Momentum distributions in the nucleus*

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The power law falloff of the nuclear form factor $F(q)$ and single particle momentum distribution $n(q)$ for large q are discussed. For $F(q)$ we show that it is q/A that must be large for the asymptotic form to be valid. In a one-dimensional model with δ function forces the form factor is shown to fall exponentially with q if q is large but q/A is not. Similar behavior is suggested for $n(q)$.

[NUCLEAR STRUCTURE Large momentum behavior of nuclear form factor and single particle momentum distribution in general and in a one-dimensional model.]

I. INTRODUCTION

"Fermi motion"—the motion of nucleons in the nucleus—is caused by nucleon confinement, interactions, and the Pauli principle. Although it is of great importance in understanding many medium energy nuclear processes, particularly those involving large momentum transfers, surprisingly little is known about the general features of the momentum distribution, particularly for large momenta. In this paper we study the nuclear form factor $F(q)$ and the one particle momentum distribution $n(q)$ in general and in a simple model to determine the large q behavior, the scale that determines whether q is "large," and the behavior for intermediate q.[1]

The single particle momentum distribution $n(q)$ is the probability density for finding a particle of momentum q (with respect to the center of mass momentum) in the nucleus. It is therefore just the square of the center of mass momentum space bound state wave function integrated over all but one momentum. The form factor $F(q)$ is the Fourier transform of the one particle position density. F and n are not the same because the densities involve the square of the wave function. $F(q)$ can also be thought of as the amplitude for giving a particle in the nucleus an additional momentum $\vec{q}$ and returning it to the nucleus so that the entire nucleus recoils with momentum $\vec{q}$. This is shown schematically in Fig. 1.

In determining how F and n behave for large q, and deciding what scale characterizes the onset of this asymptotic behavior, we use the Schrödinger equation satisfied by the bound state wave function ψ:

$$\psi = -(B+K)^{-1}V\psi = -G_0 V\psi, \tag{1}$$

where B is the binding energy, K the kinetic energy, and V the sum of pair potentials (assumed local) v_{ij}.

Consider first the form factor. The "struck" particle in Fig. 1 must share its additional momentum $\vec{q}$ with the $A-1$ other nucleons. Finally each nucleon must acquire an additional momentum of $\vec{q}/A$, so that the entire nucleus has momentum $\vec{q}$. This sharing requires at least $A-1$ iterations of the Schrödinger equation. If q is sufficiently large $G_0 \sim 1/q^2$ and $V \sim \tilde{v}(q)$ where $\tilde{v}$ is the Fourier transform of the pair-potential v. Hence we get the well known result[2] that for very large q

$$F(q) \sim [(1/q^2)\tilde{v}(q)]^{A-1}, \tag{2}$$

where we have not included complications that can arise from the Pauli principle[3] or from angular factors that introduce logarithmic terms. It is clear from the discussion that it is not q large compared with typical momenta of the problem that is needed for (2) to be valid, but q/A large. For A large there is therefore an interesting and important region of q large compared with typical momenta but q/A not large. In this regime we need not expect the polynomial falloff as implied by (2),[4] but we still expect $F(q)$ to fall rapidly as the momentum q branches out and distributes itself among the $A-1$ other particles. We show that, for a simple soluble one-dimensional model of particles interacting with δ-function forces, $F(q)$ decreases like $e^{-\alpha q}$ in this regime even though F is strictly a function of q^2. This is discussed in Sec. II while the use of Feynman graph methods to obtain the one-dimensional results is discussed in Appendix A.

