k}', \vec{q}) + \left(\frac{1}{2\pi}\right)^3 \int d^3k \frac{\mathcal{T}_{aa}(\vec{k}', \vec{k})2m_\alpha \tilde{U}_a(\vec{k}, \vec{q})}{N_\alpha \hbar^2(k_a^2 - k^2 + i\epsilon)}, \tag{20}$$

where

$$\tilde{U}_a(\vec{k}', \vec{k}) = \langle \phi_{a(1)} e^{i\vec{k}'\cdot\vec{r}_\alpha} \,|\, A_{\alpha(1)} V_{\alpha(1)} \,|\, \phi_{a(1)} e^{i\vec{k}\cdot\vec{r}_\alpha} \rangle. \tag{21}$$

Equation (20) is the CM-ACEF integral equation. It differs from the RGM integral equation, Eq. (16), by virtue of having $\tilde{U}_a N_\alpha^{-1}$ as kernel instead of M_a. $\tilde{U}_a N_\alpha^{-1}$ is simpler than M_a, it is not energy dependent, and it is not momentum dependent except insofar as $V_{\alpha(1)}$ is momentum dependent. Thus the CM-ACEF equation is simpler than the RGM yet it is based on the same physical approximations.

In Ref. 4 two versions of the ACEF were presented. Equation (18) is the single partition form of the antisymmetrized Baer-Kouri coupled equations formalism (ABKCEF). An alternative version is the antisymmetrized Kouri-Levin coupled equations formalism (AKLCEF) which gives the following equation for T_α in the single partition case:

$$T_\alpha = V_{\alpha(1)} A_{\alpha(1)} N_\alpha^{-1} G_{\alpha(1)} [A_{\alpha(1)} G_{\alpha(1)}^{-1} + T_\alpha], \tag{22}$$

where

$$G_{\alpha(1)} = (E - H_{\alpha(1)} + i\epsilon)^{-1}. \tag{23}$$

This expression is equivalent to one given by Kowalski.[6]

Applying the cluster model approximation Eq. (19) and then substituting Eq. (22) into Eq. (17) gives

$$\mathcal{T}_{aa}(\vec{k}', \vec{q}) = Z_a(\vec{k}', \vec{q}) + \left(\frac{1}{2\pi}\right)^3 \int d^3k \frac{U_a(\vec{k}', \vec{k})2m_\alpha \mathcal{T}_{aa}(\vec{k}, \vec{q})}{N_\alpha \hbar^2(k_a^2 - k^2 + i\epsilon)}, \tag{24}$$

where

$$Z_a(\vec{k}', \vec{k}) = \langle \phi_{a(1)} e^{i\vec{k}'\cdot\vec{r}_\alpha} \,|\, V_{\alpha(1)} A_{\alpha(1)} N_\alpha^{-1} G_{\alpha(1)} A_{\alpha(1)} G_{\alpha(1)}^{-1} \,|\, \phi_{a(1)} e^{i\vec{k}\cdot\vec{r}_\alpha} \rangle. \tag{25}$$

The expression for Z_a becomes easier to interpret when G_α^{-1} is eliminated as follows:

$$G_\alpha A_\alpha G_\alpha^{-1} = G_\alpha A_\alpha (G^{-1} + V_\alpha) = G_\alpha G^{-1} A_\alpha + G_\alpha A_\alpha V_\alpha = G_\alpha (G_\alpha^{-1} - V_\alpha) A_\alpha + G_\alpha A_\alpha V_\alpha = A_\alpha + G_\alpha [A_\alpha, V_\alpha] . \tag{26}$$

Thus Eq. (25) becomes

$$Z_a(\vec{k}', \vec{k}) = \langle \phi_{a(1)} e^{i\vec{k}'\cdot\vec{r}_\alpha} \,|\, V_{\alpha(1)} A_{\alpha(1)} + V_{\alpha(1)} A_{\alpha(1)} N_\alpha^{-1} G_{\alpha(1)} [A_{\alpha(1)}, V_{\alpha(1)}] \,|\, \phi_{a(1)} e^{i\vec{k}\cdot\vec{r}_\alpha} \rangle . \tag{27}$$

Equations (20) and (24) are the CM-ACEF equations. Equation (20) is the CM-ABKCEF equation, and Eq. (24) is the CM-AKLCEF equation.

IV. APPLICATION TO THE DINEUTRON-DINEUTRON SCATTERING PROBLEM

We have presented a scattering formalism that is based on the same physical approximations as the RGM but yields an integral equation having a simpler kernel than the RGM integral equation. To test whether the RGM and CM-ACEF are really equivalent, we apply them to the same physical system and compare their predictions. The system used is the two dineutron system. For this system we use as the two-body force the Brink and Boeker[7] $B1$ force. An RGM treatment of the ^2n-^2n system using the $B1$ force has been published by Giraud *et al.*[8] The $B1$ force potential is

$$V_{\alpha(1)} = V(r_{13}) + V(r_{14}) + V(r_{23}) + V(r_{24}) , \tag{28a}$$

$$V(r) = \sum_{j=1}^{2} S_j (1 - n_j + n_j P_M) \exp(-r^2/\mu_j^2) , \tag{28b}$$

where P_M is the Majorana exchange operator and

$$S_1 = -140.6 \text{ MeV}, \quad S_2 = 389.5 \text{ MeV},$$
$$\mu_1 = 1.4 \text{ fm}, \quad \mu_2 = 0.7 \text{ fm},$$
$$n_1 = 0.4864, \quad n_2 = 0.5290 .$$

The channel state wave function is taken to be

$$\phi_{\alpha(1)} = \phi(r_{12})\phi(r_{34}) , \tag{29a}$$

$$\phi(r) = \pi^{-3/4} b^{-3/2} \exp(-r^2/2b^2) \chi_0^0(\tfrac{1}{2}, \tfrac{1}{2}) , \tag{29b}$$

$$b = 2.121 \text{ fm} .$$

χ_0^0 is a singlet state spin wave function. The antisymmetrization operator is

$$A_{\alpha(1)} = 1 - P_{13} - P_{23} - P_{14} - P_{24} + P_{13}P_{24} , \tag{30}$$

and N_α is 6. The reduced mass m_α is the nucleon mass m.

Let us expand the quantities that appear in our integral equations in partial waves. We set

$$\mathcal{T}_{aa}(\vec{k}, \vec{k}') = \sum_l (2l+1) P_l(\cos\theta) \mathcal{T}_l(k, k') , \tag{31a}$$

$$U_a(\vec{k}, \vec{k}') = \sum_l (2l+1) P_l(\cos\theta) U_l(k, k') , \tag{31b}$$

and use similar expansions for $\tilde{U}_a$, M_a, and Z_a. θ is the angle between $\vec{k}$ and $\vec{k}'$. Inserting these partial wave expansions into our integral equations gives

$$\mathcal{T}_l(k', k) = U_l(k', k) + \frac{1}{2\pi^2} \int_0^\infty dp\, p^2 \frac{\mathcal{T}_l(k', p) 2m M_l(p, k)}{\hbar^2 (k_a^2 - p^2 + i\epsilon)} \tag{32}$$

for the RGM,

$$\mathcal{T}_l(k', k) = U_l(k', k) + \frac{1}{2\pi^2} \int_0^\infty dp\, p^2 \frac{\mathcal{T}_l(k', p) 2m \tilde{U}_l(p, k)}{6\hbar^2 (k_a^2 - p^2 + i\epsilon)} \tag{33}$$

for the CM-ABKCEF, and

$$\mathcal{T}_l(k', k) = Z_l(k', k) + \frac{1}{2\pi^2} \int_0^\infty dp\, p^2 \frac{U_l(k', p) 2m \mathcal{T}_l(p, k_a)}{6\hbar^2 (k_a^2 - p^2 + i\epsilon)} \tag{34}$$

for the CM-AKLCEF. The explicit expressions we find for the quantities that appear in these equations are shown in the Appendix.

The integral equations for the three cases are seen to have the same structure provided the transpose of the equation for the CM-ABKCEF case is taken. Thus in the ensuing discussion the RGM integral equation will be used, but the discussion applies to the other two equations as well.

