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Exchange effects in the theory of radioactive decay*

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A nuclear reaction theory derivation of the decay width for radioactive decay by particle emission is given. Particular attention is devoted to exchange symmetry. The result is similar in form to that derived by Fliessbach using time dependent perturbation theory except that in our expression the wave function for the emitted particle is the solution of a Schrödinger equation, whereas Fliessbach finds that this function must be calculated from a Hill-Wheeler equation.

[RADIOACTIVITY Width for radioactive decay by particle emission which includes effects of exchange symmetry is derived.]

I. INTRODUCTION

It has recently been pointed out by Fliessbach[1] that the effects of exchange symmetry have not been properly treated in earlier analyses[2] of radioactive particle decay. Fliessbach seeks to remedy this failing by a modification of the usual time dependent perturbation theory approach.

In this paper we present an alternative treatment of the problem based on nuclear reaction theory. We use the "coupled equations formalism" (CEF)[3-6] for multipartition nuclear reactions which has been developed recently. In this formalism the consequences of exchange symmetry may be made manifest in a simple way by the introduction of exchange operators into the equations of motion.[7]

To apply this reaction formalism to the calculation of the mean life of a metastable nucleus for particle emission, we calculate the width of the scattering resonance that occurs when the emitted particle collides with the daughter nucleus. For the purpose of relating the width of the scattering resonance to the wave functions of the parent and daughter nuclei and to the interaction between the emitted particle and the daughter nucleus we use the Feshbach projection operator approach[8] applied to the antisymmetrized CEF equations of motion.[9]

The result we find is similar to that derived by Fliessbach. The decay width is the square of the matrix element of the residual interaction between the emitted particle and the daughter nucleus. This matrix element is calculated with respect to the initial state parent nucleus wave function and the final state emitted particle plus daughter nucleus wave function. The difference between our result and that of Fliessbach consists in the fact that in our result the relative motion wave function in the final state is a solution of a Schrödinger equation with a nonlocal, nonsymmetric exchange potential interaction. In Fliessbach's result this wave function is a solution of the more complex Hill-Wheeler equation.

In Sec. II we present the antisymmetrized coupled equations formalism (ACEF). The Feshbach projection operator method (FPOM) is applied to these equations in Sec. III. From the resulting equations an expression for the width of the scattering resonance associated with a particular compound nucleus state is found. This is done in Sec. IV. In Sec. V a simplified expression for the decay width is derived wherein the transition matrix elements are converted to surface integrals. Our formulas are illustrated in Sec. VI where they are applied to a simple model for radioactive decay.

II. ANTISYMMETRIZED COUPLED EQUATIONS FORMALISM

Consider a system of N nucleons. For each partition $\alpha,\beta,\ldots$ of the nucleons into two sets or clusters there is a decomposition of the Hamiltonian into two terms:

$$H = H_\alpha + V_\alpha = H_\beta + V_\beta = \cdots . \tag{1}$$

H_α contains the kinetic energy and the intracluster interactions, while V_α is the sum of the intercluster interactions for partition α.

The transition operator for scattering from a partition β channel to a partition α channel is

$$T_{\alpha\beta} = V_\alpha + V_\alpha \mathcal{G} V_\beta, \tag{2}$$

where

$$\mathcal{G} = (E - H + i\epsilon)^{-1} \tag{3}$$

is the system Green's function operator. The partition γ Green's function operator is defined to be

$$G_\gamma = (E - H_\gamma + i\epsilon)^{-1} = (\mathcal{G}^{-1} + V_\gamma)^{-1}. \tag{4}$$

From the above equations one can derive the Kouri-Levin[4] CEF

$$T_{\alpha\beta} = V_\alpha \sum_{\gamma=1}^{\mathfrak{N}} W_{\alpha\gamma} G_\gamma (G_\beta^{-1} + T_{\gamma\beta}), \tag{5}$$

where $\mathfrak{N}$ is the total number of (two-cluster) partitions possible for N (distinguishable) nucleons and W is a numerical $\mathfrak{N}\times\mathfrak{N}$ matrix such that

$$\sum_{\gamma=1}^{\mathfrak{N}} W_{\gamma\beta} = 1. \tag{6}$$

The channel coupling array W can be chosen in such a way as to make the kernel of the coupled equations shown in Eq. (5) connected.[5]

In the case where the nucleons are indistinguishable the same transition operators $T_{\alpha\beta}$ cannot be used to calculate transition probabilities. The antisymmetry of the wave functions and the physical indistinguishability of many of the partitions leads to the prescription[10] that $T_{\alpha\beta}$ be replaced by the combination

$$\tilde{T}_{\alpha\beta} = N_\alpha^{-1/2} N_\beta^{1/2} \sum_{n=1}^{N_\alpha} (-1)^{\sigma_\alpha(n)} P_\alpha(n)^+ T_{\alpha_n\beta}, \tag{7}$$

where

$$N_\alpha = \frac{N!}{n_\alpha!(N-n_\alpha)!}(1 - \tfrac{1}{2}\delta_{n_\alpha N/2}), \tag{8}$$

n_α is the number of nucleons in one of the two partition α clusters, α_n is one of the N_α partitions physically indistinguishable from partition α, $P_\alpha(n)$ is the permutation operator that transforms partition α into partition α_n, and $\sigma_\alpha(n)$ is the number of nucleon pair transpositions contained in $P_\alpha(n)$.

It is possible to find a set of coupled integral equations for the antisymmetrized transition operators.[7] These read

$$\tilde{T}_{\alpha\beta} = V_\alpha A_\alpha^+ N_\alpha^{1/2} \sum_{\gamma=1}^{\mathfrak{N}'} W_{\alpha\gamma} N_\gamma^{-1/2} G_\gamma \times (A_\gamma^+ N_\gamma^{1/2} G_\beta^{-1} N_\beta^{1/2} + \tilde{T}_{\gamma\beta}), \tag{9}$$

where A_α is the partition α antisymmetrizer

$$A_\alpha = N_\alpha^{-1} \sum_{n=1}^{N_\alpha} (-1)^{\sigma_\alpha(n)} P_\alpha(n). \tag{10}$$

Equation (9) constitutes the ACEF.

