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# Isospin mixing in nuclear resonance levels

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The problem of isospin mixing in nuclear resonance levels is studied using the framework of Feshbach's unified theory of reactions. The nuclear matrix elements occurring in the theory are expressed in terms of the measured quantities. These relations would be useful in interpreting the experimental data. The unitarity limit on the phase of the mixing amplitude is derived.

[NUCLEAR REACTIONS Feshbach's theory, complex isospin mixing coefficient; obtained unitarity bound.]

## I. INTRODUCTION

The mixing of nuclear resonance levels with the same spin and parity but of different isospin $T$ has been observed in various nuclear reactions.[1] It is introduced by electromagnetic interaction and the possible isospin symmetry breaking nuclear interaction. In the analysis of such experimental data, it has always been assumed that the mixing coefficients, occurring in the mixed states, are real. It was first pointed out by Baryshavskii, Lyuboshitz, and Podgoretskii,[2] and very recently by Shanley,[3] that the mixing with real parameters is not sufficiently general and that it violates unitarity of the $S$ matrix. Following the reasoning of Kabir,[4] Shanley formulated the problem of isospin mixing with complex parameters. In order to relate the mixing parameter to the measured quantities he utilized the Bell-Steinberger[5] sum rule and other relations derived from the unitarity of the $S$ matrix. The main purpose of the present work is to formulate this problem of isospin mixing of nuclear resonance levels using the framework of Feshbach's theory[6] of nuclear reactions. We also derive the unitarity limit on the phase of the mixing amplitude. It provides an alternative formulation of the problem based on conventional theory of nuclear reactions.

## II. FORMALISM

We consider nuclear reactions in which only two composite particles are involved in the open channels. The final product nuclei may be different from the initial composite particles when the redistribution of nucleons takes place during the reaction. The Hamiltonian $H$ of all the interacting nucleons in the two composite nuclei may be written as $H = H_c + T_c + V_c$, where $H_c$ is the internal Hamiltonian of the composite nuclei indicated by the suffix $c$, $T_c$ describes their relative motion, and the interaction between the composite particles is $V_c$. The eigenstates and the corresponding eigenvalues of $H_c$ are denoted by $\varphi_{ci}$ and $\mathcal{E}_{ci}$. The two labels $c$ and $i$ still do not uniquely specify the state of the two composite particles. They may have spins $S_1$ and $S_2$ and isospins $t_1$ and $t_2$. We imagine the channel label $\alpha$ to specify the pair of composite particles $c$, their internal state $i$, and the usual spin and isospin coupling schemes

$$\{ci; S_1S_2(S)lJM, t_1t_2TT_3\}.$$

The resultant spin $S$ of the two particles is coupled to their relative angular momentum $l$ to obtain the total angular momentum $J$ and its projection $M$ on the $z$ axis. Similarly, the isospins $t_1$ and $t_2$ are coupled to give the total isospin $T$ and its projection $T_3$. In order to simplify notations, we will use the same suffix $\alpha$ to $\varphi_\alpha$, $\mathcal{E}_\alpha$, $H_\alpha$, $T_\alpha$, and $V_\alpha$ instead of $c$ and $i$. It is to be noted that $\mathcal{E}_\alpha$, $H_\alpha$, $T_\alpha$ and $V_\alpha$ depend only on the index $c$ and $i$ and not on the other quantum numbers required for the complete specification of the channel wave function. The wave function describing the relative motion of the two nuclei in channel $\alpha$ is written as $u_\alpha(\vec{r}_\alpha)$. The many-body scattering wave function $\Psi$ corresponding to the total energy $E$ satisfies the Schrödinger equation

$$(E - H)\Psi = 0\,. \tag{1}$$

In Feshbach's theory,[6] one defines a projection operator $P$ which projects out the open channels part from the total wave function $\Psi$. Then the operator $Q = 1 - P$ acts only off the open channels:

$$P\Psi = \sum_{\alpha(\text{open})} u_\alpha \varphi_\alpha\,. \tag{2}$$

The projection operators $P$ and $Q$ satisfy the following relations

$$P^2 = P, \quad Q^2 = Q, \quad \text{and} \quad PQ = QP = 0. \tag{3}$$

When $P$ acts only on the internal parts of the wave function it can be written as

$$P = \sum_{\alpha(\text{open})} |\varphi_\alpha)(\varphi_\alpha| , \tag{4}$$

where the round bra or ket indicates integration only over the internal variables and the angular bra or ket, the total integration. Using relations (3), one obtains from Eq. (1)