For the single particle momentum distribution $n(q)$, the large q behavior has not been previously discussed.[5] It is clear that in order to produce

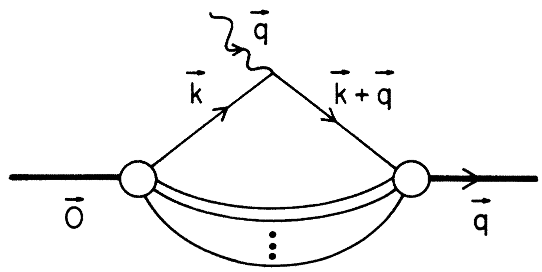

FIG. 1. Schematic representation of the form factor $F(q)$.

a particle of momentum $\vec{q}$ in the center of mass wave function, only one other nucleon need have momentum $-\vec{q}$, and hence only one iteration of (1) is needed. Since $n(q)$ involves the square of the wave function, we have $n(q)\sim[1/q^2\tilde{v}(q)]^2$ for large q. This result is obtained in more detail in Sec. III. Unfortunately we have no simple argument to determine the domain of q for which this limiting form is valid, although in a simple Hartree model we find exponential falloff for $n(q)$ as for $F(q)$. As we discuss in Appendix B, we cannot obtain a closed form for $n(q)$ in our simple one-dimensional model, and hence investigate these questions directly. In Sec. IV we present a brief summary.

II. FORM FACTOR

We now turn to a discussion of the form factor of the bound state of A particles moving in one dimension and interacting with δ function forces. The Hamiltonian is

$$H=-\sum_{i=1}^{A}\frac{\partial^2}{\partial x_i^{\,2}}-g\sum_{i<j=1}^{A}\delta(x_i-x_j),\quad g>0, \tag{3}$$

($\hbar=2m=1$). The center of mass solution for the bound state[6] is (there is only one bound state)

$$\psi(x_1,x_2\,\ldots\,x_A)=N\exp\Big(-\frac{g}{4}\sum_{i<j}|x_i-x_j|\Big), \tag{4}$$

where N is the normalization. The binding energy is

$$E_A=-\frac{g^2}{48}A(A^2-1). \tag{5}$$

In terms of ψ, the form factor can be written

$$2\pi\delta(p-q-p')F(q)=\int|\psi(x_1,x_2\,\ldots\,x_A)|^2\times e^{iqx_1}\exp\Big(i(p-p')\frac{1}{A}\sum_{i=1}^{A}x_i\Big)\prod_{i=1}^{A}dx_i\,. \tag{6}$$

If we integrate with respect to $p'-p$, we get

$$F(q)=\int|\psi(x_1,x_2\ldots x_A)|^2e^{iqx_1}\delta\Big(\frac{1}{A}\sum_{i=1}^{A}x_i\Big)\prod_{i=1}^{A}dx_i\,. \tag{7}$$

F can also be expressed in terms of the momentum space solution ϕ. As

$$F(q)=\int\phi^*(k_1,k_2\ldots k_A)\phi(k_1+q,k_2\ldots k_A)\times\delta\Big(\sum_{i=1}^{A}k_i\Big)\prod_{i=1}^{A}\frac{dk_i}{(2\pi)} \tag{8}$$

which is related to the diagrammatic discussion of F given in the Introduction. Equation (7) shows that F is the Fourier transform of the one particle x-space density defined by

$$\rho(x_1)=\int|\psi(x_1,x_2\ldots x_A)|^2\delta\Big(\frac{1}{A}\sum_{i=1}^{A}x_i\Big)\prod_{i=2}^{A}dx_i\,. \tag{9}$$

Colagero and Degasperis[6] have given a closed form expression for the density

$$\rho(x)=\sum_{n=1}^{A=1}a_n\exp(-gnA|x|/2) \tag{10}$$

with the coefficients a_n known. Therefore we have

$$F(q)=gA\sum_{n=1}^{A-1}\frac{a_n n}{q^2+(gnA/2)^2}\,. \tag{11}$$

The coefficients a_n are complicated and further direct manipulation of (11) is difficult. However, it is well known that for very large q, for any regular potential, and in any number of dimensions, F has the asymptotic behavior of Eq. (2). For the case of δ forces $\tilde{v}(q)$ is a constant and hence (2) requires that $F\sim q^{-2(A-1)}$ for large q. With (11) this can only be achieved if the a_n's are such that

$$F(q)=M\prod_{n=1}^{A-1}(q^2+n^2\lambda^2)^{-1}, \tag{12}$$

where M is a normalization constant and we have written $qA/2=\lambda$. Since the system does not saturate [see (5)], it is $(qA/2)^{-1}$ that is the size of the system. In the Appendix we show how a Feynman diagram method can be used to obtain (12) directly without recourse to the ρ of Colagero and Degasperis. Equation (12) can also be written