Equation (9) for the ACEF differs from Eq. (5) for the CEF in a very important way which is hidden by the notation. In Eq. (7) the partitions α, β, γ, ... that are referenced are an $\mathfrak{N}'$ member subset of the entire set of all $\mathfrak{N}$ possible two-cluster partitions of N distinguishable particles. The partitions in the $\mathfrak{N}'$ member subset are such that no one partition can be transformed into any other by the exchange of pairs of nucleons. Thus, for each member α of the subset there will be $N_\alpha - 1$ other members of the entire set which are excluded from the subset.

III. APPLICATION OF THE FESHBACH PROJECTION OPERATOR METHOD

By means of the ACEF we have arrived at a set of coupled integral equations for the transition operators in which exchange effects have been made manifest. For all but the simplest of systems these equations are too complex to solve in full generality by the methods presently available. Thus it is necessary to make some simplifying assumptions. The FPOM is a convenient framework for formulating approximations which incorporate into the formalism the two types of processes we know to be of major importance in many reactions: direct interactions between two-body cluster configurations and compound nucleus formation. The assumption that these two types of processes are the only ones that occur leads to a relatively simple set of equations when the FPOM is applied to the ACEF.

To apply the FPOM to the ACEF we start by writing Eq. (9) as a matrix equation in partition space:

$$\tilde{T} = VA^+ N^{1/2} W N^{-1/2} G (A^+ N^{1/2} \mathfrak{u} G^{-1} N^{1/2} + \tilde{T}). \tag{11}$$

$\mathfrak{u}$ is a matrix in which every element is the identity:

$$\mathfrak{u}_{\alpha\beta} = 1. \tag{12}$$

Let us next define

$$\tilde{Y} = \tilde{V} + \tilde{V} G \tilde{Y}, \tag{13}$$

where

$$\tilde{V} = VA^+ N^{1/2} W N^{-1/2}, \tag{14}$$

so that

$$\tilde{T} = \tilde{Y} G A^+ N^{1/2} \mathfrak{u} G^{-1} N^{1/2}. \tag{15}$$

Now define a new set of operators $\bar{Y}_{\alpha\beta}$ by the modified coupled equations

$$\bar{Y}_{\alpha\beta} = \tilde{V}_{\alpha\beta} + \sum_{\gamma=1}^{\mathfrak{N}'} \tilde{V}_{\alpha\gamma} G_\gamma P_\gamma \bar{Y}_{\gamma\beta}, \tag{16}$$

which differ from the coupled equations for $\tilde{Y}$ displayed in Eq. (13) by the presence of the projection operator P_γ which truncates the channel state expansion for G_γ.

The partition Green's function G_γ has an expansion in terms of the partition γ channel states

$$G_\gamma = \sum_c \int d^3r \int d^3r' \big| \phi_{\gamma 1}^{(c)} \phi_{\gamma 2}^{(c)} \delta(\bar{r} - \bar{r}_\gamma) \big\rangle g_c(\bar{r}, \bar{r}') \times \big\langle \phi_{\gamma 1}^{(c)} \phi_{\gamma 2}^{(c)} \delta(\bar{r}' - \bar{r}_\gamma) \big|. \tag{17}$$

$\phi_{\gamma 1}^{(c)}$ and $\phi_{\gamma 2}^{(c)}$ are particular internal motion energy eigenstates for the two clusters $\gamma 1$ and $\gamma 2$ that comprise partition γ. $\bar{r}_\gamma$ is the displacement of the

center of mass of cluster $\gamma 1$ from that of $\gamma 2$. The sum on c is over all possible pairs of internal motion energy eigenstates $\phi_{\gamma 1}^{(c)}\phi_{\gamma 2}^{(c)}$. The effect of the projection operator P_γ is to truncate this sum to just a few terms, namely those corresponding to open scattering channels.

The transition operators $\bar{Y}_{\alpha\beta}$ which are the solutions of the modified coupled equations, Eqs. (16), are a cluster model approximation to the exact transition operators $\tilde{Y}_{\alpha\beta}$ which are solutions of Eq. (13). The $\bar{Y}_{\alpha\beta}$ are similar to the transition operators that would result from a coupled reaction channel calculation, or a resonating group method calculation, or a generator coordinate method calculation.

In the spirit of the Feshbach formalism we think of the scattering reaction problem as being solved in two steps. First, Eq. (16) is solved to give $\bar{Y}$, which reflects optical potential elastic scattering and direct interaction nonelastic scattering. Then, $\bar{Y}$ is used as input in equations for $\tilde{Y}$ in which compound nucleus effects can be included. The equations for $\tilde{Y}$ are found by using Eq. (16) to eliminate $\tilde{V}$ from Eq. (13):

$$\tilde{V} = \bar{Y}(1 + GP\bar{Y})^{-1}, \tag{18}$$

$$\begin{aligned}\tilde{Y} &= \bar{Y}(1 + GP\bar{Y})^{-1}(1 + G\tilde{Y}) \\ &= (1 + \bar{Y}GP)^{-1}\bar{Y}(1 + G\tilde{Y}) \\ &= \bar{Y}(1 + G\tilde{Y}) - \bar{Y}GP\tilde{Y} \\ &= \bar{Y} + \bar{Y}GQ\tilde{Y},\end{aligned} \tag{19}$$

where

$$Q_{\alpha\beta} = (1 - P)_{\alpha\beta} = \delta_{\alpha\beta}(1 - P_\beta). \tag{20}$$

Finally, substituting the formal solution of Eq. (19) into Eq. (15) gives

$$\tilde{T} = [\bar{Y} + \bar{Y}(G^{-1} - Q\bar{Y})^{-1}QY]\, GA^+N^{1/2}\mathfrak{u}G^{-1}N^{1/2}. \tag{21}$$

At this point one may approximate $(G^{-1} - Q\bar{Y})^{-1}$ by inverting $G^{-1} - Q\bar{Y}$ in the space spanned by the configurations that are important in the makeup of the compound nucleus states in the range of excitation being considered. This then supplies the approximate compound nucleus formation contribution to $\tilde{T}$ to supplement the direct interaction contribution arising from the first term on the right-hand side of Eq. (21).