$$\left(E - H_{PP} - H_{PQ}\frac{1}{E - H_{QQ}}H_{QP}\right)P\Psi = 0 \tag{5}$$

and

$$\left(E - H_{QQ} - H_{QP}\frac{1}{E^{+} - H_{PP}}H_{PQ}\right)Q\Psi = 0 , \tag{6}$$

where

$$H_{PP} = PHP, \quad H_{QQ} = QHQ, \quad H_{PQ} = PHQ,$$

and

$$H_{QP} = QHP .$$

Substituting $P\Psi$ from Eq. (2) into Eq. (5) and integrating over the internal coordinates, one obtains the following set of coupled equations:

$$\sum_\beta \left[(E - \mathcal{E}_\alpha - T_\alpha)\delta_{\alpha\beta} - (\varphi_\alpha|V_\alpha|\varphi_\beta) - \left(\varphi_\alpha \middle| VQ\frac{1}{E - H_{QQ}}QV \middle| \varphi_\beta\right)\right]u_\beta = 0 . \tag{7}$$

Our formulation is based on Lemmer's[7] approach to nuclear reactions. The eigenstates of $H_{QQ}$ which are near $E$ will introduce rapid energy changes in the last term of Eq. (7). The contribution of these states therefore would be important at $E$, while the contribution of other states will be a slowly varying function of $E$ and can be absorbed in $V_\alpha$. Let $q$ project onto the states of $H_{QQ}$ that are important in the vicinity of $E$, and $Q'$ onto the rest with $Q = q + Q'$. The effective interaction $U_\alpha$ varying slowly with $E$ is

$$U_\alpha = V_\alpha + V_\alpha Q'(E - H_{Q'Q'})^{-1}Q'V_\alpha .$$

With this approximation Eqs. (5) and (7) become

$$[E - P(H_\alpha + T_\alpha + U_\alpha)P]P\Psi \equiv [E - H^0_{PP}]P\Psi = \left(Vq\frac{1}{E - H_{qq}}qV\right)P\Psi \tag{8}$$

and

$$\sum_\beta \left[(E - \mathcal{E}_\alpha - T_\alpha)\delta_{\alpha\beta} - (\varphi_\alpha|U_\alpha|\varphi_\beta) - \left(\varphi_\alpha \middle| Vq\frac{1}{E - H_{qq}}qV \middle| \varphi_\beta\right)\right]u_\beta = 0 . \tag{9}$$

In the direct interaction picture the fluctuating term is neglected and one has the solutions

$$\Phi^{\pm}_\alpha = \sum_{\beta(\text{open})} u^{\pm(\alpha)}_\beta \varphi_\beta . \tag{10}$$

The suffix $\alpha$ on $\Phi^{\pm}$ and $u^{\pm}_\beta$ indicates the presence of the incident plane wave in channel $\alpha$. The transition amplitude for reaching a final state $f$ from an initial state $i$, in this approximation, is

$$t_{fi} = \langle \varphi_{\alpha f} u^0_{\alpha f}|U_\alpha|\Phi^{+}_{\alpha i}\rangle . \tag{11}$$

In Eq. (11), $u^0_{\alpha f}$ is the plane wave of energy $E - \mathcal{E}_{\alpha f}$. According to our assumptions $t_{fi}$ is a slowly varying function of $E$ over the energy range of interest $\Delta E$ around $E$. The formal solution of Eq. (8) is

$$P\Psi_\alpha = \Phi^{+}_\alpha + \frac{1}{E^{+} - H^0_{PP}}PVq\frac{1}{E - H_{qq}}qV|P\Psi_\alpha\rangle . \tag{12}$$

The transition amplitude obtained from Eq. (12) is

$$T_{fi} = t_{fi} + \langle \Phi^{-}_{\alpha f}|Vq(E - H_{qq})^{-1}qV|P\Psi_{\alpha i}\rangle . \tag{13}$$

Using Eq. (8) for $P\Psi$, $T_{fi}$ can also be written as

$$T_{fi} = t_{fi} + \left\langle \Phi^{-}_{\alpha f} \middle| Vq\left(E - H_{qq} - qV\frac{1}{E^{+} - H^0_{PP}}Vq\right)^{-1}qV \middle| \Phi^{+}_{\alpha i}\right\rangle . \tag{14}$$

