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Bounds on the rate of $^1H(p, e^+\nu)^2H$ in impulse approximation

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Working in impulse approximation, we derive simple bounds on the experimentally inaccessible rate of the proton-proton reaction $^1H(p, e^+\nu)^2H$, using only a few common and physically plausible assumptions. We evaluate these bounds at the kilovolt energies of interest for hydrogen burning in stars and find that they permit at most a few percent enhancement over currently accepted impulse-approximation rates. Within the framework of the usual nonrelativistic Schrödinger-equation description of the two-nucleon interaction, this result is obtained independent of the specific features of the two-nucleon interaction at distances less than 3 fm, provided that the energy dependence of the interaction may be neglected at the low energies in question.

[NUCLEAR REACTIONS $^1H(p, e^+\nu)$; bounds on rate in impulse approximation, keV energies, arbitrary energy-independent short-range two-nucleon interaction.]

As the process which ignites thermonuclear hydrogen burning, the proton-proton reaction $^1H(p,e^+\nu)^2H$ plays a crucial role in the conventional picture of stars of the age and mass of our sun.[1] In the absence of direct laboratory measurements,[2] it is necessary to rely on theoretical calculations of the rate of this reaction—the "p-p rate," as we shall call it hereafter. Thus far, there has been no systematic study of the sensitivity of calculated p-p rates to uncertainties in the short-range behavior of the two-nucleon radial functions used in the computation. Careful scrutiny of the p-p rate calculation is now in order for two reasons: (1) The counting rate predicted by standard solar models for the ongoing Brookhaven solar neutrino experiment[3] (BSNE) is fairly sensitive to the p-p rate.[4] For example, according to Newman and Fowler,[5] if the true p-p rate should happen to be twice the value presently used, the theoretical BSNE counting rate would drop from the current 4.6 or so solar neutrino units (snu) to the 1.5 snu which represents the latest reported experimental upper limit.[3] Even a 10% increase in the p-p rate would be significant in this context, lowering the expected counting rate to about 3.8 snu.[5] (2) Recent experience with parametrizations of the unknown short-range nonlocality in the two-nucleon interaction by means of phase-shift-preserving unitary transforms[6] has shown that, even when the available empirical and phenomenological constraints on the interaction are employed, calculated cross sections and rates for a variety of processes involving inelastic collisions of two nucleons are not nearly as well determined as older computations with local potentials had suggested. The radial functions produced by unitary transforms often differ drastically from those generated by local potentials of the sort used thus far in calculations of the p-p rate. Hence, qualitative arguments which suggest that the p-p rate should be insensitive to short-distance behavior of the two-nucleon radial functions[1] lose force.

In this paper, we examine the uncertainties associated with unknown strong short-range nonlocality of the two-nucleon interaction. In particular, we inquire whether such nonlocality can give rise to a substantial enhancement of the p-p rate in impulse approximation (and, thereby, to a correspondingly large decrease in the predicted BSNE counting rate). We show that this possibility can be ruled out quite generally on the basis of a few common and plausible assumptions.

Since there are inexhaustible supplies of phase-shift-preserving unitary transforms of even the simplest type, any attempt to draw general conclusions from a finite number of explicit calculations of p-p rates with transformed radial functions is futile. Instead, we establish bounds which circumscribe the range of variation of the p-p rate for any transforms (i.e., any type of nonlocality) of a given range. To do this, we invoke the following assumptions: (1) At low energies, the two-nucleon interaction is represented sufficiently well by an Hermitian potential whose energy dependence may be neglected. (2) A reliable parametrization of the $^1S_0$ p-p phase shift at a few keV is provided by the usual Coulomb-modified effective-range formula[7] using the scattering length and effective range extracted from experimental cross sections at MeV energies. (3) The $^3S_1$ deuteron radial function is well determined for internucleon separations greater than some length R of the order of a pion

Compton wavelength. (4) The two-nucleon potential of assumption (1) reduces to a local potential for distances larger than R.