$$F(q)=\prod_{n=1}^{A-1}(1+q^2/n^2\lambda^2)^{-1}, \tag{13}$$

where we have used the normalization $F(0)=1$ to fix M. As we have emphasized elsewhere and discussed in the Introduction, this result makes explicit the fact that the asymptotic domain of F, where (2) holds is not $q^2\gg\lambda^2$, but $(q/A)^2\gg\lambda^2$. This follows in general from the fact that in order to communicate its momentum "kick" q to the other nucleons and add q to the momentum of the entire nucleus, the struck nucleon gives each of the $A-1$ others a momentum q/A on the average. For large A there is an important regime where $q^2 \gg \lambda^2 \gg (q/A)^2$. In that case the upper limit of the product in (13) can be extended to ∞ to give[7]

$$F(q) \cong \prod_{n=1}^{\infty} (1+q^2/n^2\lambda^2)^{-1} = \frac{\pi q}{\lambda}\left(\sinh\frac{\pi q}{\lambda}\right)^{-1}. \quad (14)$$

Hence, $F(q)$ in this model can be written in closed form for $(q/A-1)^2 \ll \lambda^2$. If $q^2 \gg \lambda^2$, F has a regime of exponential decrease in q, that is $F \sim qe^{-\alpha q}$, not in q^2 in spite of the fact that F is a function of q^2.

Colagero and Degasperis[6] also obtain the Hartree solution for the one-dimensional δ problem. The Hartree form for $\rho(x)$ is an essential part of that solution and from it we can easily calculate the Hartree F. It is exactly the $\sinh^{-1}$ form in (14). This is not surprising since the terms neglected in obtaining (14) are just those associated with nuclear recoil, but in the Hartree approximation the nucleus cannot recoil.

How will features of the "real world" affect the answer (13) or the approximate form (14)? If the potential is not of zero range, its form will enter and this will greatly complicate the results in a way dependent on the dynamics, but will also lead to a faster rate of falloff for F. The Pauli principle will also increase the rate of falloff as has been shown in explicit cases[3] and as is clear from general arguments. The large q behavior of F depends on the small distance behavior of the wave function and the effect of the Pauli principle is to make the wave function vanish more rapidly at small interparticle separations, which in turn means faster q decrease. These two effects together make the details of F more difficult to study, but they make F decrease more rapidly with q and may therefore extend the range of validity of the approximation (14). Of course for small q, (13) is not generally valid since the small q behavior depends entirely on the dynamical details. Finally the fact that the "real" form factor is calculated in three dimensions rather than one dimension will not affect the power law behavior but can introduce logarithmic factors that come from the angular integrals.

III. MOMENTUM DISTRIBUTION

The single particle momentum distribution $n(q)$ is the probability of finding a particle of momentum q in a given quantum mechanical state. In terms of the center of mass bound state momentum space wave function of that state (now in three dimensions) it can be written

$$n(q) = \int |\phi(\vec{q}, \vec{k}_2, \vec{k}_3 \ldots \vec{k}_A)|^2 \delta\left(\vec{q} + \sum_{i=2}^{A} \vec{k}_i\right) \prod_{i=2}^{A} \frac{d^3k_i}{(2\pi)^3} \quad (15)$$

or in terms of the r-space center of mass wave function

$$n(q) = \int e^{i\vec{q}\cdot\vec{r}'} \psi^*(\vec{r}_1 + \tfrac{1}{2}\vec{r}', \vec{r}_2, \vec{r}_3 \ldots \vec{r}_A)\psi(\vec{r}_1 - \tfrac{1}{2}\vec{r}', \vec{r}_2, \vec{r}_3 \ldots \vec{r}_A)\delta\left(\frac{1}{A}\sum_{i=1}^{A} \vec{r}_i\right) d^3r' \prod_{i=1}^{A} d^3r_i . \quad (16)$$