Rather than solve the T matrix equations given above it is more convenient to solve the associated K-matrix integral equation and then get the T matrix by means of the Heitler equation. Define the RGM K matrix to be the solution of

$$\mathcal{K}_l(k', k) = M_l(k', k) + \frac{\mathcal{P}}{2\pi^2} \int_0^\infty dp\, p^2 \frac{\mathcal{K}_l(k', p) 2m M_l(p, k)}{\hbar^2 (k_a^2 - p^2)} . \tag{35}$$

Let us use symbolic representations of Eqs. (32)

and (35):

$$\mathcal{T} = U + \mathcal{T}gM\,, \tag{36}$$

$$\mathcal{K} = M + \mathcal{K}g^{(0)}M = M + Mg^{(0)}\mathcal{K}\,. \tag{37}$$

The Heitler equation connecting $\mathcal{T}$ and $\mathcal{K}$ is then derived in the following manner:

$$\begin{aligned}\mathcal{T} &= [U + \mathcal{T}gM](1 + g^{(0)}\mathcal{K}) - \mathcal{T}g^{(0)}\mathcal{K} \\ &= U + Ug^{(0)}\mathcal{K} + \mathcal{T}(g - g^{(0)})\mathcal{K}\,.\end{aligned} \tag{38}$$

Now we revert to explicit representation and use the fact that

$$\frac{1}{k^2 - s^2 + i\epsilon} - \frac{\mathcal{P}}{k^2 - s^2} = -i\pi\delta(k^2 - s^2) \tag{39}$$

to find

$$\mathcal{T}_l(k', k) = U_l(k', k) + \frac{\mathcal{P}}{2\pi^2}\int_0^\infty dp\, p^2 \frac{U_l(k', p)2m\mathcal{K}_l(p, k)}{\hbar^2(k_2^{\,2} - p^2)} - \frac{i}{4\pi}\frac{k_a\mathcal{T}_l(k', k_a)2m\mathcal{K}_l(k_a, k)}{\hbar^2}\,. \tag{40}$$

The partial wave transition amplitude is the on-the-energy shell value of $\mathcal{T}_l$, for which Eq. (40) becomes

$$\mathcal{T}_l(k_a, k_a) = \frac{U_l(k_a, k_a) + (m\mathcal{P}/\pi^2\hbar^2)\int_0^\infty dp\, p^2[U_l(k_a, p)\mathcal{K}_l(p, k_a)/(k_a^{\,2} - p^2)]}{1 + (imk_a/2\pi\hbar^2)\mathcal{K}_l(k_a, k_a)}\,. \tag{41}$$

This would reduce to the conventional form of the Heitler equation with just $\mathcal{K}_l(k_a, k_a)$ in the numerator if we had $U_l = M_l$.

Our calculation consisted in solving the integral equations for the partial wave K-matrix amplitudes, Eq. (35). This was done by approximating the integral by a Gaussian quadrature sum and solving the resulting set of linear algebraic equations. The on-the-energy shell T-matrix partial wave amplitudes were calculated from the Heitler equation, Eq. (41). Then the total scattering cross section was determined by means of

$$\sigma_T(K_a) = \frac{m^2}{\pi\hbar^4}\sum_l (2l+1)\left|\mathcal{T}_l(k_a, k_a)\right|^2\,, \tag{42a}$$

$$K_a = \hbar^2 k_a^{\,2}/2m\,. \tag{42b}$$

The total scattering cross section as a function of K_a, the kinetic energy in the center of mass frame, is plotted in Fig. 1. The RGM, CM-ABKCEF, and CM-AKLCEF results are plotted together. The two CM-ACEF results are practically indistinguishable. For energies beyond 30 MeV the agreement between the RGM and the CM-ACEF is pretty good. Below 30 MeV the agreement is only fair.

The calculation was repeated with the sign of the Majorana exchange terms in the nucleon-nucleon potential, Eq. (28b), reversed. This produces a much more strongly attractive interaction. As a result the cluster model approximation cannot work so well. The cross sections are plotted in Fig. 2. All three formalisms predict a maximum at $K_a = 0$ and all approach the Born approximation at high values of K_a. All three cases have a broad resonance at about $K_a = 10$ MeV.

For the ACEF calculations the phase of the resonance amplitude is opposite that of the background amplitude so that there is destructive interference. For the RGM, on the other hand, the phase of the resonant amplitude sweeps through π radians across the resonance so that the interference changes from destructive to constructive. The zero energy peaks in the cross sections are not shown on the plot. These are very different, being about 8.6 b for the RGM, 66 b for the CM-ABKCEF, and 112 b for the CM-AKLCEF.

We conclude that the cross sections predicted by the three methods agree pretty well when a reasonable two-body force is used. When a very strong two-body force is used we find that the dependence on energy of the cross sections predicted by the three formalisms is still roughly the same. Below 30 MeV kinetic energy in the center of mass frame the three methods differ considerably. Above 30 MeV there is rough agreement. In the absence of the exact solution it is hard to say how significant these discrepancies are. Some light will be shed on this matter in the next section.

V. ALTERNATIVE CALCULATION OF THE RGM CROSS SECTION

Using the conventional form of the Heitler equation leads to the following expression for the partial wave transition amplitude

$$\mathcal{T}_l^K(k_a, k_a) = \frac{\mathcal{K}_l(k_a, k_a)}{1 + (imk_a/2\pi\hbar^2)\mathcal{K}_l(k_a, k_a)}\,, \tag{43}$$

and the associated cross section is

$$\sigma_K(K_a) = \frac{m^2}{\pi\hbar^4}\sum_l (2l+1)\left|\mathcal{T}_l^K(k_a, k_a)\right|^2\,. \tag{44}$$

This is what we would have gotten in place of Eqs. (41) and (42) if the driving terms in Eqs. (32)–(34) were identical with the interactions occurring in the kernels.

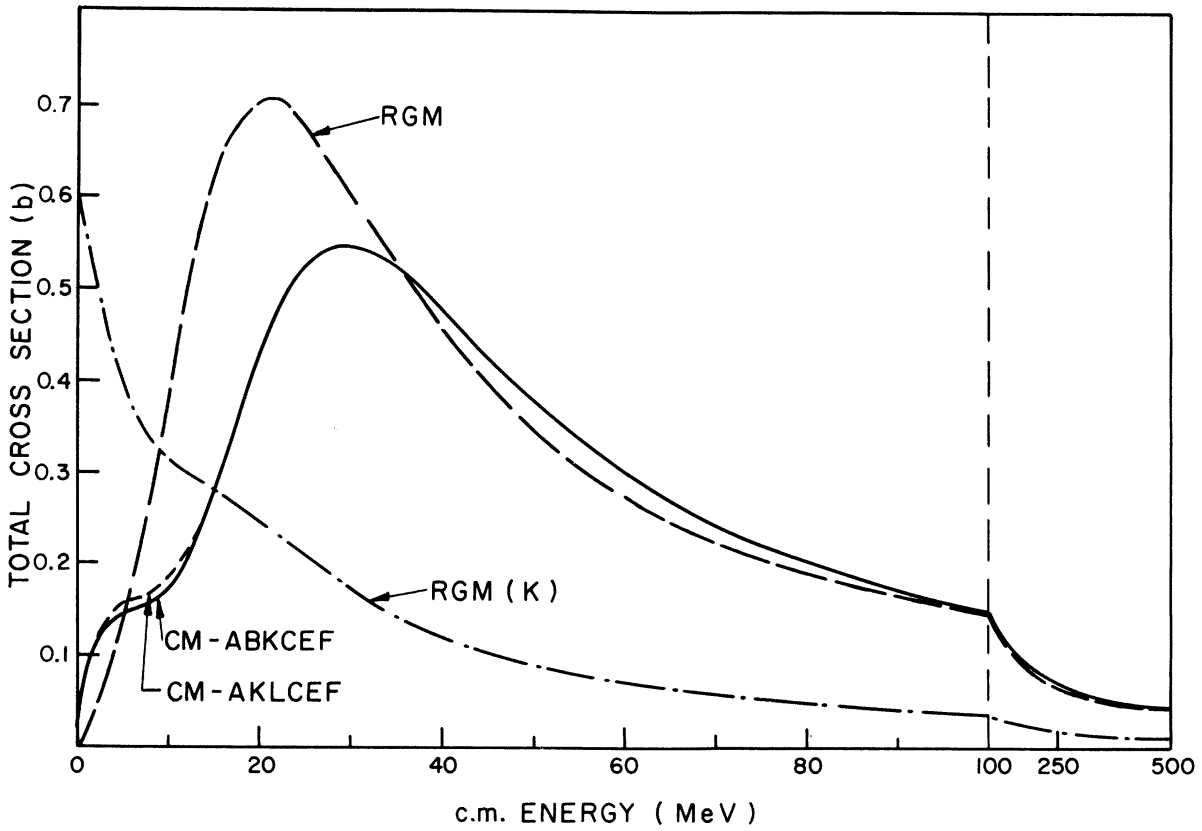

FIG. 1. Total cross section for dineutron-dineutron scattering calculated by means of the resonating group method (RGM), the cluster model-antisymmetrized Baer-Kouri coupled equations formalism (CM-ABKCEF), and the cluster model-antisymmetrized Kouri-Levin coupled equations formalism (CM-AKLCEF). The Brink and Boeker B1 two-body force is used. RGM(K) is the total cross section deduced from the asymptotic behavior of the RGM wave function.