The p-p rate is proportional to the magnitude squared of a dimensionless radial matrix element usually denoted as $\Lambda(E)$.[8] It is this quantity which we seek to bound, since it contains all the uncertainties in the calculated rate once the Gamow-Teller coupling constant and the positron Fermi function are specified. In impulse approximation, the conventional definition of $\Lambda(E)$ is

$$\Lambda(E)=\exp(i\delta)\,[kC_0(\eta)]^{-1}(\tfrac{1}{2}\gamma^3)^{1/2}\times\int_0^\infty dr\,u_d(r)\,w_{pp}(k,r)\,, \tag{1}$$

where δ is the ${}^1S_0$ p-p phase shift at relative momentum[9] $k=E^{1/2}$, $\eta=(57.62k)^{-1}$ is the usual Coulomb strength parameter, $C_0^2(\eta)=2\pi\eta\,[\exp(2\pi\eta)-1]^{-1}$, $\gamma=0.2316\ \mathrm{fm}^{-1}$ is the inverse "deuteron radius," $u_d(r)$ is the ${}^3S_1$ deuteron radial function, and $w_{pp}(k,r)$ is the ${}^1S_0$ p-p scattering radial function. The normalizations used here are

$$\int_0^\infty dr\,u_d^2(r)=1-P_D\,, \tag{2}$$

with P_D the deuteron D-state probability, and

$$w_{pp}(k,r)\underset{r\to\infty}{\sim}v(k,r)=\cos\delta\,F_0(\eta,kr)+\sin\delta\,G_0(\eta,kr)\,, \tag{3}$$

in which F_0 and G_0 are the standard regular and irregular spherical Coulomb functions of order zero.[10] Having made assumption (2) of the preceding paragraph, we regard $u_d(r)$ for $r>R$ as known. If we know the local potential which acts in the ${}^1S_0$ partial wave for $r>R$, call it $U(r)$, assumptions (3) and (4) suffice to give us $w_{pp}(k,r)$ for $r>R$ as well. Let us suppose, for the time being, that $U(r)$ is known. Later, if need be, we can study the sensitivity of our results to variations in $U(r)$. We may then isolate the uncertainties buried in the short-range radial functions by writing

$$\int_0^\infty dr\,u_d(r)\,w_{pp}(k,r)=I_{\text{int}}+I_{\text{ext}}\,, \tag{4}$$

where the known exterior contribution is

$$I_{\text{ext}}=\int_R^\infty dr\,u_d(r)\,w_{pp}(k,r) \tag{5}$$

and the interior contribution, as yet undetermined, is

$$I_{\text{int}}=\int_0^R dr\,u_d(r)\,w_{pp}(k,r)\,. \tag{6}$$

Since

$$\left|\int_0^\infty dr\,u_d(r)\,w_{pp}(k,r)\right|\le|I_{\text{int}}|+|I_{\text{ext}}|$$

and

$$|I_{\text{int}}|^2=\left|\int_0^R dr\,u_d(r)\,w_{pp}(k,r)\right|^2\le\left|\int_0^R dr\,u_d^2(r)\right|\left|\int_0^R dr\,w_{pp}^2(k,r)\right|\,,$$

we have the formal upper bound

$$\Lambda^2(E)\equiv|\Lambda(E)|^2\le[kC_0(\eta)]^{-2}(\tfrac{1}{2}\gamma^3)\,(|I_dI_{pp}|^{1/2}+|I_{\text{ext}}|)^2\,, \tag{7}$$

where

$$I_d=\int_0^R dr\,u_d^2(r) \tag{8}$$

and

$$I_{pp}=\int_0^R dr\,w_{pp}^2(k,r)\,. \tag{9}$$

Equation (2) immediately gives us I_d in terms of known quantities:

$$I_d=1-P_D-\int_R^\infty dr\,u_d^2(r)\,. \tag{10}$$

An analogous relation for I_{pp} is obtained from the standard Coulomb-modified effective-range identity[7]

$$\int_0^\infty dr\,[w_{pp}^2(k,r)-v^2(k,r)]=-\{\sin^2\delta/[2kC_0^2(\eta)]\}\frac{d}{dk}[C_0^2(\eta)\,k\cot\delta+2\eta k\,h(\eta)]\,, \tag{11}$$

