

S-matrix approach to hadronic atom level shifts

A. Deloff †

Institute for Nuclear Research, Warsaw, Poland

(Received 8 July 1975)

S-matrix perturbation theory accurate to all orders in the strong interaction potential is developed. The method presented is applied to the derivation of two important relations: (a) The complex level shift in a hadronic atom is expressed in terms of the logarithmic derivative of the regular wave function at the nuclear boundary; and (b) the complex level shift is related in a model independent way to the zero-energy hadron-nucleus scattering phase shift. It is shown that these two relations provide practical means for a fast and accurate evaluation of the complex level shifts and generalized scattering lengths.

[NUCLEAR REACTIONS S-matrix theory, hadronic atom complex level shift calculated and correlated with zero-energy phase shift.]

I. INTRODUCTION

The purpose of this paper is twofold; (i) we present a very simple method for calculating the level shifts ϵ and level broadening Γ in a hadronic atom, and (ii) we correlate the complex level shift with the zero-energy scattering phase shift in a model independent way (for a review of the experimental situation cf. Ref. 1). It is well known that in most cases conventional perturbation theory is inadequate for a hadronic atom problem and the calculated ϵ and Γ may disagree by a factor of 2 to 3 with the correct values, not to mention that ϵ may have the wrong sign. The hadron-nucleus strong interaction, which may be conveniently represented in a form of an equivalent complex potential, appreciably distorts the hadronic wave function only in a small distance of the order of the nuclear radius, but this very region turns out to be most important in the calculation of ϵ and Γ , so that perturbation theory fails badly. On the other hand, since the range of the distorting potential is small in comparison with the spread of the Coulomb wave function, ϵ and Γ are both much smaller than the Coulomb binding energy. Although this fact calls for an approximation scheme, owing to the long tail of the Coulomb potential and the presence of absorption in the strong interaction potential, all the approximate procedures known to us have appeared rather impractical, i.e., not easier than a direct method of solving the complex eigenvalue problem for the appropriate wave equation.

The idea of relating the hadronic atoms complex level shifts with the hadron-nucleus zero-energy scattering phase shifts has attracted many authors.²⁻⁷ Indeed, both the level shifts and the

phase shifts are correlated through the hadron-nucleus potential and either of these may be calculated once the potential is given. This procedure defines the desired functional dependence, but in practice, the derivation of the equivalent potential is the most difficult part of the theory of hadron-nucleus interaction. Another important question is whether the above-mentioned relationship could be formulated in a model independent way, i.e., so as to be insensitive to the form and the strength of the potential.

As emphasized in Ref. 8, this relation provides the unique opportunity of deducing the zero-energy phase shifts directly from the experimentally measured hadronic atom level shifts. For $l \geq 1$ this information is hardly accessible from the low energy scattering data, since for $k \rightarrow 0$ the s wave dominates the cross section. Another advantageous feature of the above-mentioned correlation is the theoretical possibility of describing the observed level shifts in terms of multiple scattering theory, without ever introducing the equivalent hadron-nucleus potential.

To realize the above program, however, one must make sure that the theory is adequate for very strong potentials which preclude perturbative treatment. Furthermore, the nonhermicity of the Hamiltonian and the presence of the long-range Coulomb potential introduce additional nontrivial complications. It will be shown below that the powerful S-matrix method provides an effective tool to cope with this situation.

In Sec. II we develop an S-matrix perturbation theory, somewhat similar to the methods applied about ten years ago to evaluate the SU(3) symmetry breaking effects. The binding energy is then sought as the pole of the S matrix. The binding-

energy shift and the zero-energy phase shift are expressed in terms of the logarithmic derivative of the regular wave function at the nuclear boundary. In Sec. III we derive the relation between the complex level shift and the zero-energy phase shift. This relation turns out to be model independent and practically holds for arbitrarily strong interactions. In Sec. IV we present our conclusions.

II. CALCULATION OF THE COMPLEX LEVEL SHIFT

We shall consider individually each quasibound state of a hadronic atom characterized by the quantum numbers (n, l) . The strong interaction gives rise to the shift of the total (relativistic) energy. We shall denote this shift by δE and define it as the difference

$$E = E - E_0 = -\epsilon - i\frac{1}{2}\Gamma,$$

where E and E_0 are the eigenvalues of the relativistic Hamiltonian with and without the strong interaction, respectively. Owing to the absorptive nature of the strong interaction, δE is complex and the imaginary part is related to the width Γ of the level in the usual way; the real parameter ϵ represents the shift of the binding energy.

We assume that the hadron-nucleus strong interaction potential $V(r)$ has the following properties: (a) $V(r)$ is less singular at the origin than r^{-2} ; (b) $V(r) = 0$ for $r \geq R$, where the range R is much smaller than the classical Coulomb radius; (c) $V(r)$ is regarded as a scalar in Minkowski space; (d) otherwise $V(r)$ may be a general nonlocal operator.⁹ With these assumptions the relevant Klein-Gordon equation may be written in the form

$$\psi''(k, r) + [k^2 - 2E\alpha Z/r - \lambda(\lambda + 1)/r^2 - 2\mu V(r)]\psi(k, r) = 0, \quad (1)$$

where $E = (k^2 + \mu^2)^{1/2}$, μ is the hadron-nucleus reduced mass, $(-\alpha)$ is the fine structure constant, and $\lambda = [(l + \frac{1}{2})^2 - \alpha^2 Z^2]^{1/2} - \frac{1}{2}$. Let $\varphi(k, r)$ and $f(k, r)$ be the regular and irregular solutions of Eq. (1), respectively, with their usual boundary conditions¹⁰

$$\lim_{r \rightarrow 0} [r^{-\lambda-1} \varphi(k, r)] = 1, \quad (2a)$$

$$\lim_{r \rightarrow \infty} \{\exp[-ikr + i(\alpha ZE/k) \ln(2kr)] f(k, r)\} = 1. \quad (2b)$$