IV. MEAN LIFE FOR RADIOACTIVE DECAY BY PARTICLE EMISSION

We seek an expression for the mean life of a parent nucleus (P) which decays into a daughter nucleus (D) and an emitted particle (E). Our approach is to consider the scattering of E by D. The mean life τ of P is equal to $\hbar$ divided by the width Γ of the resonance in $E + D$ scattering caused by the compound nucleus configuration P. Equation (21) will be used to estimate the width of the resonance.

It will be assumed that at the energy in question the only open channel is the $D + E$ elastic scattering channel. This means that in Eq. (21) only one partition is involved, and the matrix equation becomes an algebraic operator equation with $W = 1$ and $\mathfrak{u} = 1$:

$$\tilde{T} = \bar{Y} + \bar{Y}(E - H_{ED} - Q_{ED}\bar{Y})^{-1}Q_{ED}\bar{Y}\; G_{ED}^{(+)}A_{ED}^{+}G_{ED}^{(+)-1}N_{ED}. \tag{22}$$

The single partition form of Eq. (16) is

$$\bar{Y} = V_{ED}A_{ED}^{+}(1 + G_{ED}^{(+)}P_{ED}\bar{Y}). \tag{23}$$

In these equations V_{ED} is the residual interaction between E and D, $H_{ED} = H - V_{ED}$, $G_{ED}^{(\pm)} = (E - H_{ED} \pm i\epsilon)^{-1}$, A_{ED} is the sum of exchange operators that antisymmetrizes a product of an antisymmetric E internal motion wave function with an antisymmetric D internal motion wave function and an $E + D$ relative motion function. In addition,

$$N_{ED} = (N_E + N_D)!/N_E!N_D! = N_P!/N_E!N_D!. \tag{24}$$

P_{ED} is the projection operator that projects onto the elastic $E + D$ channel, and $Q_{ED} = 1 - P_{ED}$.

Having made the single channel assumption for $\tilde{T}$ and $\bar{Y}$ we next make the single level approximation for $(E - H_{ED} - Q_{ED}\bar{Y})^{-1}$ in Eq. (22). We suppose that we have available a good approximation Φ_P for the wave function of the parent nucleus. Then in the immediate vicinity of the scattering resonance due to the formation of P as a compound nucleus in $E + D$ scattering we can set

$$(E - H_{ED} - Q_{ED}\bar{Y})^{-1}Q_{ED} \approx \frac{Q_{ED}|\Phi_P\rangle\langle\Phi_P|Q_{ED}}{\langle\Phi_P|Q_{ED}(E - H_{ED} - \bar{Y})Q_{ED}|\Phi_P\rangle}. \tag{25}$$

The denominator on the right-hand side of Eq. (25) will have a Breit-Wigner form

$$\frac{\langle\Phi_P|Q_{ED}(E - H_{ED} - \bar{Y})Q_{ED}|\Phi_P\rangle}{\langle\Phi_P|Q_{ED}|\Phi_P\rangle} = E - E_0 - \Delta + \tfrac{1}{2}i\Gamma, \tag{26}$$

where

$$E_0 = \langle\Phi_P|Q_{ED}HQ_{ED}|\Phi_P\rangle/\langle\Phi_P|Q_{ED}|\Phi_P\rangle. \tag{27}$$

That is to say, at energies E near E_0 the level shift Δ and the width Γ are independent of small changes in E. Our task is to find Γ.

Combining Eqs. (26) and (27) gives

$$\Delta - \tfrac{1}{2}i\Gamma = \langle\Phi_P|Q_{ED}(\bar{Y} - V_{ED})Q_{ED}|\Phi_P\rangle/\langle\Phi_P|Q_{ED}|\Phi_P\rangle. \tag{28}$$

To develop this expression we introduce the scattering wave functions ψ and χ associated with the transition operator $\bar{Y}$:

$$\bar{Y}|\phi\rangle = V_{ED}A^{+}_{ED}|\psi\rangle, \tag{29}$$

$$\bar{Y}^{+}|\phi\rangle = A_{ED}V^{+}_{ED}|\chi\rangle, \tag{30}$$

$$(E - H_{ED})|\phi\rangle = 0. \tag{31}$$

The integral equations for these functions follow from that for $\bar{Y}$, Eq. (23):

$$|\psi\rangle = |\phi\rangle + G^{(+)}_{ED}P_{ED}V_{ED}A^{+}_{ED}|\psi\rangle, \tag{32}$$

$$|\chi\rangle = |\phi\rangle + G^{(-)}_{ED}P_{ED}A_{ED}V^{+}_{ED}|\chi\rangle. \tag{33}$$

Operating on these equations with $G^{(\pm)-1}_{ED}$ then gives the differential equations

$$(E - H_{ED})|\psi\rangle = P_{ED}V_{ED}A^{+}_{ED}|\psi\rangle, \tag{34}$$

$$(E - H_{ED})|\chi\rangle = P_{ED}A_{ED}V^{+}_{ED}|\chi\rangle. \tag{35}$$

Since the interaction potential in the differential equation for ψ is not Hermitian, the solutions of this equation do not form an orthogonal set. It is for this reason that the adjoint function χ is introduced. We can see that the ψ and the χ are biorthogonal sets. Consider

$$\langle\chi_{E'}|E - H_{ED}|\psi_E\rangle - \langle\chi_{E'}|E' - H^{+}_{ED}|\psi_E\rangle$$
$$= \langle\chi_{E'}|P_{ED}V_{ED}A^{+}_{ED} - V_{ED}A^{+}_{ED}P_{ED}|\psi_E\rangle = 0. \tag{36}$$

This follows from the differential equations and the fact that

$$[P_{ED}, H_{ED}] = 0 \tag{37}$$

implies

$$|\psi\rangle \equiv P_{ED}|\psi\rangle = |\Phi_E\Phi_D\psi(\bar{k}, \bar{r}_{ED})\rangle\rangle, \tag{38}$$

$$|\chi\rangle \equiv P_{ED}|\chi\rangle = |\Phi_E\Phi_D\chi(\bar{k}, \bar{r}_{ED})\rangle\rangle. \tag{39}$$

The quantities Φ_E and Φ_D are the internal motion energy eigenstates of particles E and D with eigenvalues ϵ_E and ϵ_D. The eigenvalue E of Eqs. (34) and (35) is then related to k by

$$E = \frac{\hbar^2k^2}{2\mu_{ED}} + \epsilon_E + \epsilon_D. \tag{40}$$

From Eq. (36) we get

$$(E - E')\langle\chi_{E'}|\psi_E\rangle = \langle\chi_{E'}|H_{ED} - H^{+}_{ED}|\psi_E\rangle = 0. \tag{41}$$

The hermiticity of H_{ED} used to get this result is a consequence of the fact that χ is constrained by asymptotic boundary conditions that are the time reverse of those constraining ψ. This is manifest in Eqs. (32) and (33).