Let us define an operator $\mathcal{H}_{qq}$

$$\mathcal{H}_{qq} = H_{qq} + qV\frac{1}{E^{+} - H^0_{PP}}Vq , \tag{15}$$

and its eigenvalues $E_\mu$ and the corresponding eigenfunctions $\Psi_\mu$ by

$$\mathcal{H}_{qq}\Psi_\mu = E_\mu\Psi_\mu . \tag{16}$$

Clearly, $\mathcal{H}_{qq}$ in Eq. (15) is not a Hermitian operator, therefore the $\Psi_\mu$s are not orthogonal. However, if we define another set of functions $\Psi^\alpha_\mu$ such that

$$\mathcal{H}^{\dagger}_{qq}\Psi^\alpha_\mu = E^*_\mu\Psi^\alpha_\mu ; \tag{17}$$

then $\langle \Psi^\alpha_\mu|\Psi_\gamma\rangle = 0$ for $\mu \neq \gamma$. With the normalization of $\Psi^\alpha_\mu$ chosen so that $\langle \Psi^\alpha_\mu|\Psi_\mu\rangle = 1$, we have the completeness relation

$$\sum_\mu |\Psi_\mu\rangle\langle \Psi^\alpha_\mu| = 1 . \tag{18}$$

From Eqs. (14), (16), and (18), we obtain

$$T_{fi} = t_{fi} + \sum_\mu \frac{\langle \Phi^{-}_{\alpha f}|V|\Psi_\mu\rangle\langle \Psi^\alpha_\mu|V|\Phi^{+}_{\alpha i}\rangle}{(E - E_\mu)} . \tag{19}$$

One observes from Eq. (19) that the poles of the $T$ matrix are at the eigenvalues of $\mathcal{H}_{qq}$ and that the corresponding resonance wave functions are $\Psi_\mu$. From the eigenvalue equation (16) one has

$$\langle \Psi_{\mu'}|(\mathcal{H}_{qq} - \mathcal{H}^{\dagger}_{qq})|\Psi_\mu\rangle = (E_\mu - E^*_{\mu'})\langle \Psi_{\mu'}|\Psi_\mu\rangle . \tag{20}$$

Substituting Eq. (15) into Eq. (20), one arrives at the following relation:

$$i(E_\mu - E^*_{\mu'})\langle \Psi_{\mu'}|\Psi_\mu\rangle = 2\pi \sum_{\alpha(\text{open})} \langle \Psi_{\mu'}|V|\Phi^{+}_\alpha\rangle\langle \Psi_\mu|V|\Phi^{+}_\alpha\rangle^* . \tag{21}$$

This is the Bell-Steinberger[5] sum rule used by Shanley[3] [there was a misprint of minus sign in his Eq. (6)]. In arriving at this result use is made of the fact that $V$, $H_{qq}$, $H^0_{PP}$, and $q$ are Hermitian operators. With the choice of the normalization $\langle \Psi_\mu | \Psi_\mu \rangle = 1$ and the definition of $E_\mu = E^r_\mu - \frac{1}{2} i \Gamma_\mu$, the width $\Gamma_\mu$ is obtained from Eq. (21) with $\mu' = \mu$,

$$\Gamma_\mu = 2\pi \sum_{\alpha(\text{open})} |\langle \Psi_\mu | V | \Phi^+_\alpha \rangle|^2 . \tag{22}$$

The relations derived in Eqs. (21) and (22) are valid for overlapping resonances irrespective of whether there is isospin mixing or not.

Let us consider now a special case of mixing of two resonances of isospins $T=1$ and $T=0$. Let the corresponding unmixed resonance wave functions be $\Phi_i$ $(i=1,0)$. Before we switch on the isospin breaking interaction $v_c$, one has to carry out the resonance mixing analysis as described above. In the particular case of our interest there is only one level of each value of $T$; we assume that

$$\mathcal{H}^0_{qq} \Phi_i = \eta_i \Phi_i , \tag{23}$$

where $\mathcal{H}^0_{qq}$ implies that in $\mathcal{H}_{qq}$ we have put $v_c = 0$. According to the above analysis $\langle \Phi^a_i | \Phi_j \rangle = \delta_{ij}$. When $v_c$ is switched on, the new resonance positions and the corresponding wave functions will be given by