The RGM cross section calculated in this manner is shown in Fig. 1 and is referred to as the K cross section. The K cross section is seen to be rather different from the total cross section. For the RGM we would argue that the discrepancy between σ_K and σ_T is a measure of the quality of the results since we would have $\sigma_K = \sigma_T$ if the RGM were an exact formalism.

The K cross section σ_K is the result one gets for the total cross section by calculating particle fluxes in the asymptotic region from the solution ψ_a of Eq. (8). This would be the scattering due to the interaction W_a. The total cross section σ_T is derived from the scattering amplitude calculated by inserting this same wave function into the expression shown in Eq. (4). These two ways of calculating the cross section give the same result when the exact wave function is used. The extent to which they differ is an indication of how much the RGM wave function differs from the exact one.

The RGM σ_K is much smaller than the RGM σ_T. The two have resonances at the same energy but the resonances have different characteristics. The difference between the RGM σ_T and the CM-ACEF σ_T's is less than the difference between the RGM σ_T and the RGM σ_K. On this basis one can say that the CM-ACEF agrees with the RGM within the limits of uncertainty created by neglecting excitation, rearrangement, and breakup effects.

The K cross sections calculated for the CM-ABKCEF and CM-AKLCEF are equal to each other and are very much smaller than the corresponding total cross sections. However, in contrast to the RGM, there is no basis for expecting the ACEF σ_K to equal the ACEF σ_T.

As in the RGM case, the ACEF σ_K can be interpreted in terms of a wave function. Let us define the wave function $\Psi^{BK}_{a(1)}$ by the requirement that

$$T_\alpha \phi_{a(1)} e^{i\vec{k}_a\cdot\vec{r}_\alpha} = V_{\alpha(1)} A_{\alpha(1)} \Psi^{BK}_{a(1)}, \tag{45}$$

where for T_α one substitutes the formal solution of Eq. (18):

$$V_{\alpha(1)} A_{\alpha(1)} (1 - G_{\alpha(1)} N_\alpha^{-1} A_{\alpha(1)} V_{\alpha(1)})^{-1} \Phi_{a(1)} = V_{\alpha(1)} A_{\alpha(1)} \Psi^{BK}_{a(1)}, \tag{46}$$

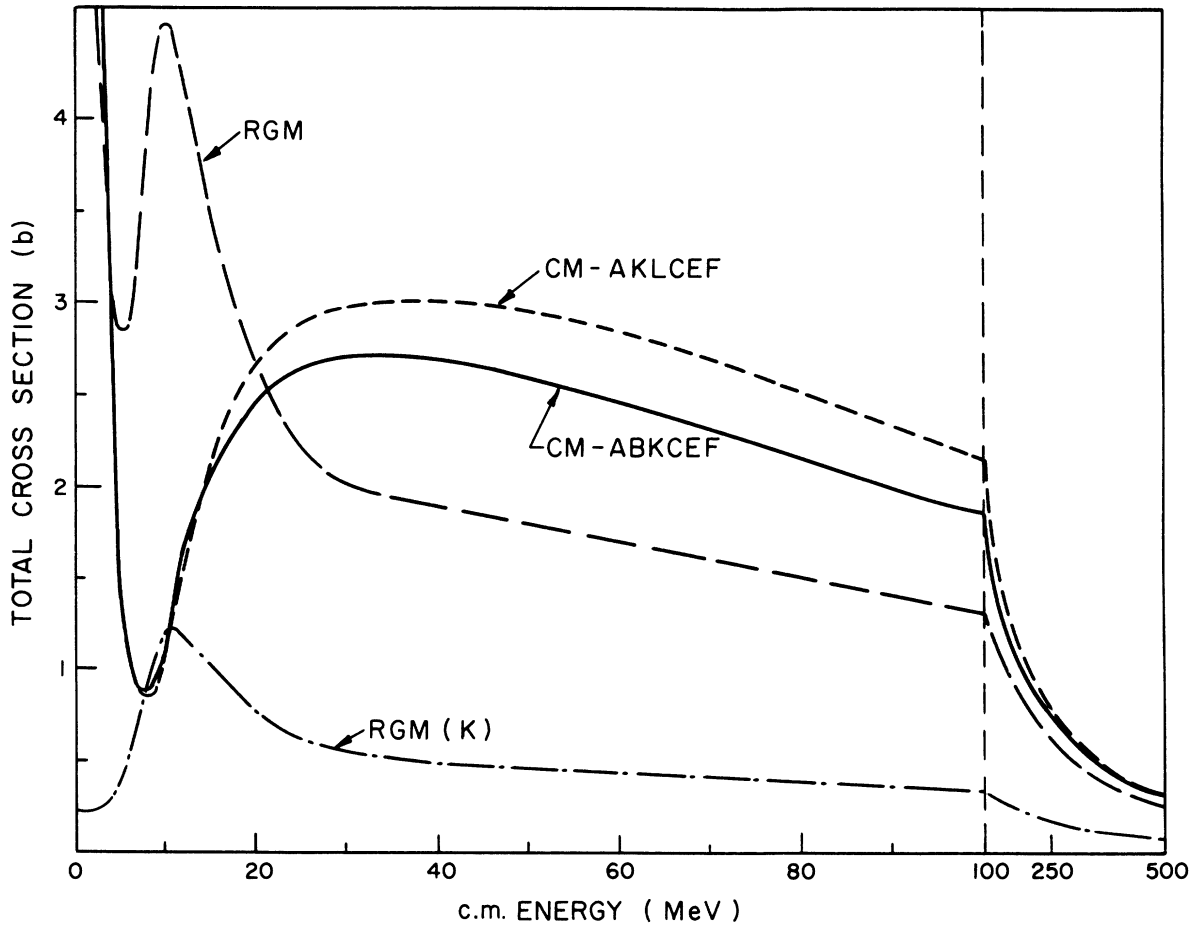

FIG. 2. Same as Fig. 1 except that the two-body B1 force potential has had the sign of the Majorana exchange potential reversed.

where

$$G_{\alpha(1)} = (E - H_{\alpha(1)} + i\epsilon)^{-1}, \tag{47}$$

and

$$\Phi_{a(1)} = \phi_{a(1)} e^{i\vec{k}_a \cdot \vec{r}_\alpha}. \tag{48}$$

Thus we can set

$$\begin{aligned}\Psi^{BK}_{a(1)} &= (1 - G_{\alpha(1)} N_\alpha^{-1} A_{\alpha(1)} V_{\alpha(1)})^{-1} \Phi_{a(1)} \\ &= \Phi_{a(1)} + G_{\alpha(1)} N_\alpha^{-1} A_{\alpha(1)} V_{\alpha(1)} \Psi^{BK}_{a(1)},\end{aligned} \tag{49}$$

where the quantities that appear need retain only terms that are not projected to zero by $A_{\alpha(1)}$. Operating on Eq. (49) by $E - H_{\alpha(1)}$ gives

$$(E - H_{\alpha(1)}) \Psi^{BK}_{a(1)} = N_\alpha^{-1} A_{\alpha(1)} V_{\alpha(1)} \Psi^{BK}_{a(1)}. \tag{50}$$

Equations (49) and (50) are the equations for the ABKCEF wave functions in the single partition case. For comparison we write the RGM counterparts to these equations:

$$\Psi^{RG}_{a(1)} = \Phi_{a(1)} + G_{\alpha(1)} W_{\alpha(1)} \Psi^{RG}_{a(1)}, \tag{51}$$

$$W_{\alpha(1)} = V_{\alpha(1)} A_{\alpha(1)} + (E - H_{\alpha(1)})(1 - A_{\alpha(1)}) \tag{52a}$$

$$= A_{\alpha(1)} V_{\alpha(1)} + (1 - A_{\alpha(1)})(E - H_{\alpha(1)}), \tag{52b}$$

$$(E - H_{\alpha(1)}) \Psi^{RG}_{a(1)} = W_{\alpha(1)} \Psi^{RG}_{a(1)}. \tag{53}$$

Using the cluster model approximation to $G_{a(1)}$ shown in Eq. (19) and the cluster model approximation to $\Psi^{RG}_{a(1)}$ shown in Eq. (5) causes Eq. (51) to become identical with the RGM equation, Eq. (8). Neither of the two equations, Eq. (50) or Eq. (53), is identical with the Schrödinger equation, Eq. (2a), however.