To study $n(q)$ for large q we change variables in (15) to obtain

$$n(q) = \int |\phi(\vec{q}, \vec{p}_2, -\vec{q}, \vec{p}_3, \vec{p}_4 \ldots \vec{p}_A)|^2 \delta\left(\sum_{i=2}^{A} \vec{p}_i\right) \prod_{i=2}^{A} \frac{d^3p_i}{(2\pi)^3} . \quad (17)$$

ϕ satisfies the homogeneous Schrödinger equation (1). If we concentrate on the v_{12} term in that equation, taking all particles to have the same mass and putting $\hbar = 2m = 1$, we obtain

$$\phi(\vec{q}, \vec{p}_2, -\vec{q}, \vec{p}_3, \vec{p}_4 \ldots \vec{p}_A) = \left[B + q^2 + (\vec{p}_2 - \vec{q})^2 + \sum_{i=3}^{A} p_3{}^2\right]^{-1} \int \langle \vec{q}, \vec{p}_2 - \vec{q} | v_{12} | \vec{k}, \vec{p}_2 - \vec{k} \rangle \times \frac{d^3k}{(2\pi)^3} \phi(\vec{k}, \vec{p}_2 - \vec{k}, \vec{p}_3, \vec{p}_4 \ldots \vec{p}_A) - G_0 \sum_{\substack{i>j \\ (\neq 1,2)}} v_{ij}\phi \quad (18)$$

subject to $\sum_{i=2}^{A} \vec{p}_2 = 0$. For very large q the first term on the right goes to $1/q^2 \tilde{v}_{12}(q) I$, where I is a finite function of the p_i's and where $\tilde{v}(q)$ is the Fourier transform of the two-body potential v_{12} assumed to be local. Hence, we obtain

$$n(q) \sim 1/q^4[\tilde{v}(q)]^2 \quad (19)$$

for large q. Thus it is the pair correlations that ultimately determine the large q behavior of n. Unfortunately except for noting that this result depends on selecting one from the $\frac{1}{2}A(A-1)$ terms in (18), we have no simple way of setting the q scale in n, that is of determining whether it is q or q/A that must be large, and therefore of deciding when the asymptotic regime is reached.

We turn now to the one-dimensional model of Sec. II to see what insight we can gain there. We see immediately from (16) that $n(q)$ is not the Fourier transform of $\rho(x)$, but involves a convolution. This happens, of course, because the densities are quadratic in the wave functions. Unfortunately we are not able to obtain a simple closed form expression for the momentum distribution $n(q)$ in the one-dimensional model. In Appendix B we discuss the technical problems in more detail and demonstrate explicitly that n satisfies (19) for this case (recall that for δ forces $\tilde{v}$ is constant) using Feynman graph methods. It is simple to get $n(q)$ for the Hartree solution of the one-dimensional problem as given by Colagero and Degasperis. We find

$$n(q)=N\left(\cosh\frac{\pi q}{\lambda}\right)^{-2}, \tag{20}$$

where N is a normalization and λ the same momenta range that enters in (12). Hence, we see again that neglecting the constraint of center of mass motion gives a simple form that for large q falls exponentially in q even though n is a function of q^2. Unfortunately we do not know over what range of q (20) is valid.

IV. SUMMARY

By using the homogeneous Schrödinger equation (1) satisfied by a many-body bound state, we discuss the large momentum form of the bound state form factor $F(q)$ (2) and the single particle momentum distribution $n(q)$ (19). For F we show that it is q/A that must be large for (2) to apply and in a simple one-dimensional model with δ function forces we show that q/A is not large compared with the momenta scale of the problem, but q is $F(q)\sim qe^{-\alpha q}$, (14). The ultimate importance of the polynomial falloff when q/A is large is associated with nuclear recoil and disappears in a Hartree model that neglects that recoil. For $n(q)$ we are not able to set the scale of "large" for q or to obtain its form in the one-dimensional model, but again the Hartree result gives exponential falloff. In the Appendixes we show how Feynman graph methods can be used to obtain F and discuss n in the one-dimensional case.

These results have application in indicating appropriate parametrizations for "Fermi motion" in a wide class of large momentum transfer medium energy nuclear physics examples.[8]

I would like to thank J. Negele for bringing Ref. 6 to my attention and R. M. Woloshyn for emphasizing the importance of the Hartree results in that work.