Equation (41) expresses the biorthogonality of ψ and χ, namely that $\langle\chi_{E'}|\psi_E\rangle = 0$ if $E' \neq E$. We can choose the normalization of these functions such that

$$\langle\chi_{E'}|\psi_E\rangle = \int d^3r_{ED}\chi(\bar{k}', \bar{r}_{ED})^*\psi(\bar{k}, \bar{r}_{ED})$$
$$= (2\pi)^3\delta(\bar{k}' - \bar{k}). \tag{42}$$

From the integral equation for $\bar{Y}$, Eq. (23), we get the formal solution

$$\bar{Y} = V_{ED}A^{+}_{ED} + V_{ED}A^{+}_{ED}\mathcal{G}_{ED}V_{ED}A^{+}_{ED}, \tag{43}$$

$$\mathcal{G}_{ED} = [G^{(+)-1}_{ED} - P_{ED}V_{ED}A^{+}_{ED}]^{-1}P_{ED}. \tag{44}$$

The Green's function operator $\mathcal{G}_{ED}$ can be expressed in terms of the wave functions ψ and χ:

$$\mathcal{G}_{ED} = \int \frac{d^3k'}{(2\pi)^3}|\Phi_E\Phi_D\psi(\bar{k}', \bar{r}_{ED})\rangle\rangle \frac{2\mu_{ED}}{\hbar^2(k^2 - k'^2 + i\epsilon)}\langle\langle\Phi_E\Phi_D\chi(\bar{k}', \bar{r}_{ED})|$$
$$= \frac{\mu_{ED}}{\hbar^2(2\pi)^3}\int_0^\infty d(k'^2)k' \int d\hat{k}'|\Phi_E\Phi_D\psi(\bar{k}', r_{ED})\rangle\rangle \left\{\frac{\mathcal{P}}{k^2 - k'^2} - i\pi\delta(k^2 - k'^2)\right\}\langle\langle\Phi_E\Phi_D\chi(\bar{k}', \bar{r}_{ED})|. \tag{45}$$

Now combining Eqs. (28), (43), and (45) gives

$$\Delta = \frac{\langle\Phi_P|Q_{ED}V_{ED}(A^{+}_{ED} - 1)Q_{ED}|\Phi_P\rangle}{\langle\Phi_P|Q_{ED}|\Phi_P\rangle}$$
$$+ \frac{\mu_{ED}}{(2\pi)^3\hbar^2}\int_0^\infty d(k'^2)k' \int d\hat{k}'\langle\Phi_P|Q_{ED}V_{ED}A^{+}_{ED}|\Phi_E\Phi_D\psi(\bar{k}', \bar{r}_{ED})\rangle\rangle \frac{\mathcal{P}}{k^2 - k'^2}\frac{\langle\langle\Phi_E\Phi_D\chi(\bar{k}', \bar{r}_{ED})|V_{ED}A^{+}_{ED}Q_{ED}|\Phi_P\rangle}{\langle\Phi_P|Q_{ED}|\Phi_P\rangle}, \tag{46}$$

$$\Gamma = \frac{\mu_{ED}k}{(2\pi)^2\hbar^2}\int d\hat{k}\langle\Phi_P|Q_{ED}V_{ED}A^{+}_{ED}|\Phi_E\Phi_D\psi(\bar{k}, \bar{r}_{ED})\rangle\rangle \frac{\langle\langle\Phi_E\Phi_D\chi(\bar{k}, \bar{r}_{ED})|V_{ED}A^{+}_{ED}Q_{ED}|\Phi_P\rangle}{\langle\Phi_P|Q_{ED}|\Phi_P\rangle}. \tag{47}$$

Equation (47), then, is our expression for the decay width for the $P \to E + D$ decay in the one-channel, one-level approximation.

The consequences of exchange symmetry for the expression for the decay width are thus (a) the wave function Φ_P for the parent nucleus is antisymmetric, (b) the wave functions Φ_E and Φ_D for the internal degrees of freedom of the decay products are antisymmetric, and (c) the exchange operator A^{+}, defined in Eq. (10), is present.

Our result is rather similar to that derived by Fliessbach[1] from time dependent perturbation theory. In our notation his expression has the following form

$$\Gamma = \frac{\mu_{ED}k}{\hbar^2(2\pi)^2}\int d\hat{k}|\langle\Phi_P|(H-E)A_{ED}|\Phi_E\Phi_D\,\mathfrak{U}(\bar{k},\bar{r}_{ED})\rangle|^2$$

$$= \frac{\mu_{ED}k}{\hbar^2(2\pi)^2}\int d\hat{k}|\langle\Phi_P|H-H^+|\Phi_E\Phi_D\mathfrak{U}(\bar{k},\bar{r}_{ED})\rangle|^2, \tag{48a}$$

where $\mathfrak{U}$ is the solution of the Hill-Wheeler equation

$$\langle\Phi_E\Phi_D\delta(\bar{r}-\bar{r}_{ED})|A_{ED}^+(H-E)A_{ED}|\Phi_E\Phi_D\mathfrak{U}(\bar{k},\bar{r}_{ED})\rangle = 0, \tag{48b}$$

In practice one does not solve this equation but merely takes $\mathfrak{U}$ to be the solution of a two-body Schrödinger with a phenomenological local E,D potential energy. Fliessbach points out that this is inconsistent with $\mathfrak{U}$ being the solution of a Hill-Wheeler equation. He suggests that one should instead approximate the quantity

$$\Omega(\bar{r}) = \int d^3s N^{1/2}(\bar{r},\bar{s})\mathfrak{U}(\bar{s}) \tag{49a}$$

by the solution to a two-body Schrödinger equation, where $N^{1/2}$ is defined by

$$\int d^3s N^{1/2}(\bar{r},\bar{s})N^{1/2}(\bar{s},\bar{t}) = N(\bar{r},\bar{t}), \tag{49b}$$

$$N(\bar{r},\bar{t}) = \langle\delta(\bar{r}-\bar{r}_{ED})\Phi_E\Phi_D|A_{ED}|\delta(\bar{t}-\bar{r}_{ED})\Phi_E\Phi_D\rangle N_{ED}. \tag{49c}$$

Thus, in this approach one must get $\mathfrak{U}$ by solving the difficult Hill-Wheeler equation or by evaluating the operator $N^{-1/2}$, which is also difficult, and using it in conjunction with a solution of a two-body Schrödinger equation.