Since

$$A_{\alpha(1)}{}^2 V_{\alpha(1)} \Psi_{a(1)} = N_\alpha A_{\alpha(1)} V_{\alpha(1)} \Psi_{a(1)}, \tag{54}$$

operating on Eq. (50) with $A_{\alpha(1)}$ yields

$$A_{\alpha(1)} (E - H_{\alpha(1)}) \Psi^{BK}_{a(1)} = A_{\alpha(1)} V_{\alpha(1)} \Psi^{BK}_{a(1)} \tag{55}$$

which is identical to Eq. (53). However, the corresponding integral equations, Eqs. (49) and (5), become inequivalent once the cluster model ap-

proximation is inserted for the Green's function operator $G_{\alpha(1)}$.

Finally, let us get the equations for the AKLCEF wave function in the single partition case. Define the wave function $\Psi^{KL}_{a(1)}$ with the help of Eq. (22) and Eq. (48):

$$V_{\alpha(1)}A_{\alpha(1)}\Psi^{KL}_{a(1)} = T_\alpha \Phi_{a(1)} = V_{\alpha(1)}A_{\alpha(1)}N_\alpha^{-1}G_{\alpha(1)} \times[A_{\alpha(1)}G_{\alpha(1)}^{-1} + T_\alpha]\Phi_{a(1)} . \quad (56)$$

Thus we can let

$$\Psi^{KL}_{a(1)} = G_{\alpha(1)}N_\alpha^{-1}A_{\alpha(1)}G_{\alpha(1)}^{-1}\Phi_{a(1)} + G_{\alpha(1)}N_\alpha^{-1}V_{\alpha(1)}A_{\alpha(1)}\Psi^{KL}_{a(1)} . \quad (57)$$

Operating on Eq. (57) with $(E - H_{\alpha(1)})$ gives

$$(E - H_{\alpha(1)})\Psi^{KL}_{a(1)} = N_\alpha^{-1}V_{\alpha(1)}A_{\alpha(1)}\Psi^{KL}_{a(1)} . \quad (58)$$

Just as in the RGM case, the K cross sections for the CM-ACEF correspond to the total cross sections for elastic scattering for the states whose wave functions are $\Psi^{BK}_{a(1)}$ and $\Psi^{KL}_{a(1)}$. However, there is an important difference between the RGM and the ACEF in this regard. The RGM integral equation

$$T_\alpha = V_{\alpha(1)}A_{\alpha(1)} + T_\alpha G_{\alpha(1)}W_{\alpha(1)} \quad (16a)$$

used to calculate the scattering cross section is derived from Eqs. (51), (4a), and (17). Thus it is based on the assumption that $\Psi^{RG}_{a(1)}$ equals the true scattering wave function $\Psi_{a(1)}$, solution of the Schrödinger equation, Eq. (2a). Thus the RGM assumes $\sigma_K = \sigma_T$. The single partition ACEF, on the other hand, is based on integral equations which are exact, except for the restriction to a single partition. The ACEF does not require $\Psi^{BK}_{a(1)}$ or $\Psi^{KL}_{a(1)}$ to be the solution of the Schrödinger equation. Thus σ_K has no special significance for the ACEF. $\Psi^{BK}_{a(1)}$ and $\Psi^{KL}_{a(1)}$ are just certain auxiliary functions which happen to produce the same result as $\Psi_{a(1)}$ when substituted into the expression for the transition amplitude Eq. (4a).

Our result that σ_K is much smaller than σ_T for the CM-AKLCEF is very interesting with respect to another facet of many-body scattering theory. There is an alternative approach we could have used to evaluate the CM-AKLCEF driving term Z_a which appears in Eqs. (24) and (25):

$$Z_a(\vec{k}', \vec{k}) = \langle \phi_{a(1)}e^{i\vec{k}'\cdot\vec{r}_\alpha} | V_{\alpha(1)}A_{\alpha(1)}N_\alpha^{-1}G_{\alpha(1)}A_{\alpha(1)}G_{\alpha(1)}^{-1} | \phi_{a(1)}e^{i\vec{k}\cdot\vec{r}_\alpha}\rangle = \sum_{n=1}^{N_\alpha}(-1)^{\sigma_\alpha(n)}\langle \phi_{a(1)}e^{i\vec{k}'\cdot\vec{r}_\alpha} | V_{\alpha(1)}A_{\alpha(1)}N_\alpha^{-1}G_{\alpha(1)}G_{\alpha(n)}^{-1} | \phi_{a(n)}e^{i\vec{k}\cdot\vec{r}_{\alpha(n)}}\rangle . \quad (59)$$

We could have made use of the results of the analysis of Lippmann[9] and set

$$G_{\alpha(1)}G_{\alpha(n)}^{-1} | \phi_{a(n)}e^{ik\cdot r_{\alpha(n)}}\rangle = \delta_{n1} | \phi_{a(1)}e^{i\vec{k}\cdot\vec{r}_\alpha}\rangle . \quad (60)$$

This would then give the result

$$Z_a(\vec{k}', \vec{k}) = \langle \phi_{a(1)}e^{i\vec{k}'\cdot\vec{r}_\alpha} | V_{\alpha(1)}A_{\alpha(1)}N_\alpha^{-1} | \phi_{a(1)}e^{i\vec{k}\cdot\vec{r}_\alpha}\rangle = N_\alpha^{-1}U_a(\vec{k}', \vec{k}) \quad (61)$$

as an alternative to Eq. (27). Equation (24) for the CM-AKLCEF transition amplitude would then have become

$$\mathcal{T} = N^{-1}U + N^{-1}Ug\mathcal{T} . \quad (62)$$

Since the driving term in this equation is the same as the interaction in the kernel, the cross section σ_T would have been identical with σ_K. We have found that σ_K is much smaller than σ_T. That means that using Eq. (27) for the driving term of the CM-AKLCEF integral equation for the transition amplitude gives a much larger cross section than using Eq. (61). We are forced to conclude that the Lippmann prescription for evaluating $G_{\alpha(1)}G_{\alpha(n)}^{-1}\phi_{a(n)}\exp(i\vec{k}\cdot\vec{r}_{\alpha(n)})$ is not valid. If it were valid, we would have found that $\sigma_K = \sigma_T$ for the CM-AKLCEF.

VI. COMPARISON WITH THE BORN APPROXIMATION

Having solved the RGM equation and the CM-ACEF equations for two examples of $^2n + ^2n$ scattering, it is of some interest to compare the resultant cross sections with those that are given by the Born approximation. One is always curious to see how well this widely used approximation compares with the result of the more arduous exact cluster model calculation.

Two types of Born approximation calculation were done. First of all, the Born approximation to the T matrix was done by setting $\mathcal{T}_l = U_l$ for the RGM and CM-AKLCEF and setting $\mathcal{T}_l = Z_l$ for the CM-AKLCEF. Secondly, the Born approximation to the K matrix was done by setting $\mathcal{K}_l = M_l$ for the RGM, $\mathcal{K}_l = \tilde{U}_l$ for the CM-ABKCEF, and $\mathcal{K}_l = U_l$ for the CM-AKLCEF. This K matrix is then used in the Heitler equation to calculate the T matrix.