APPENDIX A: FORM FACTOR BY FEYNMAN GRAPH METHODS

In this Appendix we obtain (13) for (7) with (4) by interpreting (7) as a Feynman graph. We note that the correlation factors in the wave function $\exp(-\lambda|x_i-x_j|)$ are one-dimensional Green's functions, which is not surprising since they come from solving a problem with δ function interactions. Using the identity

$$e^{-\lambda|x|}=2\lambda\int\frac{dk}{2\pi}\,\frac{e^{ikx}}{k^2+\lambda^2} \tag{A1}$$

we see that each $e^{-\lambda|x|}$ factor corresponds to a propagator of "mass" $i\lambda$. If we use (4) and (A1) in (6) and do the x integrals we obtain

$$2\pi\delta(p-q-p')F(q)=g^{A(A-1)/2}N^2\int\prod_{i<j=1}^{A}\frac{dk_{ij}}{(2\pi)}\,\frac{1}{(k_{ij}{}^2+\frac{1}{2}g^2)}\times 2\pi\delta\left(q-Q+\sum_{j=2}^{A}k_{1j}\right)\prod_{n=2}^{A}2\pi\delta\left(Q-\sum_{j=n+1}^{A}k_{nj}+\sum_{i=1}^{n-1}k_{in}\right), \tag{A2}$$

where we have defined $p-p'=AQ$. Equation (A2) represents a Feynman graph in which there are A points or vertices. At point 1 a momentum $q-Q\ [=(A-1)Q]$ enters while a momentum Q flows out at each of the other $A-1$ points $2,3,\ldots A$. Each pair of points is connected by a propagator $(k_{ij}{}^2+\frac{1}{2}g^2)^{-1}$. The δ functions insure momentum conservation at each vertex. There are $A-1$ conditions among the k_{ij} imposed by the δ functions (the remaining δ is over all momentum conservation $q=AQ$), and they can be used to eliminate $A-1$ k_{ij}. For example if we use the δ functions to eliminate the $A-1$ k_{1j} we get (canceling the overall conserving δ but now remembering that $q=AQ$):

$$F(q)=g^{A(A-1)/2}N^2\int\prod_{\substack{i<j\\ i=2}}^{A}\frac{dk_{ij}}{2\pi}\,\frac{1}{(k_{ij}{}^2+\frac{1}{2}q^2)}\prod_{n=2}^{A}\left[\left(Q+\sum_{i=2}^{n-1}k_{in}-\sum_{i=n+1}^{A}k_{ni}\right)^2+\frac{1}{2}q^2\right]^{-1}. \tag{A3}$$

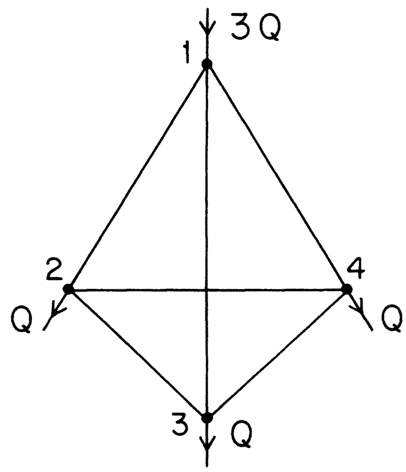

FIG. 2. Feynman graph for the four-body form factor corresponding to Eq. (A3).

The remaining k integrals can now be done by contour integration (since k runs from $-\infty$ to ∞). The integrand has only simple poles in the k's and the result of the successive k integrations therefore leads to a meromorphic function of Q (that is a ratio of polynomials). In particular no square root or logarithmic functions can develop. In fact it must be a ratio of polynomials in Q^2 since F is a function of Q^2. For very large Q, $F \sim Q^{-2(A-1)}$ as can be seen from (A3). Some caution is required here. The initial impulse is to say that for Q large, the last product factor in (A3) becomes $Q^{-2(A-1)}$ and since the remaining integral factor converges, that proves the result. The difficulty is that even for Q large some k_{st} in the bracket can also be of order Q so that the factor $Q+\sum_{i=2}^{n-1} k_{in} - \sum_{i=n+1}^{A} k_{ni}$ is of order 1, in which case that k will make one of the $k_{ij}^2 + \frac{1}{2}q^2$ terms of order Q^2. Since all contributions enter with a positive sign, they cannot cancel and hence the leading large Q behavior of (A3) is as expected and is given by (2) [recall $\tilde{v}(q) = \text{constant}$ for δ forces]. However, the complication discussed above makes it very difficult to extract the coefficient of $Q^{-2(A-1)}$.