In our approach the functions ψ and χ appear in place of $\mathfrak{U}$. These functions are solutions of Eqs. (34) and (35) which are two-body Schrödinger equations with nonlocal nonsymmetric exchange potential interactions.

V. TRANSFORMATION TO SURFACE INTEGRALS

Our expression for the decay width, Eq. (47), involves matrix elements of the residual interaction $Q_{ED}V_{ED}A_{ED}^+$ between the emitted particle E and the daughter nucleus D. This expression can be simplified somewhat by transforming the matrix elements of $Q_{ED}V_{ED}A_{ED}^+$ to surface integrals.

Let us write Eq. (47) in the form

$$\Gamma = \Lambda\int d\hat{k}M(\bar{k})^*N(\bar{k}), \tag{50}$$

$$\Lambda = \mu_{ED}k/\{(2\pi\hbar)^2\langle\Phi_P|Q_{ED}|\Phi_P\rangle\}, \tag{51}$$

$$M(\bar{k}) = \langle\Phi_P|Q_{ED}A_{ED}V_{ED}^+|\Phi_E\Phi_D\chi(\bar{k},\bar{r}_{ED})\rangle, \tag{52}$$

$$N(\bar{k}) = \langle\Phi_P|Q_{ED}V_{ED}A_{ED}^+|\Phi_E\Phi_D\psi(\bar{k},\bar{r}_{ED})\rangle. \tag{53}$$

Now we make use of the differential equations for ψ and χ,

$$(E-H_{ED}-P_{ED}V_{ED}A_{ED}^+)|\Phi_E\Phi_D\psi(\bar{k},\bar{r}_{ED})\rangle = 0, \tag{54}$$

$$(E-H_{ED}-P_{ED}A_{ED}V_{ED}^+)|\Phi_E\Phi_D\chi(\bar{k},\bar{r}_{ED})\rangle = 0, \tag{55}$$

and the fact that Φ_P is an antisymmetric approximate eigenfunction of $H = H_{ED}+V_{ED}$ to simplify the matrix elements M and N:

$$M(\bar{k}) = \langle\Phi_P|V_{ED}^+ - P_{ED}A_{ED}V_{ED}^+|\Phi_E\Phi_D\chi(\bar{k},\bar{r}_{ED})\rangle$$
$$= \langle\Phi_P|(E-H_{ED}^+)-(E-H_{ED})|\Phi_E\Phi_D\chi(\bar{k},\bar{r}_{ED})\rangle$$
$$= \langle\Phi_P|H_{ED}-H_{ED}^+|\Phi_E\Phi_D\chi(\bar{k},\bar{r}_{ED})\rangle, \tag{56}$$

$$N(\bar{k}) = \langle\Phi_P|V_{ED}A_{ED}^+ - P_{ED}V_{ED}A_{ED}^+|\Phi_E\Phi_D\psi(\bar{k},\bar{r}_{ED})\rangle$$
$$= \langle\Phi_P|(E-H_{ED}^+)A_{ED}^+-(E-H_{ED})|\Phi_E\Phi_D\psi(\bar{k},\bar{r}_{ED})\rangle$$
$$= \langle\Phi_P|H_{ED}-H_{ED}^+A_{ED}^+|\Phi_E\Phi_D\psi(\bar{k},\bar{r}_{ED})\rangle. \tag{57}$$

Here we have used the fact that V_{ED} is Hermitian.

Now we observe that

$$H_{ED} = H_E + H_D - \frac{\hbar^2}{2\mu_{ED}}\nabla_{r_{ED}}{}^2, \tag{58}$$

and that H_E and H_D are Hermitian in these matrix elements. In consequence,

$$N(\bar{k}) = N_d(\bar{k}) + N_e(\bar{k}), \tag{59}$$

$$N_e(\bar{k}) = \langle\Phi_P|H_{ED}^+(1-A_{ED}^+)|\Phi_E\Phi_D\psi(\bar{k},\bar{r}_{ED})\rangle, \tag{60}$$

$$N_d(\bar{k}) = \frac{\hbar^2}{2\mu_{ED}}\langle\xi(\bar{r})|\vec{\nabla}_r{}^2 - \overleftarrow{\nabla}_r{}^2|\psi(\bar{k},\bar{r})\rangle, \tag{61}$$

$$M(\bar{k}) = \frac{\hbar^2}{2\mu_{ED}}\langle\xi(\bar{r})|\vec{\nabla}_r{}^2 - \overleftarrow{\nabla}_r{}^2|\chi(\bar{k},\bar{r})\rangle, \tag{62}$$

$$\xi(\bar{r}) = \langle\Phi_E\Phi_D\delta(\bar{r}-\bar{r}_{ED})|\Phi_P\rangle. \tag{63}$$

The term $N_e(\bar{k})$ is purely exchange contribution. It does not simplify, but $N_d(\bar{k})$ and $M(\bar{k})$ can be converted to surface integrals by Green's theorem:

$$N_d(\bar{k}) = \frac{\hbar^2R^2}{2\mu_{ED}}\int d\hat{R}\left[\psi(\bar{k},\bar{R})\frac{\partial}{\partial R}\xi(\bar{R})^* - \xi(\bar{R})^*\frac{\partial}{\partial R}\psi(\bar{k},\bar{R})\right], \tag{64}$$

$$M(\bar{k}) = \frac{\hbar^2R^2}{2\mu_{ED}}\int d\hat{R}\left[\chi(\bar{k},\bar{R})\frac{\partial}{\partial R}\xi(\bar{R})^* - \xi(\bar{R})^*\frac{\partial}{\partial R}\chi(\bar{k},\bar{R})\right].$$

The radius R must be greater than the range of V_{ED}.