The results of the Born approximation calcula-

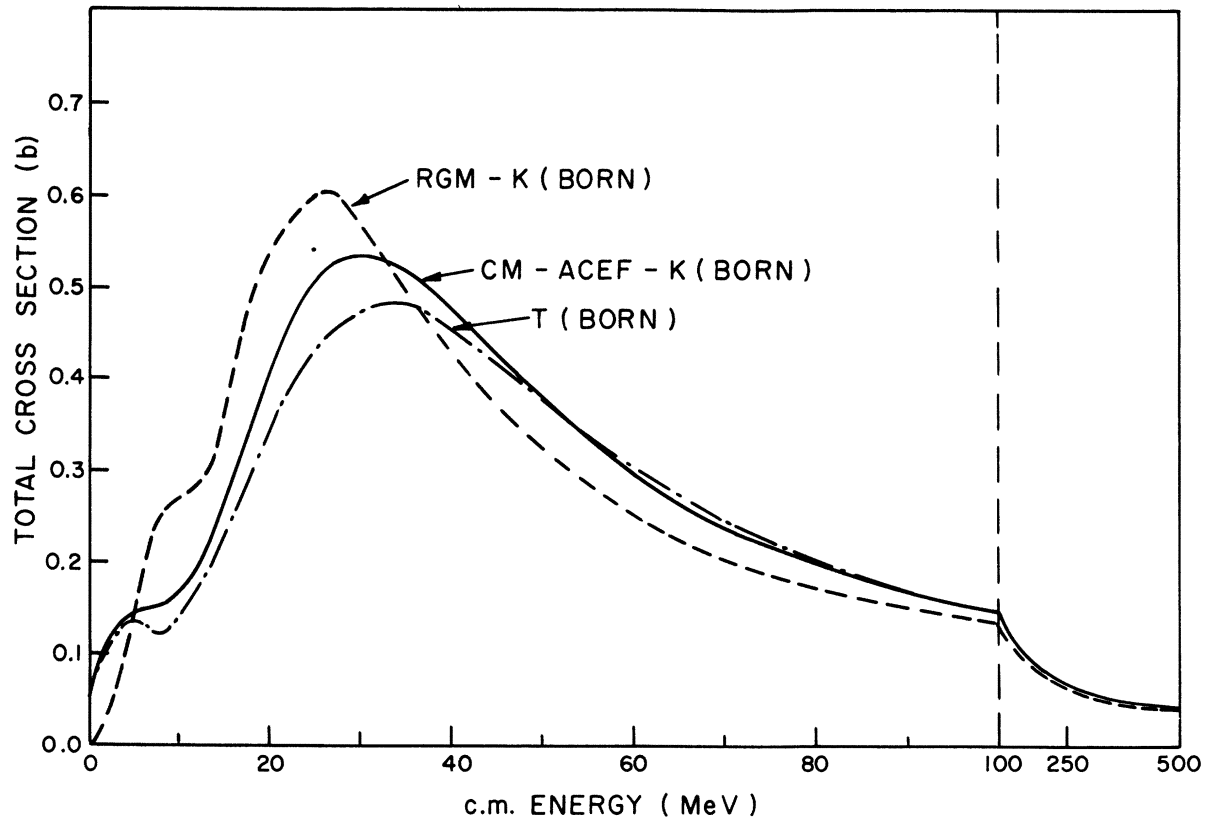

FIG. 3. Total cross section for dineutron-dineutron scattering calculated by means of the Born approximation to the resonating group method (RGM) or cluster model-antisymmetrized coupled equations formalism (CM-ACEF) T matrix [T(BORN)], or calculated by means of the Born approximation to the RGM K matrix [RGM-K(BORN)] or CM-ACEF K matrix [CM-ACEF-K(BORN)]. The Brink and Boeker B1 two-body force is used.

tions of the total cross section for $^2n+^2n$ scattering using the Brink and Boeker[6] $B1$ force are shown in Fig. 3. The two forms of the Born approximation for the T matrix give results that are essentially identical. The two CM-ACEF Born approximation to the K-matrix calculations also give indistinguishable results for the total cross section. The Born approximation is seen to work quite well for this case at least for the CM-ACEF. For the CM-ACEF the K matrix Born is less than 3% in error while the T-matrix Born error is always less than 19%. For the RGM the K-matrix Born approximation error is greater than 43% at some energies while the T-matrix Born approximation is still worse.

The calculations were repeated with a reversal of the sign of the Majorana exchange term in the two-body force. The results are shown in Fig. 4. With the strength of the two-body interaction thus increased we find that the Born approximation works less well. The overall energy dependence is reproduced except for the sharp peak in the RGM cross section. The two forms of the T-matrix Born approximation again lead to the same cross section. In the vicinity of about 40 MeV the K-matrix Born approximation is in error by about 8% for the CM-ABKCEF and by about 15% for the CM-AKLCEF.

The definition we have chosen for the K matrix was motivated by the desire to simplify the calculations for the transition amplitude. Our K matrix is different from that used by Kouri, Levin, and Sandhas[10] for the coupled equations formalism. In particular, we cannot show that any real approximation to our K matrix will necessarily lead to a unitary collision matrix. Nevertheless, we find that making the Born approximation for our K matrix does give a great improvement over the same approximation for the T matrix.

VII. MULTIPARTITION AND COMPOUND NUCLEUS EFFECTS

We have seen how the CM-ACEF provides an alternative to the RGM which is based on the same

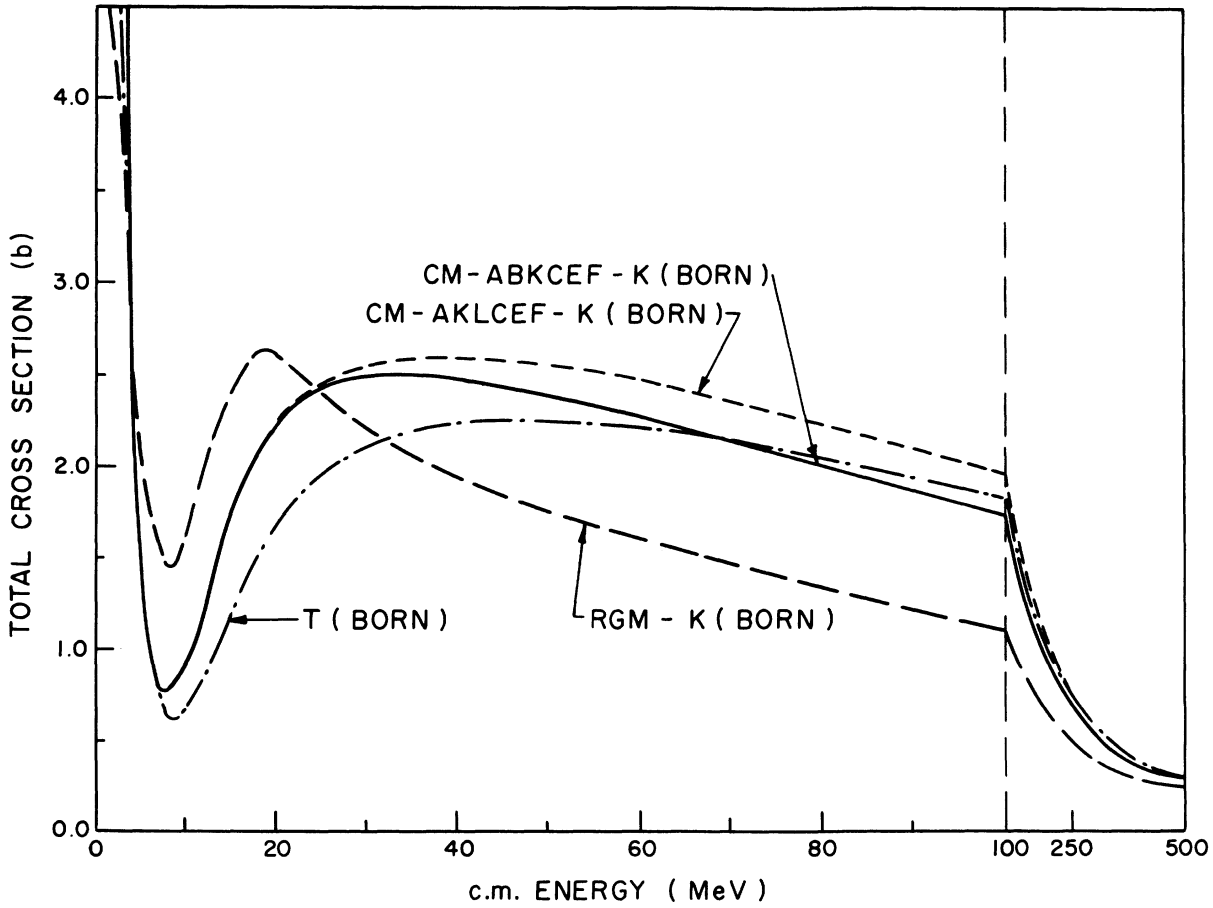

FIG. 4. Same as Fig. 3 except that the two-body $B1$ force potential has had the sign of the Majorana exchange potential reversed.

approximations but leads to equations that are somewhat simpler in form. The discussion was limited to the single channel case. When the more realistic multipartition case is considered, then the greater simplicity of the CM-ACE becomes even more striking.