Although the integral (A3) can be done, it is very complicated. However, it is straightforward to find all its singularities by the methods of Landau.[9] Since we know $F(0)=1$, and that F is meromorphic with asymptotic behavior $Q^{-2(A-1)}$, if we can show that it has only $A-1$ simple poles in Q^2 and locate them, we have found F. It then has the form

$$F(q) = \prod_{n=1}^{A-1} (1 + Q^2/Q_n{}^2)^{-1}, \qquad (A4)$$

where $-Q_n{}^2$ is the location of the nth pole of F.

In fact these are precisely the conditions given by (11) with $Q_n{}^2 = \frac{1}{2}(gn)^2$, $n = 1, \ldots A-1$, but we can also obtain this result directly by Landau analysis of (A3). Rather than introduce the Feynman parametrization of the denominators in (A3) and proceed with a complicated algebraic analysis, let us take a simple example, the four-body case, and discuss the problem graphically. The four-body form factor graph corresponding to (A3) is shown in Fig. 2, where each internal line corresponds to a propagator of "mass" $m = iq/(2)^{1/2}$. The leading singularities of this graph correspond to the reduced diagrams obtained by bringing vertices together. This corresponds to setting the Feynman parameter associated with the line to zero or in the electrical analog to short circuiting the line. The inequivalent reduced graphs obtained from Fig. 2 are shown in Figs. 3(a)–3(c). They all correspond to threshold singularities, and these come at

$$(3Q^2)^2 = (3m^2), \quad Q^2 = m^2, \quad \text{for Fig. 3(a)}$$
$$(2Q^2)^2 = (4m)^2, \quad Q^2 = 4m^2, \quad \text{for Fig. 3(b)} \qquad (A5)$$
$$Q^2 = (3m)^2, \quad Q^2 = 9m^2, \quad \text{for Fig. 3(c)}.$$

For the A particle graph the generalization is now clear. There are $A-1$ reduced graphs corresponding to bringing $0, 1, 2, \ldots A-2$ of the Q legs up to the $(A-1)$ Q vertex and contracting the remaining Q legs at the bottom. Thus there are exactly $A-1$ reduced graphs. If n legs are left at the bottom in a particular reduced graph, there are $n + (A-1-n)n$ intermediate lines. Hence, the singularity comes at $Q^2 = (A-n)^2 m^2$, $n = 1, 2, \ldots A-1$. That there are no other singularities follows from a detailed analysis of the Landau equations, which is very technical in general and is not strictly required here given (11). That the singularities are all poles follows from the fact that the function is meromorphic; that they are all simple poles follows from the fact that all threshold singularities in one dimension are simple

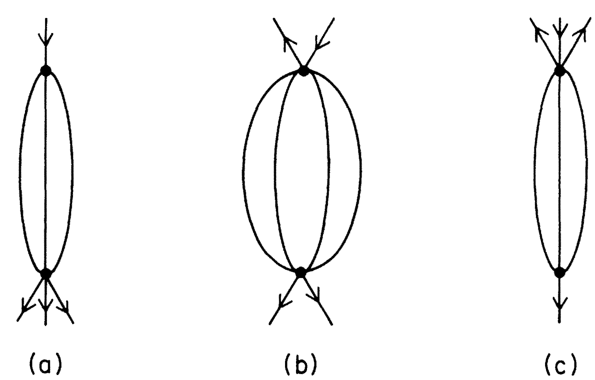

FIG. 3. Contractions of the graph of Fig. 2.