Next let us make partial wave expansions of ψ, χ, and ξ. We assume for simplicity's sake that the particles E and D have zero spins:

$$\psi(\bar{k},\bar{R}) = 4\pi\sum i^L Y_L^M(\hat{k})^* Y_L^M(\hat{R}) f_L(k,R)/R, \tag{65}$$

$$\chi(k,R)=4\pi\sum i^{L}Y_{L}^{M}(\hat{k})^{*}Y_{L}^{M}(\hat{R})g_{L}(k,R)^{*}/R, \quad (66)$$

$$\xi(\bar{R})=\xi_{\lambda}(R)Y_{\lambda}^{\mu}(\hat{R})/R. \quad (67)$$

Substituting these expressions into Eq. (64) gives

$$N_{d}(\bar{k})=\frac{\hbar^{2}i^{\lambda}}{2\mu_{ED}}Y_{\lambda}^{\mu}(\hat{k})^{*}\left[f_{\lambda}(k,R)\frac{\partial}{\partial R}\xi_{\lambda}(R)-\xi_{\lambda}(R)\frac{\partial}{\partial R}f_{\lambda}(k,R)\right], \quad (68)$$

$$M(\bar{k})=\frac{\hbar^{2}i^{\lambda}}{2\mu_{ED}}Y_{\lambda}^{\mu}(\bar{k})^{*}\left[g_{\lambda}(k,R)^{*}\frac{\partial}{\partial R}\xi_{\lambda}(R)-\xi_{\lambda}(R)\frac{\partial}{\partial R}g_{\lambda}(k,R)^{*}\right]. \quad (69)$$

Substituting these expressions into Eq. (50) and neglecting the exchange term N_e gives

$$\Gamma=\frac{\hbar^{2}km_{\lambda}n_{\lambda}}{\mu_{ED}(4\pi)^{2}q}, \quad (70)$$

$$m_{\lambda}=g_{\lambda}(k,R)\frac{\partial}{\partial R}\xi_{\lambda}(R)-\xi_{\lambda}(R)\frac{\partial}{\partial R}g_{\lambda}(k,R), \quad (71)$$

$$n_{\lambda}=f_{\lambda}(k,R)\frac{\partial}{\partial R}\xi_{\lambda}(R)-\xi_{\lambda}(R)\frac{\partial}{\partial R}f_{\lambda}(k,R), \quad (72)$$

$$q=1-\int_{0}^{\infty}dr\,\xi_{\lambda}(r)^{2}. \quad (73)$$

This result is the same as that of Fliessbach[1] except that he has $q=1$ and $f_{\lambda}=g_{\lambda}=$ the solution of a Hill-Wheeler equation. Earlier treatments[2] had $q=1$ and $f_{\lambda}=g_{\lambda}=$ the solution of a Schrödinger equation with an optical potential interaction. For us f_{λ} and g_{λ} are solutions of adjoint Schrödinger equations with nonlocal exchange potential interactions.

To get this simple result we have neglected the exchange term N_e. The justification of this step is not clear at this point. However, N_e is found to vanish in the model calculation done in the next section.

VI. APPLICATION TO A SIMPLE MODEL

We will next consider a simple model which exhibits radioactive decay via nucleon emission. It also exhibits effects due to exchange symmetry. Thus it serves to illustrate some of the consequences of exchange symmetry for radioactive decay.

The model consists of two nucleons bound in s states to an infinite mass square-shaped potential well. The initial state Φ_P will consist of a $(2s,3s)$ configuration. Then, due to a small residual interaction V between the two nucleons, a transition occurs in which one nucleon descends to the $1s$ state and the other goes up into an unbound state. The process is shown schematically in Fig. 1.

The Hamiltonian for this model is

$$H=H_{0}+V, \quad (74)$$

$$H_{0}=-\frac{\hbar^{2}}{2\mu}\left(\frac{1}{r_{1}}\frac{\partial^{2}}{\partial r_{1}^{2}}r_{1}+\frac{1}{r_{2}}\frac{\partial^{2}}{\partial r_{2}^{2}}r_{2}\right)-\frac{\hbar^{2}}{2\mu}v^{2}[\Theta(R-r_{1})+\Theta(R-r_{2})], \quad (75)$$

$$V=-\frac{\hbar^{2}}{2\mu}w^{2}\Theta(R-r_{1})\Theta(R-r_{2}). \quad (76)$$

The residual interaction V is seen to be a three-body square well force. This choice leads to matrix elements that are easy to evaluate.

In this model the parent nucleus wave function is

$$\Phi_{P}=\frac{\phi_{2}(r_{1})\phi_{3}(r_{2})-\phi_{2}(r_{2})\phi_{3}(r_{1})}{r_{1}r_{2}\sqrt{2}\,(4\pi)}, \quad (77)$$

$$\phi_{n}(r)=A_{n}\sin(\lambda_{n}r), \quad r<R$$
$$=A_{n}\sin(\lambda_{n}R)\exp[-\tau_{n}(r-R)], \quad r>R, \quad (78)$$

$$A_{n}^{-2}=\frac{R}{2}-\frac{\sin 2\lambda_{n}R}{4\lambda_{n}}+\frac{\sin^{2}\lambda_{n}R}{2\tau_{n}}, \quad (79)$$

$$\tan\lambda_{n}R=\lambda_{n}/\tau_{n}, \quad (80)$$

$$\tau_{n}^{2}=v^{2}-\lambda_{n}^{2}. \quad (81)$$

The final state wave functions are

$$\Phi_{E}\Phi_{D}\psi(\bar{k},\bar{r})=\frac{\phi_{1}(r_{1})}{r_{1}\sqrt{4\pi}}\frac{\psi(r_{2})}{r_{2}\sqrt{4\pi}}, \quad (82)$$

$$\Phi_{E}\Phi_{D}\chi(\bar{k},\bar{r})=\frac{\phi_{1}(r_{1})}{r_{1}\sqrt{4\pi}}\frac{\chi(r_{2})}{r_{2}\sqrt{4\pi}}. \quad (83)$$

