Two-nucleon exchange cuts in the Amado model for n-d scattering

Michael G. Fuda

Department of Physics and Astronomy, State University of New York, Buffalo, New York 14160

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A study is carried out of some of the singularities that occur in the second Born amplitude for elastic n-d scattering as described by the Amado model. It is shown that for real energies this amplitude is analytic in the momentum transfer plane except for a right hand cut, and that the discontinuity across the leading edge of the cut depends only on the on-shell, two-nucleon t matrix. This right hand cut leads to a left hand cut in the energy plane for the partial wave amplitudes. A simple formula which involves only the on-shell t matrix is derived for the discontinuity across the leading edge of the s-wave left hand cut. The possible implications of these results for the model dependence of three-nucleon calculations are discussed.

[NUCLEAR REACTIONS Cuts in second Born amplitudes for elastic n-d scattering; relation between discontinuities and on-shell, two-nucleon amplitudes.]

I. INTRODUCTION

One of the questions that continually arises in connection with the three-nucleon system, is the importance of the off-shell behavior of the two-nucleon t matrix in determining the various three-nucleon observables. As is well known, the work of Faddeev[1] focused attention on this question, since in his formalism, and others developed in the same spirit,[2] the off-shell, two-particle t matrix appears explicitly. In spite of the explicit appearance of the t matrix in these three-particle formalisms, it is difficult to determine the importance of off-shell effects by directly examining the equations. For this reason, many authors,[3-5] have carried out exploratory numerical calculations to determine the sensitivity of three-nucleon observables to the two-nucleon input. In Ref. 3 various separable interactions have been fitted to the same low energy two-nucleon parameters; in Refs. 4 and 5 phase-equivalent interactions have been employed.

It is, of course, not necessary to use Faddeev-like formalisms to study off-shell effects. In particular, Brayshaw[6] has used his three-particle boundary condition model to carry out three-nucleon calculations with fixed two-nucleon phase shifts. Brayshaw[6] has come to the strong conclusion that no off-shell information can be obtained from n-d elastic scattering and deuteron breakup, which is not already implicit in the value of the n-d doublet scattering length.

The importance of the doublet scattering length in determining the low energy n-d scattering parameters was first shown by Barton and Phillips[7] in a dispersion theory approach to n-d scattering. In their analysis only the one-nucleon exchange contribution to the left hand cut (LHC) of the s-wave elastic scattering amplitude was included explicitly, and inelasticity effects were neglected. A phenomenological parameter was introduced to account for the omitted portion of the LHC. By adjusting this parameter to the doublet scattering length, they were able to obtain a good description of the low energy variation of the effective range quantity $k\cot\delta$, where δ is the s-wave doublet phase shift. Using a similar approach, as well as separable potential calculations, it has recently[8] been shown that the low energy pole in $k\cot\delta$ and its residue are closely correlated with the doublet scattering length. It is important to note that in the dispersion theory calculations of Refs. 7 and 8, the only two-nucleon input is the binding energy of the deuteron and the asymptotic normalization of its wave function. In a sense this is on-shell information, since it can be obtained by analytic continuation of the on-shell, nucleon-nucleon amplitude to negative energies; a continuation which is readily carried out by means of effective range theory.

The treatment of the LHC in Refs. 7 and 8 is not complete enough to account for the correlation that is known to exist between the doublet scattering length and the triton binding energy.[9] This is not surprising, since the triton energy falls very close to the junction of the one- and two-nucleon exchange cuts.

The two-nucleon exchange cut, as well as, inelasticity effects have been included in the N/D calculation of Avishai, Ebenhöh, and Rinat.[10] Their input for the two-nucleon exchange cut can be obtained from the Amado[11] model for n-d scattering, and as such contains off-shell information. By employing a phenomenological pole to account for the neglected portion of the LHC, they were able to obtain a reasonable relationship between the triton binding energy and the doublet scattering length.