poles. To see this consider the phase space in one dimension for producing n equal mass particles of total momentum p_0:

$$\rho(s') = \prod_{i=1}^{n} \left[\delta^+(p_i^{\,2} - m^2)\, dp_i \right] \delta\left(p_0 - \sum_{i=1}^{n} p_i\right), \quad s' = p_0^{\,2}. \tag{A6}$$

The threshold behavior can be obtained from the dispersion relation

$$a(s) = \frac{1}{\pi} \int \frac{\rho(s')ds'}{s'-s} = \frac{(2m)^{-n}\, 2(nm)}{(nm)^2 - s}. \tag{A7}$$

Hence, we have established (13).

APPENDIX B: MOMENTUM DISTRIBUTION BY FEYNMAN GRAPH METHODS

In this Appendix we discuss application of the Feynman graph methods of Appendix A to the calculation of the momentum distribution for the one-dimensional problem with δ function forces described in Sec. II. We have in terms of (16) and (4)

$$n(q) = N^2 \int e^{iqx'}\, \delta\left(\frac{1}{A} \sum_{i=1}^{A} x_i\right) dx' \prod_{i=1}^{A} dx_i \, \exp - \tfrac{1}{4} g \left[\sum_{i=2}^{A} \left(|x_1 + \tfrac{1}{2}x' - x_i| + |x_1 - \tfrac{1}{2}x' - x_i| \right) + 2 \sum_{\substack{i<j \\ j,i=2}} |x_i - x_j| \right]. \tag{B1}$$

If we introduce the momentum representation for the δ function

$$\delta\left(\frac{1}{A} \sum_{i=1}^{A} x_i\right) = \int \frac{dk}{2\pi} \exp\left(i \frac{1}{A} \sum_{i=1}^{A} x_i\right) \tag{B2}$$

and the momentum representation for the correlation functions (A1), we obtain for (B1), after doing the x integrals:

$$n(q) = N^2 \frac{g^{(A-1)(3A-4)}}{2^{2(A-1)}} \int \frac{dk}{2\pi} \prod_{\substack{i>j \\ j=2}}^{A} \left(\frac{dk_{ij}}{2\pi} \frac{1}{(k_{ij}^{\,2} + \frac{1}{4}g^2)} \right) \prod_{i=2}^{A} \left(\frac{dk_{1i}}{2\pi} \frac{dk'_{1i}}{2\pi} \frac{1}{(k_{1i}^{\,2} + \frac{1}{16}g^2)(k'^{2}_{1i} + \frac{1}{16}g^2)} \right)$$
$$\times 2\pi\delta\left[2q + \sum_{i=2}^{A} (k'_{1i} - k_{1i})\right] 2\pi\delta\left[\frac{k}{A} + \sum_{i=2}^{A} (k'_{1i} + k_{1i})\right]$$
$$\times \prod_{n=2}^{A} \delta\left(\frac{k}{A} - k_{1n} - k'_{1n} + \sum_{i=n+1}^{A} k_{ni} - \sum_{j=2}^{n-1} k_{jn}\right). \tag{B3}$$

This complicated expression represents a Feynman graph with $A-1$ vertices $2, 3, \ldots A$ and two vertices 1 and 1′. Each pair of vertices of the set $2, 3, \ldots A$ is connected by a propagator $(k_{ij}^{\,2} + \frac{1}{4}g^2)^{-1}$. From each point 1 (or 1′) to each point $2, 3, \ldots A$ there is a propagator $(k_{1i}^{\,2} + \frac{1}{16}g^2)^{-1}$ [or $(k'^{2}_{1i} + \frac{1}{16}g^2)^{-1}$]. Momentum q enters at point 1 and leaves at 1′. The various δ functions insure momenta conservation at the vertices. The extra k integral removes the overall momentum conserving δ function one normally associates with a Feynman graph. Graphs for the three- and four-particle cases are shown in Fig. 4. For the three-particle case, explicit calculation gives

$$n(q) = \frac{N'(q^2 + 13g^2)}{(q^2 + \frac{1}{4}g^2)^2 (q^2 + g^2)}, \tag{B4}$$

where N' is a constant. From (B4) we see that there is a double pole at $q^2 = -\frac{1}{4}g^2$, a single pole at $q^2 = -g^2$, but a zero at $q^2 = -13g^2$ so that the entire result is $\sim 1/q^4$ for large q. This large q behavior is expected from (19) recalling that $\tilde{v}(q)$ is a constant for δ function forces. The double pole corresponds to the reduced graph obtained