$$(\mathcal{H}^0_{qq} + v_c)\Psi_i = E_i \Psi_i . \tag{24}$$

This equation gives the resonance positions

$$E_{0,1} = \tfrac{1}{2}\{\eta'_0 + \eta'_1 \pm [(\eta'_0 - \eta'_1)^2 + 4V_{10}V_{01}]^{1/2}\} , \tag{25}$$

where

$$\eta'_i = \eta_i + \langle \Phi^a_i | V_c | \Phi_i \rangle .$$

The corresponding wave functions are

$$\Psi_0 \propto \left( \Phi_0 + \frac{V_{10}}{E_0 - \eta'_1} \Phi_1 \right) ,$$
$$\Psi_1 \propto \left( \Phi_1 + \frac{V_{01}}{E_1 - \eta'_0} \Phi_0 \right) . \tag{26}$$

It is to be noted that the complex number $V_{10} = \langle \Phi^a_1 | v_c | \Phi_0 \rangle = V_{01} = \langle \Phi^a_0 | v_c | \Phi_1 \rangle$ from time-reversal invariance. It also follows from Eq. (25) that $E_0 + E_1 = \eta'_0 + \eta'_1$. From this relation and with the definition of the mixing parameter

$$\epsilon = V_{10}/(E_0 - \eta'_1) ,$$

the normalized wave functions become

$$\Psi_0 = [\Phi_0 + \epsilon \Phi_1]/(1 + |\epsilon|^2)^{1/2}$$
$$\Psi_1 = [\Phi_1 - \epsilon \Phi_0]/(1 + |\epsilon|^2)^{1/2} . \tag{27}$$

These wave functions were derived by Shanley[3] using the Bell-Steinberger[5] sum rule and other relations derived from the unitarity of the $S$ matrix.

Let us consider a situation in which all the open channels have good isospin, $T=0$ or 1. In doing this, we neglected the mixing in the continuum states. Further, let us also neglect the isospin-violating part of $V$ in Eq. (21). One then obtains from Eqs. (21), (22), and (27)

$$i(E_0 - E^*_1)\langle \Psi_1 | \Psi_0 \rangle = \frac{2\pi \sum_{\alpha(\text{open})} (\epsilon |\langle \Phi_1 | V | \Phi^+_\alpha \rangle|^2 - \epsilon^* |\langle \Phi_0 | V | \Phi^+_\alpha \rangle|^2)}{(1 + |\epsilon|^2)^{1/2}} . \tag{28}$$

$$\langle \Psi_1 | \Psi_0 \rangle = 2i \frac{\text{Im}\epsilon}{(1 + |\epsilon|^2)^{1/2}} . \tag{29}$$

$$(\Gamma_1 - \Gamma_0) = \frac{2\pi \sum_{\alpha(\text{open})} (|\langle \Phi_1 | V | \Phi^+_\alpha \rangle|^2 - |\langle \Phi_0 | V | \Phi^+_\alpha \rangle|^2)}{1 + |\epsilon|^2} . \tag{30}$$

The phase of the mixing amplitude $\epsilon = |\epsilon| e^{i\varphi_\epsilon}$ is obtained from the real part of Eq. (28) and Eqs. (29) and (30);

$$\tan\varphi_\epsilon = \frac{(\Gamma_1 - \Gamma_0)(1 + |\epsilon|^2)}{2(E^r_1 - E^r_0)(1 - |\epsilon|^2)} . \tag{31}$$

From Eqs. (29), (30), and the imaginary part of Eq. (28), one also obtains the matrix elements (unperturbed widths due to only nuclear interactions)

$$2\pi \sum_{\alpha(T=1)} |\langle \Phi_1 | V | \Phi^+_\alpha \rangle|^2 = \Gamma_1 - \Gamma_0 |\epsilon|^2$$
$$2\pi \sum_{\alpha(T=0)} |\langle \Phi_0 | V | \Phi^+_\alpha \rangle|^2 = \Gamma_0 - \Gamma_1 |\epsilon|^2 . \tag{32}$$