One of the main purposes of the present work, is to study the model dependence of the two-nucleon exchange cut as obtained from the Amado model.[11] Throughout the present work two-nucleon exchange will refer to the second Born term obtained by iterating the Amado[11] equations. This two-nucleon exchange amplitude also includes in a phenomenological way, through the vertex functions, some of the effects of pion exchange. It has already been shown[12] that the amplitudes obtained from the Amado model have the usual analytic properties for satisfying dispersion relations. In particular, the singularities in the complex energy plane lie along the real axis, and consist of right hand cuts (RHC's) associated with unitarity, and LHC's associated with exchange processes. Here we shall see that the discontinuity across the portion of the two-nucleon exchange cut adjacent to the one-nucleon exchange cut is model independent, in the sense that it depends only on the analytic continuation of the on-shell, two-nucleon t matrix to negative energies. The extent of the model-independent part of the cut depends on the inverse range of the vertex functions used to describe the deuteron and the singlet virtual bound state. This combined with what is known about the one-nucleon exchange cut[7,8] shows that in the Amado model a substantial portion of the LHC in the partial wave elastic scattering amplitudes depends only on on-shell, two-nucleon information. This model independence is probably crucial in explaining the correlation between the triton energy and the doublet scattering length,[9] although this remains to be proven.

The earliest separable potential calculations[11,13] of n-d elastic scattering showed that the low energy angular distributions could be calculated quite accurately with simple models for the two-nucleon interaction. Part of the explanation for this certainly has to do with the fact that one-nucleon exchange accounts for most of the scattering in the partial waves other than s waves.[14] As pointed out above one-nucleon exchange is essentially a model-independent process. Here we shall isolate within the framework of the Amado[11] model, the part of the two-nucleon exchange contribution to the angular distribution, which depends only on the on-shell, two-nucleon t matrix. We shall do this by examining the analytic structure of the elastic scattering amplitude in the complex t plane, where t is the negative of the square of the momentum transfer. We shall find that the second Born amplitude is analytic in the complex t plane for all physical energies, except for a RHC beginning at $16\alpha^2$, α being the deuteron wave number. Moreover, it turns out that the discontinuity across the small t end of the cut depends only on the on-shell two-nucleon t matrix.

The outline of the paper is as follows. In Sec. II the analytic structure of the two-nucleon exchange amplitude in the complex t plane is determined, and a model-independent expression for the discontinuity across the small t end of the RHC is derived. A partial wave analysis of the results of Sec. II is shown in Sec. III to lead to a LHC for the partial wave amplitude in the complex energy plane. A simple model-independent expression for the discontinuity of the s-wave amplitude across the low energy end of the LHC is derived. Section IV gives a discussion of the results.

II. CUT IN THE t PLANE

For the two-nucleon transition operator, we shall use a spin-dependent s-wave separable interaction of the form

$$t_n(s) = |g_n\rangle \Delta_n(s) \langle g_n| , \quad n = 1, 2 , \tag{2.1}$$

where $n = 1$ and 2 refer to the triplet and singlet states, respectively. Here s is a complex energy parameter, and

$$\Delta_n^{-1}(s) = -\lambda_n^{-1} + \langle g_n | (H_0 - s)^{-1} | g_n \rangle . \tag{2.2}$$

The strength of the interaction is determined by λ_n; H_0 is the kinetic energy operator. The triplet form factor $|g_1\rangle$ can be expressed in terms of the deuteron wave function $|B\rangle$ by means of the relation

$$|g_1\rangle = (-\alpha^2 - H_0) |B\rangle , \tag{2.3}$$

where α^2 is the deuteron binding energy. We are working in units in which $\hbar^2$ divided by the nucleon mass is one. The parameter λ_1 is adjusted so that

$$\Delta_1^{-1}(-\alpha^2) = 0 , \tag{2.4}$$

which implies that

$$\Delta_1^{-1}(s) \xrightarrow[s \to -\alpha^2]{} s + \alpha^2 . \tag{2.5}$$

The three-particle equations that arise from the interaction (2.1) are most easily obtained from the Alt, Grassberger, and Sandhas[2] version of the Faddeev equations. The resulting equations, which were first obtained by Amado[11] using other methods, are given by

$$X_{nm}(\vec{q}, \vec{k}; s) = Z_{nm}(\vec{q}, \vec{k}; s) + \sum_{r=1}^{2} \int Z_{nr}(\vec{q}, \vec{q}\,'; s) d\vec{q}\,' \Delta_r(s - \tfrac{3}{4}q'^2) X_{rm}(\vec{q}\,', \vec{k}; s) , \tag{2.6}$$

where

$$Z_{nm}(\vec{q},\vec{q}';s)=\frac{J_{nm}}{2\pi}\frac{g_n(|\tfrac{1}{2}\vec{q}+\vec{q}'|)g_m(|\tfrac{1}{2}\vec{q}'+\vec{q}|)}{s-q^2-\vec{q}\cdot\vec{q}'-q'^2}. \quad (2.7)$$