The differential equations for ψ and χ follow from Eqs. (34) and (35). For this case the equations for ψ and χ turn out to be identical because V is symmetric in r_1 and r_2. For ψ,

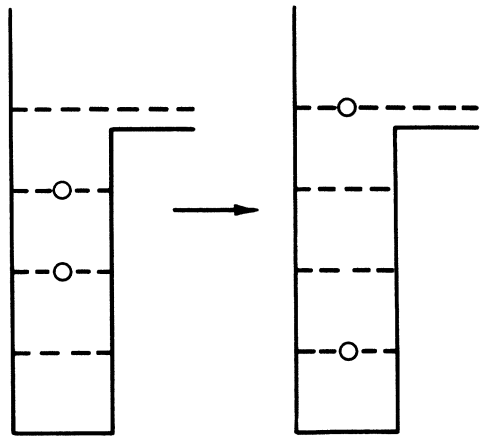

FIG. 1. Initial and final states of a simple model for radioactive decay.

$$\left\{k^2+\frac{d^2}{dr^2}+a^2\Theta(R-r)\right\}\psi(r)=b^2\Theta(R-r)\phi_1(r), \quad (84)$$

$$k^2=\frac{2\mu E}{\hbar^2}+\tau_1^2, \quad (85)$$

$$a^2=v^2+\frac{w^2}{2}\int_0^R dr\,\phi_1(r)^2, \quad (86)$$

$$b^2=\frac{w^2}{2}\int_0^R dr\,\phi_1(r)\psi(r). \quad (87)$$

The functions ψ and χ should be normalized so that for $r>R$,

$$\psi(r)=\chi(r)=\frac{1}{k}\frac{\sin(kr+\delta)}{\sqrt{4\pi}}. \quad (88)$$

Substitution of these expressions into Eq. (47) gives

$$\Gamma=\frac{\hbar^2 k}{2\mu(4\pi)^2}(c_{12}d_3-c_{13}d_2)^2, \quad (89)$$

$$c_{ij}=\int_0^R dr\,\phi_i(r)\phi_j(r), \quad (90)$$

$$d_j=w^2\int_0^R dr\,\phi_j(r)\psi(r). \quad (91)$$

Let us compare this with what is given by Eqs. (50), (68), and (69). Neglecting N_e we get

$$\Gamma=\frac{\hbar^2 k}{2\mu(4\pi)^2}(c_{12}\delta_3-c_{13}\delta_2)^2, \quad (92)$$

$$\delta_j=\psi(R)\frac{d}{dR}\phi_j(R)-\phi_j(R)\frac{d}{dR}\psi(R). \quad (93)$$

Equations (89) and (92) would agree if d_j were equal to δ_j.

Suppose we define $\hat{\psi}_j$:

$$\left\{-\tau_j^2+\frac{d^2}{dr^2}+[v^2+w^2]\Theta(R-r)\right\}\hat{\psi}_j=0. \quad (94)$$

Then by using Green's theorem we get

$$\hat{\psi}_j(R)\frac{d}{dR}\phi_j(R)-\phi_j(R)\frac{d}{dR}\hat{\psi}_j(R)=w^2\int_0^R dr\,\hat{\psi}_j(r)\phi_j(r). \quad (95)$$

Thus we see that $\delta_j=d_j$ would hold if Eq. (94) in some sense were a good approximation to Eq. (84).

Let us next examine the neglected factor N_e:

$$\begin{aligned}N_e&=\langle\Phi_P|H_{ED}^{+}(1-A_{ED}^{+})|\Phi_E\Phi_D\psi(\bar{k},\bar{r}_{ED})\rangle\\&=\left\langle\frac{\phi_2(r_1)\phi_3(r_2)-\phi_2(r_2)\phi_3(r_1)}{\sqrt{2}\,(4\pi)r_1r_2}\right|\left(-\frac{\hbar^2}{2\mu}\right)\left(\frac{1}{r_1}\frac{\partial^2}{\partial r_1^2}r_1+\frac{1}{r_2}\frac{\partial^2}{\partial r_2^2}r_2+v^2\Theta(R-r_1)+v^2\Theta(R-r_2)\right)\tfrac{1}{2}P_{12}\left|\frac{\phi_1(r_1)}{r_1\sqrt{4\pi}}\frac{\psi(r_2)}{r_2\sqrt{4\pi}}\right\rangle\\&=-\frac{\hbar^2}{4\mu}(\tau_2^2+\tau_3^2)\left\langle\frac{\phi_2(r_1)\phi_3(r_2)-\phi_2(r_2)\phi_3(r_1)}{\sqrt{2}\,(4\pi)r_1r_2}\right|\frac{\phi_1(r_2)}{r_2\sqrt{4\pi}}\frac{\psi(r_1)}{r_1\sqrt{4\pi}}\Big\rangle\\&=0.\end{aligned} \quad (96)$$

Here we have used the fact the Φ_P has been approximated by an eigenstate of H_{ED}, and we have used the orthogonality of the ϕ_j's. Thus N_e vanishes as a result of the approximation we have made for Φ_P.

One should not conclude that the vanishing of N_e implies the validity of Eq. (92). Equation (92) is equivalent to Eq. (89) when N_e vanishes only if Φ_P is an eigenfunction of H rather than of H_{ED}, as has been assumed for our model calculation.