If only one isospin value, say $T=0$, is possible for one of the decay channels $\alpha$, it follows from the use of Eq. (27) that

$$|\epsilon|^2 = \Gamma^1_\alpha / \Gamma^0_\alpha .$$

From the eigenvalue equation for $E_i$ and the definitions of $\Psi_i$ in Eq. (27), one easily obtains the mixing interaction matrix element

$$V_{10} = \frac{\epsilon(E_0 - E_1)}{(1 + \epsilon^2)} = \frac{|\epsilon| |E_0 - E_1| e^{i(\alpha_1 + \alpha_0)}}{1 + |\epsilon|^4 + 2|\epsilon|^2 \text{Cos} 2\varphi_\epsilon} , \tag{33}$$

where

$$\tan \alpha_{1,0} = 2A|\epsilon|^2 [1 - |\epsilon|^2 \mp A^2(1 + |\epsilon|^2)]^{-1}$$

and

$$A = \tfrac{1}{2}(\Gamma_1 - \Gamma_0)/(E^r_1 - E^r_0) . \tag{34}$$

From Eqs. (33) and (34) one observes that the interaction matrix element is real to order $|\epsilon|^2$. This was also found by Shanley[3] in the numerical calculation of the $\alpha$ decay of $^8$Be $2^+$ level.

Let us consider now a general case where particle decay channels of both $T=0$ and $T=1$ are open, and the electromagnetic decay widths are not negligible. We will still assume that isospin is conserved in the particle decays, i.e., the continuum states have definite isospin, and that the interaction causing the decay conserves $T$. Following an approach similar to that of Gien,[8] we derive the unitarity bound on the phase of the isospin mixing amplitude $\varphi_\epsilon$. The relation in Eq. (21) can be rewritten as

$$-2\mathrm{Im}\epsilon|E_0-E_1^*| = 2\pi \sum_{\alpha(T=1)} \epsilon e^{-i\eta}|\langle\Phi_1|V|\Phi_\alpha^+\rangle|^2 - \sum_{\alpha(T=0)} \epsilon^* e^{-i\eta}|\langle\Phi_0|V|\Phi_\alpha^+\rangle|^2 + (1+|\epsilon|^2)^{1/2}(\Gamma_\gamma^1\Gamma_\gamma^0)^{1/2}e^{i\varphi_\gamma}\,. \tag{35}$$

The phase $\eta$ in Eq. (35) is defined from $E_0-E_1^* = |E_0-E_1^*|e^{i\eta}$, $\tan\eta = \frac{1}{2}(\Gamma_1+\Gamma_0)/(E_1^r-E_0^r)$; and the phase $\varphi_\gamma$ comes from the electromagnetic decay matrix elements of the levels 1 and 0 with the electromagnetic decay widths $\Gamma_\gamma^1$ and $\Gamma_\gamma^0$, respectively. The following equation is derived by taking the imaginary part of Eq. (35):

$$2\pi|\epsilon|\mathrm{Sin}(\varphi_\epsilon-\eta)\sum_{\alpha(T=1)}|\langle\Phi_1|V|\Phi_\alpha^+\rangle|^2 + (1+|\epsilon|^2)^{1/2}(\Gamma_\gamma^1\Gamma_\gamma^0)^{1/2}\mathrm{Sin}(\varphi_\gamma-\eta) + 2\pi|\epsilon|\mathrm{Sin}(\varphi_\epsilon+\eta)\sum_{\alpha(T=0)}|\langle\Phi_0|V|\Phi_\alpha^+\rangle|^2 = 0\,. \tag{36}$$

The relations between the total widths are obtained from Eq. (20):

$$(\Gamma_1-\Gamma_0-\Gamma_\gamma^1+\Gamma_\gamma^0) = 2\pi\left(\frac{1-|\epsilon|^2}{1+|\epsilon|^2}\right)\left(\sum_{\alpha(T=1)}|\langle\Phi_1|V|\Phi_\alpha^+\rangle|^2 - \sum_{\alpha(T=0)}|\langle\Phi_0|V|\Phi_\alpha^+\rangle|^2\right), \tag{37}$$

$$(\Gamma_1+\Gamma_0-\Gamma_\gamma^1-\Gamma_\gamma^0) = 2\pi\left(\sum_{\alpha(T=1)}|\langle\Phi_1|V|\Phi_\alpha^+\rangle|^2 + \sum_{\alpha(T=0)}|\langle\Phi_0|V|\Phi_\alpha^+\rangle|^2\right). \tag{38}$$