The quantities J_{nm} are spin-isospin recoupling coefficients, and are given by

$$J_{11}=J_{22}=\tfrac{1}{4};\quad J_{12}=J_{21}=-\tfrac{3}{4}\quad \text{(doublet)},$$

$$J_{11}=-\tfrac{1}{2}\ \text{(quartet)}. \quad (2.8)$$

The form factors or vertex functions are normalized according to the relation

$$\langle\vec{p}|g_n\rangle=g_n(p)/(4\pi)^{1/2}, \quad (2.9)$$

where it has been assumed that the momentum states have the normalization

$$\langle\vec{p}|\vec{p}'\rangle=\delta(\vec{p}-\vec{p}'). \quad (2.10)$$

The on-shell momentum k is given by

$$s=E+i\epsilon=-\alpha^2+\tfrac{3}{4}k^2+i\epsilon\quad(0<\epsilon\ll 1) \quad (2.11)$$

with E being the total three body energy. The partial wave amplitudes are defined by the equation

$$X_{Lnm}(q,k;s)=2\pi\int_{-1}^{1}dxP_L(x)X_{nm}(\vec{q},\vec{k};s),\quad x=\hat{q}\cdot\hat{k} \quad (2.12)$$

and have the on-shell normalization

$$X_{L11}(k,k;s)=-\frac{3}{2\pi}\frac{e^{i\delta_L(k)}\sin\delta_L(k)}{k}, \quad (2.13)$$

where δ_L is the phase shift for the Lth partial wave.

Here we shall consider the on-shell second Born term obtained by iterating (2.6); i.e.,

$$X_{11}^{(2)}(\vec{k}',\vec{k};s)=\sum_{r=1}^{2}\int Z_{1r}(\vec{k}',\vec{q};s)d\vec{q}\,\Delta_r(s-\tfrac{3}{4}q^2)Z_{r1}(\vec{q},\vec{k};s),\quad |\vec{k}'|=|\vec{k}|=k. \quad (2.14)$$

We shall assume that the form factors $g_n(p)$ are real analytic functions of p^2 with a left hand cut beginning at $p^2=-\gamma^2$. This allows us to write

$$g_n(p)=\int_\gamma^\infty\frac{d\beta\sigma_n(\beta)}{p^2+\beta^2}. \quad (2.15)$$

Most of the phenomenological form factors are of this type, and in particular, the well known Yamaguchi[15] form factor is obtained if the weight function $\sigma_n(\beta)$ is taken to be a δ function. If the transition operator (2.1) is interpreted as the unitary pole approximation[16] to the transition operator which arises from a superposition of Yukawa potentials, then the form factors will have the analytic structure indicated by (2.15).[17] Thus (2.15) is quite general. If the representation (2.15) is inserted into (2.14), angular integrations of the type shown below are encountered

$$\int d\Omega_{\vec{q}}[(q^2+\tfrac{1}{4}k^2+\alpha^2+\vec{q}\cdot\vec{k}')(q^2+\tfrac{1}{4}k^2+\beta^2+\vec{q}\cdot\vec{k}')(\tfrac{1}{4}q^2+k^2+\beta'^2+\vec{q}\cdot\vec{k}')$$
$$\times(q^2+\tfrac{1}{4}k^2+\alpha^2+\vec{q}\cdot\vec{k})(q^2+\tfrac{1}{4}k^2+\beta''^2+\vec{q}\cdot\vec{k})(\tfrac{1}{4}q^2+k^2+\beta'''^2+\vec{q}\cdot\vec{k})]^{-1}. \quad (2.16)$$

In order to handle this integral we make two partial fraction expansions, one in $\vec{q}\cdot\vec{k}'$, and one in $\vec{q}\cdot\vec{k}$. When these expansions are multiplied together, put into (2.14), and the symmetry in $\vec{k}'$ and $\vec{k}$ is exploited, we obtain

$$X_{11}^{(2)}(\vec{k}',\vec{k};s)=\sum_{r=1}^{2}\frac{J_{1r}{}^2g_1{}^2(i\alpha)}{4\pi^2}\int\frac{d\vec{q}\,\tau_r(s-\tfrac{3}{4}q^2)}{(q^2+\tfrac{1}{4}k^2+\alpha^2+\vec{q}\cdot\vec{k}')(q^2+\tfrac{1}{4}k^2+\alpha^2+\vec{q}\cdot\vec{k})}+\text{five other terms}. \quad (2.17)$$