Finally, it might be interesting to see what the Hill-Wheeler equation, Eq. (49), for the final state wave function would be in this model. The Hill-Wheeler final state wave function is

$$\Psi=\frac{1}{\sqrt{2}}\left[\frac{\phi_1(r_1)}{r_1\sqrt{4\pi}}\frac{\mathfrak{u}(r_2)}{r_2\sqrt{4\pi}}-\frac{\phi_1(r_2)}{r_2\sqrt{4\pi}}\frac{\mathfrak{u}(r_1)}{r_1\sqrt{4\pi}}\right], \quad (97)$$

where $\mathfrak{u}$ is a solution of

$$\left\{k^2+\frac{d^2}{dr^2}+\alpha^2\Theta(R-r)\right\}\mathfrak{u}(r)=\beta^2\Theta(R-r)\phi_1(r)+\gamma^2\phi_1(r), \quad (98)$$

$$\alpha^2=v^2+w^2\int_0^R dr\,\phi_1(r)^2, \quad (99)$$

$$\beta^2=w^2\int_0^R dr\,\phi_1(r)\mathfrak{u}(r), \quad (100)$$

$$\gamma^2=\int_0^\infty dr\,\phi_1(r)[k^2+d^2/dr^2+v^2\Theta(R-r)]\mathfrak{u}(r). \quad (101)$$

This model illustrates how the decay width is calculated from the wave function of the parent nucleus and the wave function for the final state. The wave function for the relative motion in the final state is the solution of a Schrödinger equation with a nonlocal exchange potential interaction. Although this equation, Eq. (84), resembles the Hill-Wheeler equation, Eq. (98), there is a vital difference between the two due to the presence of the term $\gamma^2\phi_1$ in Eq. (98). This term cannot be regarded as an interaction term because it does not depend on the interaction strength w and because it depends on the value of the wave function $\mathfrak{u}$ over all space rather than just the interaction region

$0<r<R$. The solution of the Hill-Wheeler equation lacks uniqueness inasmuch as $\mathfrak{u}+C\phi_1$ is a solution if $\mathfrak{u}$ is a solution, where C is an arbitrary constant. Equation (84) is not afflicted by the existence of such redundant solutions.

We see from our model that the surface integral version of the theory displayed in Eq. (70) does not necessarily agree with the more exact volume integral version shown in Eq. (47), even when the exchange term N_e [Eq. (60)] is negligible. We have noted that the two versions will agree if $\hat{\psi}_j$ of Eq. (94) is equal to ψ of Eq. (80) for $0<r<R$. This would indeed be the case if $v^2 \gg k^2$, τ_j^2, w^2, that is, if the square well potential binding the nucleons to the parent nucleus is very deep.

For comparison let us apply the Fliessbach[1] approach to this model. If we start with Eq. (48), convert it to a surface integral, and apply it to our model we find

$$\Gamma = \frac{\hbar^2 k}{4\pi^2 \mu_{ED}} \left| \left\langle \Phi_P \left| \vec{\nabla}_{r_{ED}}{}^2 - \vec{\nabla}_{r_{ED}}{}^2 \right| \Phi_E \Phi_D \frac{\mathfrak{u}(r_{ED})}{\sqrt{4\pi} r_{ED}} \right\rangle \right|^2$$
$$= \frac{\hbar^2 k}{2\mu(4\pi)^2} \left(\mathfrak{u}(R) \frac{d}{dR} G(R) - G(R) \frac{d}{dR} \mathfrak{u}(R) \right)^2 , \tag{102}$$

where $\mathfrak{u}(r)/r$ is the regular solution of the Hill-Wheeler equation Eq. (98) and $G(R)$ is defined by

$$G(R) = c_{21}\phi_3(R) - c_{31}\phi_2(R) . \tag{103}$$

This expression for the decay width is identical to Eq. (92) except that $\mathfrak{u}$ is a solution of Eq. (98) instead of Eq. (84).

Following Fliessbach we seek to replace the function $\mathfrak{u}/r\sqrt{4\pi}$ by Ω defined by Eq. (49a). For our model the operator N defined by Eq. (49c) is given by

$$N(\bar{r},\bar{r}') = \left\langle \frac{\delta(r_2 - r)}{r^2 4\pi} \frac{\phi_1(r_1)}{r_1\sqrt{4\pi}} \left| 1 - P_{12} \right| \frac{\delta(r_2 - r')}{r'^2 4\pi} \frac{\phi_1(r_1)}{r_1\sqrt{4\pi}} \right\rangle$$
$$= \delta(\bar{r} - \bar{r}') - \frac{\phi_1(r)}{r\sqrt{4\pi}} \frac{\phi_1(r')}{r'\sqrt{4\pi}} . \tag{104}$$

Thus we find

$$N = 1 - \Lambda_1 = \sum_{j \neq 1} \Lambda_j , \tag{105}$$

where Λ_j is the projection operator onto the jth square well eigenstate.

It is clear that the Hill-Wheeler equation is just the result of projecting the Schrödinger equation onto the portion of Hilbert space spanned by antisymmetric cluster states. Having assumed a particular cluster state solution in our model, N is the projector that enforces antisymmetrization.

Given the definitions

$$\omega(r) = \int dr' n^{1/2}(r,r')\mathfrak{u}(r') , \tag{106}$$

$$n(r,r') = \delta(r - r') - \phi_1(r)\phi_1(r') = N(\bar{r},\bar{r}')4\pi r r' \tag{107}$$

Fliessbach would replace Eq. (102) by

$$\Gamma = \frac{\hbar^2 k}{2\mu(4\pi)^2} \left(\omega(R) \frac{d}{dR} g(R) - g(R) \frac{d}{dR} \omega(R) \right) , \tag{108}$$

where

$$g(R) = \int dr\, n^{-1/2}(R,r) G(r) . \tag{109}$$

Equation (108) is an approximation to

$$\Gamma = \frac{\hbar^2 k}{2\mu(4\pi)^2} \int dr\, \omega(r) \left(n^{-1/2}(R,r) \frac{d}{dR} G(R) - G(R) \frac{d}{dR} n^{-1/2}(R,r) \right) . \tag{110}$$

Equation (110) results from using Eq. (106) to eliminate $\mathfrak{u}$ from Eq. (102).

Strictly speaking $n^{-1/2}$ does not exist since $n^{1/2} = n$ is a projection operator. One must agree to restrict $\mathfrak{u}$ and ω by a supplementary condition requiring them to be orthogonal to ϕ_1. In that case n becomes the identity operator and ω becomes identical with $\mathfrak{u}$. What has been gained by this exercise is the appearance of a supplementary condition to be used in conjunction with the Hill-Wheeler equation. This supplementary condition selects a particular solution $\mathfrak{u}$ from among the family of redundant solutions of the Hill-Wheeler equation. Approximating this solution by a solution to a two-body Schrödinger equation that fulfills the supplementary condition will no longer be inconsistent with the assumptions used to derive Eq. (48).

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