First let us discuss multichannel effects. Instead of limiting ourselves to a single channel as heretofore, we now include several channels belonging to the same partition. We thus allow for inelastic as well as elastic scattering. The cluster model approximation to the RGM function then becomes

$$A_\alpha \Psi_a \approx A_\alpha \sum_{a'} \phi_{a'} \psi_{a'a}. \tag{63}$$

We have dropped the (1) from the subscripts a and α but its presence, which has the effect of excluding physically indistinguishable partitions, will be understood. Using Eq. (63) in place of Eq. (5) leads to a set of coupled integral equations for the elastic and inelastic transition amplitudes. Thus in place of Eq. (16) we find

$$\begin{aligned} \mathcal{T}_{a'a}(\vec{k}', \vec{k}) &= U_{a'a}(\vec{k}', \vec{k}) \\ &+ \sum_{a''} \left(\frac{1}{2\pi}\right)^3 \int d^3p \, \frac{\mathcal{T}_{a'a''}(\vec{k}', \vec{p}) 2m_\alpha M_{a''a}(\vec{p}, \vec{k})}{\hbar^2(k_a^2 - p^2 + i\epsilon)} . \end{aligned} \tag{64}$$

The same sort of generalization of the CM-ACEF equations is achieved by replacing Eq. (19) by

$$G_\alpha = \sum_{a'} \left(\frac{1}{2\pi}\right)^3 \int d^3k \, \frac{|\phi_{a'} e^{i\vec{k}\cdot\vec{r}_\alpha}\rangle 2m_\alpha \langle \phi_{a'} e^{i\vec{k}\cdot\vec{r}_\alpha}|}{\hbar^2(k_a^2 - k^2 + i\epsilon)} . \tag{65}$$

Next we consider the multichannel case where rearrangement scattering is permitted as well as inelastic and elastic scattering. The cluster model approximation to the RGM wave function is taken to be

$$A_\alpha \Psi_a \approx \sum_\beta A_\beta \sum_b \phi_b \psi_{ba}. \tag{66}$$

Substituting this into Eq. (7) leads to

$$\chi_{ba}(\vec{k}) = \delta_{\beta\alpha}\delta_{ba}\delta(\vec{k}-\vec{k}_a) + \sum_\gamma \sum_c \left(\frac{1}{2\pi}\right)^3 \int d^3k' \frac{2m_\beta M_{bc}(\vec{k},\vec{k}')\chi_{ca}(\vec{k}')}{\hbar^2(k_b^2 - k^2 + i\epsilon)} - \sum_{\gamma\neq\beta} \sum_c \left(\frac{1}{2\pi}\right)^3 \int d^3k' \frac{2m_\beta N_{bc}(\vec{k},\vec{k}')\chi_{ca}(\vec{k}')}{\hbar^2(k_b^2 - k^2 + i\epsilon)} \tag{67}$$

in place of Eq. (11). In Eq. (67)

$$\chi_{ba}(\vec{k}) = \left(\frac{1}{2\pi}\right)^3 \int d^3r e^{-i\vec{k}\cdot\vec{r}} \psi_{ba}(\vec{r}), \tag{68}$$

$$M_{bc}(\vec{k},\vec{k}') = \langle \phi_b e^{i\vec{k}\cdot\vec{r}_\beta} | W_\gamma | \phi_c e^{i\vec{k}'\cdot\vec{r}_\gamma} \rangle, \tag{69}$$

and

$$N_{bc}(\vec{k},\vec{k}') = \langle \phi_b e^{i\vec{k}\cdot\vec{r}_\beta} | E - H_\gamma | \phi_c e^{i\vec{k}'\cdot\vec{r}_\gamma} \rangle. \tag{70}$$

Thus in the multipartition generalization the RGM picks up the nonorthogonality interaction $(E - H_\gamma)$ in addition to the RGM interaction W_γ. Finally, in place of Eq. (16) we find

$$\mathcal{T}_{ba}(\vec{k}',\vec{k}) = U_{ba}(\vec{k}',\vec{k}) + \sum_\gamma \sum_c \left(\frac{1}{2\pi}\right)^3 \int d^3p \frac{\mathcal{T}_{bc}(\vec{k}',\vec{p}) 2m_\gamma [M_{ca}(\vec{p},\vec{k}) - (1-\delta_{\gamma\alpha}) N_{ca}(\vec{p},\vec{k})]}{\hbar^2(k_c^2 - p^2 + i\epsilon)}, \tag{71}$$

where

$$U_{ba}(\vec{k}',\vec{k}) = \langle \phi_b e^{i\vec{k}'\cdot\vec{r}_\beta} | V_\beta A_\alpha | \phi_a e^{i\vec{k}\cdot\vec{r}_\alpha} \rangle. \tag{72}$$

To derive the multipartition form of the CM-ABKCEF we start with the multipartition generalization of Eq. (18) given in Ref. (4):

$$T_{\beta\alpha} = N_\beta^{1/2} V_\beta N_\alpha^{-1/2} A_\alpha + \sum_\gamma T_{\beta\gamma} N_\gamma^{-1/2} G_\gamma W_{\gamma\alpha} N_\alpha^{-1/2} A_\alpha V_\alpha, \tag{73}$$

where W is a numerical matrix called the channel coupling array. The transition amplitude is related to the transition operator $T_{\beta\alpha}$ by

$$\mathcal{T}_{ba}(\vec{k}',\vec{k}) = \langle \phi_b e^{i\vec{k}'\cdot\vec{r}_\beta} | T_{\beta\alpha} | \phi_a e^{i\vec{k}\cdot\vec{r}_\alpha} \rangle. \tag{74}$$

Substituting the channel state expansion for G_γ, Eq. (65), into Eq. (73), and using Eq. (74) gives

$$\mathcal{T}_{ba}(\vec{k}',\vec{k}) = N_\beta^{1/2} U_{ba}(\vec{k}',\vec{k}) N_\alpha^{-1/2} + \sum_\gamma N_\gamma^{-1/2} W_{\gamma\alpha} N_\alpha^{-1/2} \sum_c \left(\frac{1}{2\pi}\right)^3 \int d^3p \frac{\mathcal{T}_{bc}(\vec{k}',\vec{p}) 2m_\gamma \tilde{U}_{ca}(\vec{p},\vec{k})}{\hbar^2(k_c^2 - p^2 + i\epsilon)}, \tag{75}$$

where

$$U_{ca}(\vec{k}',\vec{k}) = \langle \phi_c e^{i\vec{k}'\cdot\vec{r}_\gamma} | A_\alpha V_\alpha | \phi_a e^{i\vec{k}\cdot\vec{r}_\alpha} \rangle. \tag{76}$$

For the multipartition form of the CM-AKLCEF one starts with the multipartition generalization of Eq. (22) found in Ref. (3),

$$T_{\beta\alpha} = N_\beta^{-1/2} V_\beta A_\beta^+ \sum_\gamma W_{\beta\gamma} N_\gamma^{-1/2} G_\gamma [N_\gamma^{-1/2} A_\gamma^+ G_\alpha^{-1} N_\alpha^{1/2} + T_{\gamma\alpha}]. \tag{77}$$

Combining this with Eqs. (65) and (74) gives

$$\mathcal{T}_{ba}(\vec{k}',\vec{k}) = \sum_\gamma N_\beta^{-1/2} W_{\beta\gamma} N_\gamma^{-1/2} \sum_c \left(\frac{1}{2\pi}\right)^3 \int d^3p \frac{Y_{bc}(\vec{k}',\vec{p}) 2m_\gamma}{\hbar^2(k_c^2 - p^2 + i\epsilon)} [X_{ca}(\vec{p},\vec{k}) + \mathcal{T}_{ca}(\vec{p},\vec{k})], \tag{78}$$

where

$$Y_{bc}(\vec{k}',\vec{k}) = \langle \phi_b e^{i\vec{k}'\cdot\vec{r}_\beta} | V_\beta A_\beta^+ | \phi_c e^{i\vec{k}\cdot\vec{r}_\gamma} \rangle, \tag{79}$$

and

$$X_{ca}(\vec{k}',\vec{k}) = N_\gamma^{-1/2} \langle \phi_c e^{i\vec{k}'\cdot\vec{r}_\gamma} | A_\gamma^+ G_\alpha^{-1} | \phi_a e^{i\vec{k}\cdot\vec{r}_\alpha} \rangle N_\alpha^{1/2}$$

$$= N_\gamma^{-1/2} \langle \phi_c e^{i\vec{k}'\cdot\vec{r}_\gamma} | \frac{\hbar^2(k_c^2 - k'^2 + i\epsilon)}{2m_\alpha} A_\gamma^+ + (A_\gamma^+ V_\alpha - V_\gamma A_\gamma^+) | \phi_a e^{i\vec{k}\cdot\vec{r}_\alpha} \rangle N_\alpha^{1/2}. \tag{80}$$

Equations (71), (75), and (78) are our multipartition forms of the RGM, CM-ABKCEF, and CM-AKLCEF. The ACEF equations are seen to be simpler than the RGM equation in that only direct and exchange matrix elements of the residual interactions $V_\alpha, V_\beta, \ldots$ are required whereas the RGM requires the matrix elements of $E-H_\alpha$, $E-H_\beta, \ldots$ as well. The approximation on which these treatments are based results from the restriction of the range of the channel indices $a, b, c, \ldots$ to a finite number of values. If the sums over channels c included all possible channel states, then Eqs. (71), (75), and (78) would be exact.