Here τ_r is the on-shell two-particle t matrix given by

$$\tau_r(p^2)=g_r{}^2(p)\Delta_r(p^2+i\epsilon), \quad (2.18)$$

and $g_1{}^2(i\alpha)$ is related to the deuteron wave number α and effective range ρ by[8]

$$g_1{}^2(i\alpha)=\frac{4\alpha}{\pi(1-\alpha\rho)}. \quad (2.19)$$

Thus the term written explicitly in (2.17) is model independent in the sense that it depends only on the on-shell, two-nucleon t matrix, and its analytic continuation to negative energies.

We shall now examine the analytic structure of the second Born term in the momentum transfer variable defined by

$$t = -|\vec{k}' - \vec{k}|^2 = 2k^2(\cos\theta - 1)\,. \tag{2.20}$$

In particular we shall show that each of the six terms in (2.17) gives rise to a right hand cut in t, and most importantly, the branch point arising from the model-independent term lies closest to the origin of the t plane. Each of the six terms in (2.17) gives rise to an angular integral of the form

$$I(\lambda_1, \lambda_2, \cos\theta) = \int \frac{d\Omega_{\vec{q}}}{(\lambda_1 + \hat{q}\cdot\hat{k}')(\lambda_2 + \hat{q}\cdot\hat{k})}, \tag{2.21}$$

where

$$\lambda_i = (qk)^{-1}\left(a_i q^2 + \frac{k^2}{4a_i} + \sigma_i^2\right). \tag{2.22}$$

Minimizing λ_i with respect to q leads to

$$\lambda_i \geq \left(1 + \frac{4a_i\sigma_i^2}{k^2}\right)^{1/2} > 1\,, \tag{2.23}$$

thus for real, positive k^2, the integral in (2.21) is well defined. Following Goldberger and Watson's[18] analysis of the second Born term for the Yukawa potential, it is straightforward to show that

$$I = 8\pi k^2 \int_{T^+}^{\infty} \frac{dt'}{(t'-t)[(t'-T^+)(t'-T^-)]^{1/2}}, \tag{2.24}$$

with

$$T^{\pm} = 2k^2\{\lambda_1\lambda_2 - 1 \pm [(\lambda_1^2 - 1)(\lambda_2^2 - 1)]^{1/2}\}. \tag{2.25}$$

If we minimize T^+ with respect to q we find that the minima occur at

$$q^2 = \frac{k^2}{4a^2} + \frac{\sigma_1\sigma_2}{a}, \quad a_1 = a_2 = a\,,$$

$$q^2 = \frac{\sigma_1 + 8\sigma_2}{4\sigma_1 + 2\sigma_2}k^2 + 2\sigma_1\sigma_2, \quad a_1 = \tfrac{1}{4}, \quad a_2 = 1\,,$$

and that the minima are

$$T^+_{\min} = 4a(\sigma_1 + \sigma_2)^2, \quad a_1 = a_2 = a\,,$$
$$T^+_{\min} = \tfrac{9}{4}k^2 + (2\sigma_1 + \sigma_2)^2, \quad a_1 = \tfrac{1}{4}, \quad a_2 = 1\,. \tag{2.26}$$

Upon examining the various pairs of denominators that occur in (2.17), we find that the model-independent term has a right hand cut in t beginning at $16\alpha^2$; the model-dependent cut with the smallest branch point arises from the third and sixth denominators in (2.16), and the branch point occurs at $4\gamma^2$. If we use (2.24) for each of the six terms in (2.17), and interchange the order of integration in each term, we find that

$$X_{11}^{(2)}(\vec{k}', \vec{k}; s) = \int_{16\alpha^2}^{\infty} \frac{dt'}{t'-t}\rho(k^2, t')\,, \tag{2.27}$$

where

$$\rho(k^2, t) = \sum_{r=1}^{2} \frac{J_{1r}^{\,2} g_1^{\,2}(i\alpha)}{2\pi\sqrt{t}} \times \int_{a_-^2(k^2,t)}^{a_+^2(k^2,t)} \frac{dq^2\,\tau_r(s - \frac{3}{4}q^2)}{[(a_+^2 - q^2)(q^2 - a_-^2)]^{1/2}}, \quad 16\alpha^2 \leq t \leq 4\gamma^2\,, \tag{2.28}$$

with

$$a_\pm^2(k^2, t) = \tfrac{1}{8}t + \tfrac{1}{4}k^2 - \alpha^2 \pm \tfrac{1}{8}[(t + 4k^2)(t - 16\alpha^2)]^{1/2}. \tag{2.29}$$