{where $h(\eta)=\eta^2\sum_{m=1}^\infty[m(m^2+\eta^2)]^{-1}-\ln\eta-0.57722$}, which is valid for any energy-independent potential, local or nonlocal. [This is the point at which we need assumption (1). For energy-dependent potentials, the right-hand side of Eq. (11) is modified by a term in which the unknown short-range potential appears explicitly.] Now rearrangement of Eq. (11) gives I_{pp} in terms of those quantities we have assumed to be known:

$$I_{pp}=\int_0^R dr\,v^2(k,r)-\int_R^\infty dr\,[w_{pp}^2(k,r)-v^2(k,r)]-\{\sin^2\delta/[2kC_0^2(\eta)]\}\frac{d}{dk}[C_0^2(\eta)\,k\cot\delta+2\eta k\,h(\eta)]\,. \tag{12}$$

At the energies of interest, assumption (2) allows us to simplify Eq. (12) to

$$I_{pp}=\int_0^R dr\, v^2(k,r)-\int_R^\infty dr\,[w_{pp}{}^2(k,r)-v^2(k,r)] -\{\sin^2\delta/[2C_0{}^2(\eta)]\}\, r_{pp}^C\,, \qquad (13)$$

where r_{pp}^C is the Coulomb-modified p-p effective range and δ is obtained from[7] $C_0{}^2 k\cot\delta+2\eta k h(\eta) =(-1/a_{pp}^C)+\frac{1}{2}r_{pp}^C k^2$, in which a_{pp}^C is the Coulomb-modified p-p scattering length.

Equations (7), (10), and (13) provide a simple recipe for evaluating an upper bound on $\Lambda^2(E)$ in impulse approximation. The input required is minimal: the p-p effective-range parameters, the $^3S_1$ deuteron radial function $u_d(r)$ for $r>R$, and the local potential $U(r)$ which, for $r>R$, represents the nuclear $^1S_0$ proton-proton interaction. [$U(r)$ does not appear explicitly on the right-hand side of Eq. (13), but it is needed to generate $w_{pp}(k,r)$ for $r>R$ from its asymptotic form $v(k,r)$. The latter function is fixed by the choice of δ.] As a byproduct, whenever $I_{\text{ext}}>|I_d I_{pp}|^{1/2}$—in practice, this means for R less than 5 fm or so—we also have a lower bound

$$\Lambda^2(E)\geq[kC_0(\eta)]^{-2}(\tfrac{1}{2}\gamma^3)(I_{\text{ext}}-|I_d I_{pp}|^{1/2})^2\,. \qquad (14)$$

We have evaluated these bounds using the $^3S_1$ deuteron radial function of McGee[11] (for which $P_D=0.06$) to represent $u_d(r)$ for $r>R$ and the $^1S_0$ Reid soft-core potential[12] as $U(r)$, with the range of nonlocality R running from 1 to 5 fm. Two sets of effective-range parameters were used: those recommended by Noyes[13] as the best empirical values, and the rather different ones calculated by Reid from his potential.[12] Table I presents our results at center-of-mass energies 0.75 keV and 6 keV. The latter energy corresponds to the Gamow peak for the p-p reaction at the center of the sun.[1] For comparison, we show in Table II the values of $\Lambda^2(E)$ at these energies calculated using Eq. (1) with $w_{pp}(k,r)$ obtained by numerical solution of the radial equation with the $^1S_0$ Reid soft-core potential for all r and either the Reid soft core ($P_D=0.065$) or McGee ($P_D=0.06$) $u_d(r)$.[14] Our numbers agree with other impulse-approximation estimates of recent vintage[15] except for one apparent discrepancy: Gari and Huffman[16] report $\Lambda^2(E=0)=7.23$ for Reid soft core w_{pp} and u_d. This exceeds our upper bound for $R=1$ fm with the Reid effective-range parameters and also disagrees with our own calculations with these same radial functions, as may be seen from the second line of Table II. The entries of Table II are all consistent with the values adopted as best estimates by Bahcall and May [Eq. (22) of their paper cited in Ref. 15], namely $\Lambda^2(E)=(7.08\pm0.18)[1+2.2E(\text{MeV})]$.