Substituting Eq. (38) into Eq. (36), one has

$$\mathrm{Sin}\,\varphi_\epsilon\mathrm{Cos}\,\eta(\Gamma_1+\Gamma_0-\Gamma_\gamma^1-\Gamma_\gamma^0) - \mathrm{Cos}\,\varphi_\epsilon\mathrm{Sin}\,\eta(\Gamma_1-\Gamma_0-\Gamma_\gamma^1+\Gamma_\gamma^0)\left(\frac{1+|\epsilon|^2}{1-|\epsilon|^2}\right) = (1+|\epsilon|^2)^{1/2}(\Gamma_\gamma^1\Gamma_\gamma^0)^{1/2}\mathrm{Sin}(\eta-\varphi_\gamma)/|\epsilon|\,. \tag{39}$$

Let us define the phase $\eta'$ and the magnitude $R$ by

$$\tan\eta' = \frac{(1+|\epsilon|^2)}{(1-|\epsilon|^2)}\frac{(\Gamma_1-\Gamma_0-\Gamma_\gamma^1+\Gamma_\gamma^0)}{(\Gamma_1+\Gamma_0+\Gamma_\gamma^1+\Gamma_\gamma^0)}\tan\eta$$

and

$$R^2 = \mathrm{Sin}^2\eta(\Gamma_1-\Gamma_0-\Gamma_\gamma^1+\Gamma_\gamma^0)^2\left(\frac{1+|\epsilon|^2}{1-|\epsilon|^2}\right)^2 + \mathrm{Cos}^2\eta(\Gamma_1+\Gamma_0-\Gamma_\gamma^1-\Gamma_\gamma^0)^2\,. \tag{40}$$

The following inequality can easily be derived from Eqs. (39) and (40):

$$|\mathrm{Sin}(\varphi_\epsilon-\eta')| \leq (1+|\epsilon|^2)^{1/2}(\Gamma_\gamma^1\Gamma_\gamma^0)^{1/2}/R|\epsilon|\,. \tag{41}$$

For the known values of the partial widths and resonance positions, Eq. (41) limits the values of $\varphi_\epsilon$. It is interesting to note that when the $\Gamma_\gamma^i$ are very small as compared with the total widths $\Gamma_i$, one obtains from Eqs. (40) and (41) the expression for $\tan\varphi_\epsilon$ which is the same as that derived in Eq. (31).

In principle all the terms appearing in the scattering amplitude expression in Eq. (19) are measurable. The slowly varying direct part of $T_{fi}$ is usually determined from the elastic scattering data off the resonance using the optical model potential. This is then used in the analysis of other reactions involving the same composite particles. The extraction of the resonance amplitudes of $T_{fi}$ requires knowledge of the resonance energies and their total widths. The complete resonance amplitude cannot be determined from this information alone. The measurement of the partial decay widths gives the magnitude of the nuclear matrix elements occurring in the resonance part of $T_{fi}$ in Eq. (19). The determination of the phases of these matrix elements requires additional data that depends on the interference of these resonance amplitudes. In a special case of mixing of two resonances of isospins $T=1$ and $T=0$, the number of measurable parameters in the theory depends on the number of open channels. If only one channel $\alpha$ of $T=0$ is open besides the elastic channel, the measurable resonance parameters are $E_0^r$, $\Gamma_0$, $E_1^r$, $\Gamma_1$, the two partial decay widths of the $T=0$ channel, and the phases of the nuclear matrix elements. Using this information, the complex mixing amplitude is calculated from Eq. (31) and $|\epsilon|^2 = \Gamma_\alpha^1/\Gamma_\alpha^0$. From the known value of $\epsilon$, the Coulomb matrix element can be calculated from Eq. (33). The nuclear matrix elements corresponding to $T=1$ and $T=0$ are determined from Eq. (32).

## III. CONCLUSION

The problem of isospin mixing in nuclear resonance levels is formulated using Feshbach's theory of nuclear reactions. Some of our results are identical with those derived by Shanley using the well-known results in high energy physics. The Bell-Steinberger sum rule and other relations derived from unitarity of the $S$ matrix in high energy physics follow from Feshbach's theory. The nuclear matrix elements occurring in the reaction model are expressed in terms of the measured quantities. These relations would be useful in the interpretation of the experimental data. In many cases the $\gamma$ widths of the resonance levels are of the same order of magnitude as the isospin forbidden $\alpha$-decay widths or the particle emission widths. In the above formulation (as well as that by Shanley) the effect of $\gamma$-decay channels is not included. In some cases different $T$ channels are also open. The above formulas cannot be used directly to analyse such data. Using the unitarity relations, the limit on the value of the phase angle $\varphi_\epsilon$ is obtained in terms of the resonance parameters.

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