Thus the second Born term is analytic in the t plane except for a right hand cut beginning at $16\alpha^2$, and the discontinuity across the leading edge of the cut is model independent. It is important to check that the argument of the on-shell t matrix in (2.28) never reaches the left hand singularities described by (2.15). A little calculation shows that

$$E - \tfrac{3}{4}q^2 \geq -\gamma^2 + \tfrac{1}{4}(\gamma^2 - 4\alpha^2), \quad 16\alpha^2 \leq t \leq 4\gamma^2,$$
$$k^2 \geq -4\alpha^2, \tag{2.30}$$

thus only well defined on shell t matrix elements enter into (2.28). Throughout we assume that $\gamma^2 > 4\alpha^2$. We shall use the results of this section in the next section to derive the model-independent part of the discontinuity across the LHC of the s-wave projection of (2.27).

III. LEFT HAND CUT DISCONTINUITY

From (2.12) and (2.27), it follows that

$$X_{L11}^{(2)}(k, k; s) = \frac{2\pi}{k^2}\int_{16\alpha^2}^{\infty} dt\, Q_L\left(1 + \frac{t}{2k^2}\right)\rho(k^2, t)\,, \tag{3.1}$$

where Q_L is the well known Legendre function of the second kind. The logarithmic singularity in Q_L leads to a LHC in the k^2 plane beginning at $-4\alpha^2$; the discontinuity across the cut is given by

$$\operatorname{Im} X_{L11}^{(2)}(k, k; s) = \frac{\operatorname{Disc} X_{L11}^{(2)}(k, k; s)}{2i} = \frac{\pi^2}{k^2}\int_{16\alpha^2}^{-4k^2} dt\, P_L\left(1 + \frac{t}{2k^2}\right)\rho(k^2, t)\,, \quad k^2 \leq -4\alpha^2\,, \tag{3.2}$$

where the discontinuity is taken to be the function above the cut minus the function below the cut. Since ρ is model independent for $t \leq 4\gamma^2$, it follows that the discontinuity across the left hand cut is model independent for $k^2 \geq -\gamma^2$. We shall now manipulate (2.28) and (3.2) in order to obtain a simple formula for the discontinuity in the $L=0$ amplitude.

From (2.1), (2.2), (2.5), (2.15), and (2.18) it follows that the on-shell t matrix is analytic in the complex energy plane except for the right hand unitarity cut, a LHC, and a possible bound state pole. This allows us to write

$$\tau_r(E) = \frac{1}{\pi}\int_{-\infty}^{-\gamma^2} \frac{dE' \, \mathrm{Im}\tau_r(E'+i\epsilon)}{E'-E} + \delta_{r1}\frac{g_1{}^2(i\alpha)}{E+\alpha^2} + \frac{1}{\pi}\int_0^{\infty} \frac{dE' \, \mathrm{Im}\tau_r(E'+i\epsilon)}{E'-E} . \tag{3.3}$$

Putting this relation into (2.28), and doing the q^2 integrations by contour integration leads to the result

$$\rho(k^2,t) = \sum_{r=1}^{2} \frac{J_{1r}{}^2 g_1{}^2(i\alpha)}{2\pi\sqrt{t}} \left\{ -\int_{4/3(E+\gamma^2)}^{\infty} dq^2 \frac{\mathrm{Im}\tau_r(s-\frac{3}{4}q^2)}{[(q^2+\frac{1}{4}k^2+\alpha^2)^2 - k^2q^2 - \frac{1}{4}q^2 t]^{1/2}} - \frac{4\pi}{3}\delta_{r1}\frac{g_1{}^2(i\alpha)}{[(\frac{5}{4}k^2+\alpha^2)^2 - k^4 - \frac{1}{4}k^2 t]^{1/2}} \right.$$
$$\left. + \int_{-\infty}^{4E/3} dq^2 \frac{\mathrm{Im}\tau_r(s-\frac{3}{4}q^2)}{[(q^2+\frac{1}{4}k^2+\alpha^2)^2 - k^2q^2 - \frac{1}{4}q^2 t]^{1/2}} \right\}, \quad 16\alpha^2 \leq t \leq 4\gamma^2 . \tag{3.4}$$