As Table I shows, the upper bound provided by Eqs. (7), (10), and (13) is remarkably stringent.[17] To what degree do these numbers reflect specific features of the models we used for $U(r)$ and $u_d(r)$? For distances greater than 3 fm, all meson-theory potentials and "realistic" phenomenological two-nucleon potentials are indistinguishable since the universal one-pion-exchange tail dominates in this range. Similarly, all commonly used $^3S_1$ deuteron radial functions are virtually identical at these distances.[18] Thus, the entries in the last three columns of Table I are essentially model independent. They reflect the constraints imposed by the empirical $^1S_0$ phase shift, the deuteron D-

TABLE I. Upper and lower bounds on $\Lambda^2(E)$ in impulse approximation for center-of-mass energies $E=0.75$ keV and $E=6$ keV.

	R [a] (fm)	1.0	2.0	3.0	4.0	5.0
		$E=0.75$ keV				
Noyes parameters [b]	Upper bound	7.053	7.072	7.112	7.220	7.429
	Lower bound	6.659	4.137	1.949	0.642	0.067
RSC parameters [c]	Upper bound	7.066	7.074	7.109	7.212	7.418
	Lower bound	6.579	4.093	1.928	0.634	0.065
		$E=6$ keV				
Noyes parameters [b]	Upper bound	7.146	7.165	7.205	7.314	7.526
	Lower bound	6.745	4.186	1.967	0.643	0.065
RSC parameters [c]	Upper bound	7.159	7.166	7.201	7.306	7.514
	Lower bound	6.664	4.142	1.946	0.636	0.064

[a] R is the radius inside of which the two-nucleon potential is left unspecified.
[b] Effective-range parameters from Ref. 13: $a_{pp}^C=-7.823$ fm, $r_{pp}^C=2.794$ fm.
[c] Effective-range parameters from Ref. 12: $a_{pp}^C=-7.78$ fm, $r_{pp}^C=2.72$ fm.

TABLE II. $\Lambda^2(E)$ calculated from Eq. (1) using the Reid soft-core $^1S_0$ p-p radial function with the Reid soft-core or McGee $^3S_1$ deuteron radial function.

Center-of-mass energy E		0.75 keV	6 keV
RSC + McGee [a]	$\Lambda^2(E)$	7.063	7.156
RSC + RSC [b]	$\Lambda^2(E)$	6.947	7.038

[a] Calculated with the $^3S_1$ deuteron radial function of Ref. 11 (P_D = 0.06).
[b] Calculated with the $^3S_1$ deuteron radial function of Ref. 12 (P_D = 0.065).

state probability, and the longest-ranged (and presumably best understood) components of the nucleon-nucleon interaction. In impulse approximation, then, there is no possibility that the p-p rate might be enhanced by more than a few percent, no matter how exotic a nonlocality characterizes the "true" short-range two-nucleon interaction, provided that the energy dependence of the interaction may be neglected at the extremely low energies of astrophysical interest. On the other hand, even with the unknown nonlocality confined within 1 fm, our lower bounds permit a substantial *decrease* in $\Lambda^2(E)$, so that it is conceivable that the discrepancy between the predicted and (un)observed BSNE counting rate is somewhat worse than current estimates of the p-p rate have led us to believe. We shall not pursue this possibility here.

To understand why our upper bounds leave little room for increasing $\Lambda^2(E)$, it is only necessary to recall that they are based on the Schwartz inequality written above Eq. (7). This inequality becomes an equality if $w_{pp}(k,r)$ is some constant multiple of $u_d(r)$ over the interval $r=0$ to $r=R$. For commonly used local potentials, these two functions are in fact very similar in shape at short distances.