Although extending the range of the channel sums causes Eq. (71) to become exact in the sense that it is based on a formal solution of the exact Schrödinger equation, it creates a situation in which Eq. (71) becomes invalid as an integral equation because the kernel ceases to be connected. The ACEF equations, Eqs. (75) and (78), do not suffer from this shortcoming.

The multipartition reaction formalisms presented above represent a generalization of the coupled channels reaction formalism to include exchange effects. These formalisms provide a basis for a complete description of direct interaction processes in nuclear reactions. To include compound nucleus effects it would be necessary to extend the range of the channel states sums by a large amount beyond those channels which are open. A more attractive alternative for attempting to include compound nucleus effects is to regard the truncation of the channel state sums in the representations of the Green's function operators $G_\alpha, G_\beta, \ldots$ as the action of a Feshbach projection operator. Then the results of the solution of the resulting finite set of coupled integral equations could be used as input to a Feshbach formalism[11] for compound nucleus effects. We will sketch below how this may be done. Our method will be somewhat different from that published earlier[12] in that we apply the formalism to the K matrix rather than directly to the T matrix.

Let us transcribe our coupled integral equations into symbolic form

$$\mathcal{T} = U + \mathcal{T} g \mathcal{O} \quad \text{(RGM)}, \tag{81}$$

$$\mathcal{T} = \mathcal{U} + \mathcal{T} g \mathcal{W} \quad \text{(CM-ABKCEF)}, \tag{82}$$

$$\mathcal{T} = \mathcal{Y} g (X + \mathcal{T}) \quad \text{(CM-AKLCEF)}. \tag{83}$$

All quantities in the above equations are to be regarded as matrices in channel space and operators in momentum space. The associated K matrices are defined by

$$\mathcal{K} = \mathcal{O} + \mathcal{O} g^{(0)} \mathcal{K} \quad \text{(RGM)}, \tag{84}$$

$$\mathcal{K} = \mathcal{W} + \mathcal{W} g^{(0)} \mathcal{K} \quad \text{(CM-ABKCEF)}, \tag{85}$$

$$\mathcal{K} = \mathcal{Y} + \mathcal{Y} g^{(0)} \mathcal{K} \quad \text{(CM-AKLCEF)}. \tag{86}$$

Let us set

$$g = g^{(0)} - i\pi\Delta. \tag{87}$$

Then following the procedure that was used in Eq. (38) we get the following generalized Heitler equations:

$$\mathcal{T} = U(1 + g^{(0)}\mathcal{K}) - i\pi \mathcal{T}\Delta\mathcal{K} \quad \text{(RGM)}, \tag{88}$$

$$\mathcal{T} = \mathcal{U}(1 + g^{(0)}\mathcal{K}) - i\pi \mathcal{T}\Delta\mathcal{K} \quad \text{(CM-ABKCEF)}, \tag{89}$$

$$\mathcal{T} = (1 + \mathcal{K} g^{(0)})\mathcal{Y} g X - i\pi \mathcal{K}\Delta\mathcal{T} \quad \text{(CM-AKLCEF)}. \tag{90}$$

The Heitler equations give us the T matrix once the K matrix is known. Since all the K-matrix equations have the same structure, we continue the discussion using the RGM equation. We rewrite Eq. (84) to read

$$\hat{\mathcal{K}} = \mathcal{O} + \mathcal{O} g^{(0)} P \hat{\mathcal{K}}. \tag{91}$$

P is a projection operator which explicitly indicates that the channel state sums in $g^{(0)}$ have been truncated. The circumflex over $\hat{\mathcal{K}}$ indicates that it is the cluster model approximation to the exact $\mathcal{K}$ which is the solution to Eq. (84).

The relationship between $\hat{\mathcal{K}}$ and $\mathcal{K}$ is given by

$$\begin{aligned}\mathcal{K} &= (1 + \hat{\mathcal{K}} g^{(0)} P)(\mathcal{O} + \mathcal{O} g^{(0)} \mathcal{K}) - \hat{\mathcal{K}} \mathcal{O} g^{(0)} P \mathcal{K} \\ &= \hat{\mathcal{K}} + \hat{\mathcal{K}} g^{(0)} (1 - P)\mathcal{K} \\ &= \hat{\mathcal{K}} + \hat{\mathcal{K}} g^{(0)} Q \mathcal{K}.\end{aligned} \tag{92}$$

The formal solution to Eq. (92) is

$$\begin{aligned}\mathcal{K} &= (1 - \hat{\mathcal{K}} g^{(0)} Q)^{-1} \hat{\mathcal{K}} \\ &= \hat{\mathcal{K}} + \hat{\mathcal{K}} (1 - g^{(0)} Q \hat{\mathcal{K}})^{-1} g^{(0)} Q \hat{\mathcal{K}}.\end{aligned} \tag{93}$$

By making substitutions of Eq. (91) into Eq. (93) the following expression may be found:

$$\begin{aligned}\mathcal{K} = \hat{\mathcal{K}} + (\hat{\mathcal{K}} g^{(0)} P + 1)\mathcal{O}[1 - g^{(0)} Q \mathcal{O} - g^{(0)} Q \mathcal{O} g^{(0)} P \mathcal{O} \\ - g^{(0)} Q \mathcal{O} g^{(0)} P \hat{\mathcal{K}} g^{(0)} P \mathcal{O}]^{-1} \\ \times g^{(0)} Q \mathcal{O} [1 + g^{(0)} P \hat{\mathcal{K}}].\end{aligned} \tag{94}$$

In these expressions the quantity

$$\begin{aligned}(1 - g^{(0)} Q \hat{\mathcal{K}})^{-1} g^{(0)} Q = [1 - g^{(0)} Q \mathcal{O} - g^{(0)} Q \mathcal{O} g^{(0)} P \mathcal{O} \\ - g^{(0)} Q \mathcal{O} g^{(0)} P \hat{\mathcal{K}} g^{(0)} P \mathcal{O}]^{-1} g^{(0)} Q\end{aligned} \tag{95}$$

is to be evaluated in the basis of those states projected onto by $Q = 1 - P$. This set of states is then to be approximated by some basis that is deemed to provide a good representation of the states of the compound nucleus that are important for the nuclear reaction being treated. We will refer to

these states as compound nucleus states. The states projected onto by P are the channel states.

Equation (91) for $\hat{\mathfrak{K}}$ is solved in the channel state basis. The advantage of using Eq. (94) instead of Eq. (93) to calculate $\mathfrak{K}$ from $\hat{\mathfrak{K}}$ is that in Eq. (94) only matrix elements of $\hat{\mathfrak{K}}$ with respect to channel states are required, and these are just what is provided by solving Eq. (91). In addition to these matrix elements one must calculate the matrix elements of Θ in the compound nucleus basis and the matrix elements of Θ connecting compound nucleus and channel states.

VIII. SUMMARY AND CONCLUSIONS

When the cluster model approximation is made for the partition Green's function operator G_α in the antisymmetrized coupled equations formalism (ACEF), a many-body scattering formalism results which appears to have the same physical content as the resonating group method (RGM). The resulting cluster model-antisymmetrized coupled equations formalism (CM-ACEF) yields an integral equation for the transition amplitude in which the driving term and the interaction factor in the kernel are nonsymmetric, energy independent functions which differ from each other.