The integration over t in (3.2) can be done by elementary methods and gives rise to the expression

$$\mathrm{Im}X_{011}^{(2)}(k,k;s) = \frac{2\pi i}{k^2}\sum_{r=1}^{2} J_{1r}{}^2 g_1{}^2(i\alpha)\left[\int_{4/3(E+\gamma^2)}^{\infty} \frac{dq^2}{q}\mathrm{Im}\tau_r(s-\tfrac{3}{4}q^2)\ln\left(\frac{q+\frac{1}{2}k-i\alpha}{q-\frac{1}{2}k+i\alpha}\right) - \frac{4\pi}{3}\delta_{r1}g_1{}^2\left(\frac{i\alpha}{k}\right)\ln\left(\frac{\frac{3}{2}k-i\alpha}{\frac{1}{2}k+i\alpha}\right)\right.$$
$$\left. + \int_{-\infty}^{4E/3}\frac{dq^2}{q}\mathrm{Im}\tau_r(s-\tfrac{3}{4}q^2)\ln\left(\frac{q+\frac{1}{2}k-i\alpha}{q-\frac{1}{2}k+i\alpha}\right)\right], \quad -\gamma^2 \leq k^2 \leq -4\alpha^2 . \tag{3.5}$$

By using (3.3), it is straightforward to show that (3.5) can be simplified to give

$$\mathrm{Im}X_{011}^{(2)}(k,k;s) = -\frac{4\pi^2 i}{k^2}\sum_{r=1}^{2} J_{1r}{}^2 g_1{}^2(i\alpha)\int_0^{1/2k-i\alpha} dq\,\tau_r(E-\tfrac{3}{4}q^2), \quad -\gamma^2 \leq k^2 \leq -4\alpha^2 . \tag{3.6}$$

This is our final result for the model-independent part of the s-wave amplitude's discontinuity across its LHC.

Since the route that has led to (3.6) is rather circuitous, it is natural to ask if there is some other way to derive the result. The author has been able to do so by using the techniques employed by Greben and Kok.[19] In their approach the integral representation for the partial wave second Born term is analytically continued above and below the LHC. In so doing it is necessary to deform the path of integration in one way above the LHC, and in another way below the LHC. This ultimately leads to the discontinuity (3.6). The advantage of Greben and Kok's[19] approach is that it is not necessary to use explicitly the decomposition (3.3) for the on-shell t matrix. The method employed in the present work has the advantage of showing the connection between the RHC of the full amplitude in the t plane, and the LHC in the partial wave amplitude.

It is interesting to note that the discontinuity (3.6) vanishes at the right end of the LHC. This might lead one to believe that this discontinuity is small compared to the discontinuity across the one-nucleon exchange cut,[7,8] however simple numerical calculations show that this is not the case. It is important to check that the argument of the on-shell t matrix element in (3.6) never reaches the left hand singularities described by (2.15). It is straightforward to show that

$$\begin{aligned} &-\gamma^2 + \tfrac{1}{4}(\gamma^2 - 4\alpha^2) \leq E - \tfrac{3}{4}q^2 \leq -4\alpha^2 , \\ &-\gamma^2 \leq k^2 \leq -4\alpha^2 , \end{aligned} \tag{3.7}$$

so the integral in (3.6) is well defined.

IV. DISCUSSION

We have succeeded in isolating, within the framework of the Amado equations,[11] the model-independent contribution to the discontinuity across the RHC in t of the full second Born amplitude, and the model-independent contribution to the discontinuity across the LHC in energy of the partial wave second Born amplitude. As pointed out above we mean by a model-independent contribution one which depends only on the on-shell two-nucleon t matrix, and its analytic continuation. These results should be useful in analyzing the n-d angular distributions that arise from elastic scattering, and in trying to understand the correlations that exist among the low-energy, three-nucleon observables. For example, in carrying out a phase shift analysis of n-d elastic scattering, the high partial wave phase shifts can be determined in a model-independent way from the one-nucleon exchange amplitude.[14] By combining this amplitude with the on-shell contribution to the second Born term [see (2.27) and (2.28)], it should be possible to extend the model independence to lower partial waves and higher energies. We are presently carrying out calculations to test this idea.