Although our analysis clarifies the status of impulse-approximation calculations of the p-p rate, it leaves two major questions unanswered. First, by how much do exchange currents alter the impulse-approximation results? According to Gari and Huffman,[16] they increase $\Lambda^2(E)$ by about 10% over its impulse-approximation value when Reid soft-core radial functions are used to calculate their contributions. This figure may be changed appreciably if unitary-transformed radial functions are used instead.[19] Second, we have relied on the usual Coulomb-modified effective-range parametrization to extrapolate δ from the lowest energies at which p-p elastic-scattering cross sections have been measured—380 keV to several MeV—down to the experimentally inaccessible range of a few keV. A better extrapolation is provided by the Coulomb- and vacuum-polarization-modified effective-range expansion of Heller.[20] Either way, the reliability of the extrapolation bears some investigation.

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[2]The minuscule cross section predicted by standard weak-interaction theory suggests that such measurements will never be made. See, e.g., C. L. Sarazin, Nuovo Cimento 26B, 94 (1975).

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[5]M. J. Newman and W. A. Fowler, Phys. Rev. Lett. 36, 895 (1976). The numbers we quote were estimated from Fig. 3 of the preprint version of this reference [California Institute of Technology Orange Aid Report No. OAP-435, 1975 (unpublished)], which takes up a full page and is thus much easier to read than Fig. 3 of the published paper.

[6]A sampling of papers applying unitary transforms in nuclear physics may be found in footnote 3 of A. W. Sâenz and W. W. Zachary, J. Math. Phys. 17, 409 (1976).

[7]See, e.g., M. A. Preston, *Physics of the Nucleus* (Addison-Wesley, Reading, Mass., 1962), pp. 628 and 629.

[8]E. E. Salpeter, Phys. Rev. 88, 547 (1952).

[9]We use units in which $\hbar$ and the proton mass are both taken as unity. Momenta are in fm^{-1} and energies in fm^{-2} in these units.

[10]M. Abramowitz, in *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. Stegun, National Bureau of Standards Applied Mathematics Series No. 55 (U.S. Government Printing Office, Washington, D.C., 1964), p. 537.

[11]I. J. McGee, Phys. Rev. 151, 772 (1966). The entries of Table I of this reference were altered to give the radial function the correct asymptotic decay (by chang-

ing α from 0.2338 to 0.2316 fm^{-1}) and to make the radial function vanish at $r=0$ (by changing C_4 from -8.9651 to -8.96508). Our $P_D=0.06$.

[12]R. V. Reid, Jr., Ann. Phys. (N.Y.) 50, 411 (1968).

[13]H. P. Noyes, Annu. Rev. Nucl. Sci. 22, 465 (1972).

[14]The $^1S_0$ radial function was obtained numerically by means of the Numerov algorithm, with the potential cut off at 15 fm, using a mesh of 3×10^{-3} fm. Doubling the number of sample points and increasing the cutoff radius to 20 fm changed the radial function only in the fifth significant figure. The Reid soft-core $^3S_1$ deuteron radial function was obtained by piecewise Lagrange interpolation of the entries of Table XVI of Ref. 12 for r less than 10.01 pion Compton wavelengths. Beyond this point, following Reid, we used his asymptotic form.

[15]J. N. Bahcall and R. M. May, Ap. J. 155, 501 (1969); J. E. Brolley, Sol. Phys. 20, 249 (1971).

[16]M. Gari and A. H. Huffman, Ap. J. 178, 543 (1972). Note that with the radial-function normalizations cited by these authors, their Eq. (10) for $\Lambda(O)$ is incompatible with the numbers they give in their Table 2.

[17]In fact, the reader who compares the upper bounds calculated with Noyes parameters and $R=1$ fm (the first entries in the first and fifth rows of Table I) with the impulse-approximation numbers of the first row of Table II (RSC +McGee) may well wonder how it is that the latter numbers should actually *exceed* the former. This inconsistency is only apparent; the bounds we give are bounds for a prescribed phase shift, and the use of the Noyes parameters spoils the phase-shift equivalence of the $^1S_0$ radial functions. The numerical solution of the radial equation used in the impulse-approximation calculation necessarily corresponds to the RSC effective-range parameters, which are quite different from those of Noyes.

[18]See, e.g., Fig. 1 of D. W. L. Sprung, in *Few Body Problems in Nuclear and Particle Physics*, edited by R. J. Slobodrian, B. Cujec, and K. Ramavataram (Les Presses de l'Université Laval, Québec, 1975), p. 491.

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