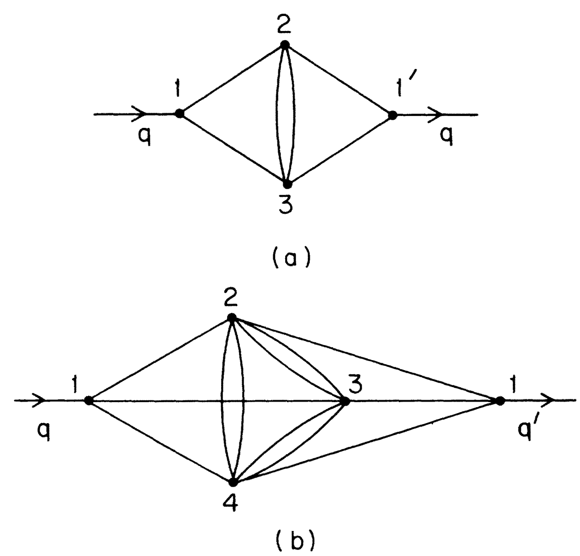

FIG. 4. Feynman graph for the momentum distribution $n(q)$ corresponding to Eq. (B3) for the (a) three-body and (b) four-body cases. The double lines correspond to propagators of "mass squared" $-\frac{1}{4}g^2$, the single lines to $-\frac{1}{16}g^2$.

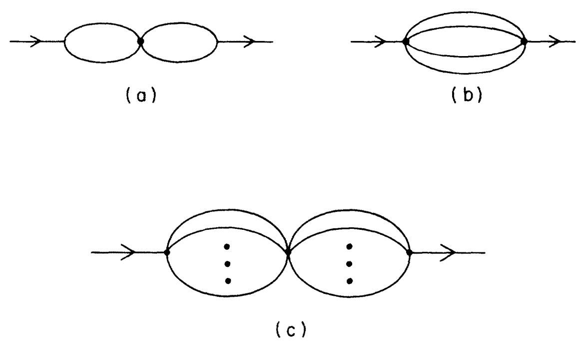

FIG. 5. Contractions of the graphs for $n(q)$. (a) and (b) are the contractions of the three-body graph Fig. 4(a). (c) is the leading contraction of the general A-body graph.

by joining points 2 and 3 in Fig. 4(a) while the other pole comes from collapsing the lines 1-2 and 1′-3. These reduced graphs are shown in Figs. 5(a) and 5(b). It is clear that in general $n(q)$ will have a double pole at $q^2 = -[(A-1)\frac{1}{4}g^2]^2$ from collapsing the $A-2$ central points together as shown in Fig. 5(c). This is always its first singularity. There are then further poles (n is meromorphic for the same reason as F) that come from the many ways of collapsing the k_{1i} and k'_{1i} lines. It is a simple matter to locate all these poles. They come for q^2 more negative than the double pole, but since $n(q) \sim 1/q^4$ for very large q there are just as many zeros and we have not found a way to locate them. We cannot determine if, as in the three-body case, the zeros all come for q^2 more negative than the last pole. If they did that would show that the $1/q^4$ region is only important for $(q/A)^2$ large. However, we have no general form for $n(q)$.

To show that indeed (B3) goes like q^{-4} for large q we note that the momentum q entering at 1 need only be carried to the center by one propagator, for instance k_{1i}. If it is then carried out by k'_{1i}, none of the k_{ij} $(j > i > 1)$ are involved. Hence, only two propagators are of order $1/q^2$ and $n \sim q^4$. As we discussed in Appendix A, care is needed in this argument and in particular it cannot be used simply to extract the coefficient of q^{-4}, but because all propagators are positive there is no possibility of cancellation and the $1/q^4$ behavior is established.

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