It is relevant here to discuss what is known in general about the analytic structure in the complex t plane of the elastic n-d scattering amplitude. The first Born term singularities can be located easily by evaluating (2.7) on shell ($q=q'=k,\ n=m=1$). A trivial analysis shows that there is a simple pole at $t=-2\alpha^2-\frac{9}{2}k^2$, whose residue is determined by the deuteron's binding energy and asymptotic normalization [see (2.19)]. This pole contribution is the one-nucleon exchange amplitude referred to above, and lies just to the left of the physical region ($-4k^2 \leqslant t \leqslant 0$). If one assumes that the vertex function is given by (2.15), then it is straightforward to show that there is a LHC in t beginning at $t=-2\gamma^2-\frac{9}{2}k^2$. We are presently analyzing the singularities in t that arise from the third Born term in the Amado model.[11] Hartle and Sugar[17] have studied the analytic structure in the momentum transfer plane of the three-particle scattering amplitudes that describe collisions in which two of the initial and final particles are in a bound state. They consider a system of spinless, nonrelativistic particles which interact via two-body central potentials, which can be written as a superposition of Yukawa potentials. They find that the amplitudes are analytic inside a Lehmann ellipse in the cosine of the scattering angles for all real energies. For real energies below the breakup threshold, the amplitudes are analytic in the momentum transfer plane except for left and right hand cuts. It will be interesting to see if these results carry over to the Amado model[11]; they probably do.

As pointed out in the Introduction, the result obtained here for the two-nucleon exchange cut combined with what is known about the one-nucleon exchange cut show that a substantial portion of the LHC discontinuity for the partial wave amplitudes is model independent. In order to further discuss the LHC it is useful to introduce a dimensionless energy parameter defined by[7,8]

$$z=\frac{3k^2}{4\alpha^2}\,. \tag{4.1}$$

In terms of this parameter the one-nucleon exchange cut[7,8] extends from $z=-\frac{1}{3}$ to $z=-3$; the model-independent part of the two-nucleon exchange cut extends from [see (3.6)] $z=-3$ to $z=-3\gamma^2/(4\alpha^2)$. If we assume that the vertex functions [see (2.15)] are constructed according to the unitary pole approximation,[16] and that the longest range part of the two-nucleon potential is given by one-pion exchange, then it follows from Ref. 17 that

$$\gamma=\mu+\alpha\,, \tag{4.2}$$

where μ is the inverse pion Compton wavelength ($0.7\ \text{fm}^{-1}$). For the triplet state α is the deuteron wave number ($\alpha=0.232\ \text{fm}^{-1}$); for the singlet state α is the antideuteron wave number ($\alpha_s=-0.040\ \text{fm}^{-1}$). Thus in this framework the model-independent part of the two-nucleon exchange cut terminates at

$$z=-\frac{3(\mu+\alpha_s)^2}{4\alpha^2}=-6.09\,. \tag{4.3}$$

From the point of view of particle exchange, this is the branch point arising from two-pion exchange.[7]

By using (2.7), (2.12), and (2.15), it is easy to show that besides the one-nucleon exchange cut, the on-shell, partial wave Born term has a branch point at

$$z=-\frac{(\mu+\alpha)^2}{3\alpha^2}=-5.39\,. \tag{4.4}$$

This corresponds to the anomalous branch point of the nucleon-deuteron vertex function with one nucleon off shell.[7,20] Thus the part of the discontinuity across the LHC which can be obtained from the on-shell, two-nucleon t matrix extends from $z=-\frac{1}{3}$ to $z=-5.39$; beyond this point off-shell information is needed. It is interesting to note that the unitary pole approximation just discussed, when used in the Amado model, leads to an n-d elastic scattering amplitude with a low energy, LHC structure which agrees with that obtained from the more fundamental point of view of particle exchanges.[7,20,21]

It is natural to inquire into the possible contribution of the exchange of three or more nucleons to the LHC discontinuity. According to Ref. 19 there are no such contributions. The third and higher Born terms contribute through the vertex functions, but not through the vanishing of the denominator in (2.7). The singularities associated with the vanishing of this denominator have been called rescattering singularities by Rubin, Sugar, and Tiktopoulos.[22] They show that the rescattering singularities of order n should vanish if n is greater than or equal to the maximum number of classical binary contact collisions of three particles. For the equal mass case n is 3.

At the present time we are carrying out N/D calculations of the type reported in Ref. 8 in order to see if inclusion of the model-independent part of the two-nucleon exchange cut will lead to a reasonable correlation between the triton binding energy and the doublet scattering length.

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