We compared the RGM and the CM-ACEF by applying them to a one-channel analysis of ^2n-^2n scattering. The RGM and the CM-ACEF were found to give quite similar results for the total elastic cross section. We noted that the RGM wave function has rather poor quality since the transition amplitude that one gets by calculating the particle flux from the asymptotic part of the wave function differs considerably from the transition amplitude one gets by using the wave function to calculate the matrix element of the ^2n-^2n interaction with respect to the final state.

The Born approximation to the RGM and the CM-ACEF was compared with the exact solutions of the RGM and CM-ACEF equations. The Born approximation worked better for the CM-ACEF than for the RGM. When the Born approximation was made for the K matrix instead of for the T matrix, the results were especially good.

Our numerical results showed that the relation

$$G_\alpha G_\beta^{-1}\Phi_b = \delta_{\alpha\beta}\Phi_b$$

when $G_\beta = (E - H_\beta + i\epsilon)^{-1}$ and $G_\beta^{-1}\Phi_b = i\epsilon\Phi_b$, which has been commonly used in many-body reaction theory, is not valid.

Finally, we showed how the CM-ACEF may be extended so as to treat multipartition and compound nucleus formation effects.

We conclude that the CM-ACEF is an attractive alternative to the RGM. The physical approximation basis is the same for the two formalisms, while the CM-ACEF equations are somewhat simpler than those of the RGM.

Note added in proof: Our numerical results for ^2n-^2n scattering are quite different from those of Giraud *et al.*[6] although our parameters are the same as theirs. Dr. Giraud has suggested that the difference stems from the fact that in our calculation the ^2n binding energy was chosen to be the expectation value of the two-nucleon Hamiltonian with respect to the approximate internal motion wave function $\phi(r)$ of Eq. (29b). Giraud *et al.* used a different value for the ^2n binding energy.

The authors are grateful to E. F. Redish for much help and advice and to K. L. Kowalski for helpful discussions.

APPENDIX

$$U_l(k',k) = \sum_{j=1}^{2}\{[1-0.5n_j]I_j^l(k',k) - [0.5+0.5n_j]J_j^l(k',k) - [0.5-n_j]K_j^l(k',k)\}, \tag{A1}$$

$$I_j^l(k',k) = 4\pi^{3/2}\mu_j^3 S_j i^l[1+(-1)^l] j_l\left(\frac{ikk'(b^2+2\mu_j^2)}{4}\right)\exp\left[-\frac{(k^2+k'^2)(b^2+2\mu_j^2)}{8}\right], \tag{A2}$$

$$J_j^l(k',k) = \frac{32\pi^{3/2}\mu_j^3 b^3 S_j}{(2\mu_j^2+4b^2)^{3/2}} i^l[1+(-1)^l] j_l\left(\frac{ikk'b^4}{2\mu_j^2+4b^2}\right)\exp\left[-\frac{(k^2+k'^2)b^2(b^2+\mu_j^2)}{4\mu_j^2+8b^2}\right], \tag{A3}$$

$$K_j^l(k',k) = \frac{64\pi^{3/2}\mu_j^3 b^3 S_j}{(2\mu_j^2+3b^2)^{3/2}}\delta_{l0}\exp\left[-\frac{k^2b^2}{4} - \frac{k'^2b^2}{4}\left(\frac{2\mu_j^2+b^2}{2\mu^2{}_j+3b^2}\right)\right], \tag{A4}$$

$$\tilde{U}_l(k',k) = U_l(k,k'), \tag{A5}$$

$$M_l(k',k) = \tilde{U}_l(k',k) + \hat{M}_l(k',k), \tag{A6}$$

$$\hat{M}_l(k',k) = -(2\pi)^3\frac{\delta(k-k')(-1)^l}{4\pi kk'}\frac{\hbar^2}{2m}(k_a^2-k^2) + \bar{M}_l(k',k), \tag{A7}$$

$$\bar{M}_l(k',k)=\delta_{l0}\Bigg\{2(2\pi)^{3/2}b^3\frac{\hbar^2}{2m}[k_a^2-k'^2-k^2+3b^{-2}]\exp\Big[-\frac{b^2(k'^2+k^2)}{4}\Big]$$
$$-\sum_{j=1}^{2}\frac{64\pi^{3/2}\mu_j^3b^3S_j[0.5-n_j]}{(2\mu^2{}_j+3b^2)^{3/2}}\exp\Big[-\frac{k^2b^2}{4}-\frac{k'^2b^2}{4}\Big(\frac{2\mu_j^2+b^2}{2\mu_j^2+3b^2}\Big)\Big]$$
$$+\sum_{j=1}^{2}\frac{32\pi^{3/2}\mu_j^3b^3S_j[0.5-n_j]}{(2\mu_j^2+2b^2)^{3/2}}\exp\Big[-\frac{b^2(k'^2+k^2)}{4}\Big]\Bigg\}, \tag{A8}$$

$$Z_l(k',k)=U_l(k',k)+\frac{1}{2\pi^2}\int dp\,p^2U_l(k',p)2m\,\frac{[U_l(k,p)-U_l(p,k)]}{6\hbar^2(k_a^2-p^2+i\epsilon)}\,. \tag{A9}$$

Note that the interaction U_l, $\tilde{U}_l$, and Z_l vanish for odd values l. Thus the CM-ACEF predicts only even-l scattering. The same consequence follows from the presence of the Dirac δ function term in $\hat{M}_l$ shown in Eq. (A7). Substituting Eqs. (A6) and (A7) into Eq. (32) gives

$$[1+(-1)^l]\mathcal{T}_l(k',k)=U_l(k',k)+\frac{1}{2\pi^2}\int_0^\infty dp\,p^2\,\frac{\mathcal{T}_l(k',p)2m[\tilde{U}_l(p,k)+\bar{M}_l(p,k)]}{\hbar^2(k_a^2-p^2+i\epsilon)}\,. \tag{A10}$$

Thus for the even-l partial waves the RGM partial wave transition amplitude is given by

$$\mathcal{T}_l(k',k)=0.5U_l(k',k)+\frac{1}{2\pi^2}\int_0^\infty dp\,p^2\,\frac{\mathcal{T}_l(k',p)m[\tilde{U}_l(p,k)+\bar{M}_l(p,k)]}{\hbar^2(k_a^2-p^2+i\epsilon)}\,. \tag{A11}$$

The odd-l partial wave amplitudes are undetermined by these equations but we know from symmetry considerations that they must be set equal to zero.

---

*Work supported by the National Science Foundation.

[1]J. A. Wheeler, Phys. Rev. 52, 1083 (1937); D. L. Hill and J. A. Wheeler, Phys. Rev. 89, 1102 (1953); J. Griffin and J. A. Wheeler, *ibid.* 108, 311 (1957); K. Wildermuth and W. McClure, *Cluster Representation of Nuclei* (Springer, Berlin, 1966); Y. C. Tang, in *Proceedings of the International Conference on Clustering Phenomena in Nuclei, Bochum, Germany, 1969* (International Atomic Energy Agency, Vienna 1969), p. 109.

[2]H. Friedrich, Nucl. Phys. A224, 537 (1974).

[3]E. F. Redish and P. C. Tandy, in Proceeding of the Second International Conference on Clustering Phenomena in Nuclei, April 1975 [USERDA Publication ORO-4856-26 (unpublished)], p. 146.

[4]W. Tobocman, Phys. Rev. 12, 1146 (1975).

[5]W. Tobocman, *Theory of Direct Nuclear Reactions* (Oxford, London, 1961), Appendix; M. L. Goldberger and K. M. Watson, *Collision Theory* (Wiley, New York, 1964), Chap. 4.

[6]B. Giraud, J. C. Hocquenghem, and A. Lumbroso, Phys. Rev. C 7, 2274 (1973).

[7]D. M. Brink and E. Boeker, Nucl. Phys. A91, 1 (1967).

[8]K. L. Kowalski, Nuovo Cimento X30, 266 (1963).

[9]B. A. Lippmann, Phys. Rev. 102, 264 (1956).

[10]D. J. Kouri, F. S. Levin, and W. Sandhas, Phys. Rev. C 13, 1825 (1976).

[11]H. Feshbach, Ann. Phys. (N.Y.) 5, 357 (1958); 19, 257 (1962). W. M. MacDonald, Nucl. Phys. 54, 393 (1964); 56, 636 (1964).

[12]W. Tobocman, Phys. Rev. C 12, 741 (1975).