We denote by $\varphi^C(k, r)$ and $f^C(k, r)$ the purely Coulomb wave functions. They become the solutions of Eq. (1) if $V(r)$ is switched off and they obey the boundary conditions (2). The Jost function is defined as the Wronskian¹⁰

$$F(k) = W[f(k, r), \varphi(k, r)] \quad (3)$$

or

$$F(k) = W[f^C(k, r), \varphi(k, r)]_{r \geq R},$$

because for $r \geq R$ one has $f(k, r) = f^C(k, r)$. Since the binding energy corresponds to a zero of the Jost function, we are looking for the solution of the equation $F(k) = 0$. Introducing a new function $H(k) \equiv -F(k)/[f^C(k, R)\varphi(k, R)]$, this equation in view of (3) can be written as

$$H(k) \equiv \varphi'(k, R)/\varphi(k, R) - f^{C'}(k, R)/f^C(k, R) = 0. \quad (4)$$

We are going to solve Eq. (4) by the familiar Newton-Raphson iterative procedure. This method refines the initial value of the root κ by setting $k = \kappa + \delta k$ and expanding $H(k)$ in a Taylor series around κ

$$H(k) = H(\kappa) + \delta k \dot{H}(\kappa) + \dots,$$

where the dot denotes differentiation with respect to k . Neglecting higher order terms, one has

$$\delta k = -H(\kappa)/\dot{H}(\kappa). \quad (5)$$

The value $\kappa + \delta k$ may be next used as a new initial value and the whole procedure is repeated until desired accuracy is attained. This form of the Newton-Raphson method guarantees quadratic convergence.¹¹

In order to bring formula (5) to a more tractable form we have to evaluate $H(\kappa)$. Using the definition (4), one finds

$$\dot{H}(\kappa) = [\varphi(\kappa, R)]^{-2} W[\varphi(\kappa, R), \dot{\varphi}(\kappa, R)] - [f^C(\kappa, R)]^{-2} W[f^C(\kappa, R), \dot{f}^C(\kappa, R)]. \quad (6)$$

We can get rid of the dotted quantities by making use of the wave equation. Differentiating (1) with respect to k one obtains the following equation for $\dot{\varphi}(k, r)$:

$$\dot{\varphi}''(k, r) + [k^2 - 2E\alpha Z/r - \lambda(\lambda + 1)/r^2 - 2\mu V(r)]\dot{\varphi}(k, r) = -2k(1 - \alpha Z/Er)\varphi(k, r). \quad (7)$$

Multiplying Eqs. (1) and (7) by $\dot{\varphi}(k, r)$ and $\varphi(k, r)$, respectively, and subtracting, one has

$$\frac{d}{dr} W[\varphi(k, r), \dot{\varphi}(k, r)] = -2k(1 - \alpha Z/Er)[\varphi(k, r)]^2, \quad (8a)$$

and similarly

$$\frac{d}{dr} W[f^C(k, r), \dot{f}^C(k, r)] = -2k(1 - \alpha Z/rE)[f^C(k, r)]^2. \quad (8b)$$

The Wronskians $W[\varphi, \dot{\varphi}]$ and $W[f^C, \dot{f}^C]$ are now readily obtained by integrating (8a) and (8b) in the limits $(0, R)$ and (R, ∞) , respectively. Inserting the resulting expressions into (6), the formula for $\dot{H}(\kappa)$ takes the form

$$\dot{H}(\kappa) = -2\kappa \left\{ \int_0^R (1 - \alpha Z/rE) [\varphi(\kappa, r)/\varphi(\kappa, R)]^2 dr + \int_R^\infty (1 - \alpha Z/rE) [f^C(\kappa, r)/f^C(\kappa, R)]^2 dr \right\}. \quad (9)$$

Since the range R is small according to our assumptions, the dominant contribution comes from the second integral; thus to a good approximation we have¹²

$$\dot{H}(\kappa) \simeq -2\kappa \int_0^\infty [f^C(\kappa, r)/f^C(\kappa, R)]^2 dr.$$

Inserting the above expression into (5) we obtain our final formula for δk

$$\delta k = W[f^C(\kappa, R), \varphi(\kappa, R)] f^C(\kappa, R) \times \left\{ 2\kappa \varphi(\kappa, R) \int_0^\infty [f^C(\kappa, r)]^2 dr \right\}^{-1}. \quad (10)$$

This equation establishes our iterative scheme.

It is obvious that the number of iterations required to achieve the desired accuracy depends crucially on how well we have chosen the initial guess, i.e., how far we start from the true solution. As we have already mentioned, we know from experiment that the energy shifts are smaller by several orders of magnitude than the Coulomb binding energies. This indicates that if we were to take the Coulomb energy as the initial value very little refinement would actually be needed and the first iteration should provide sufficient accuracy. Indeed, owing to the quadratic convergence, the relative error incurred is of the order of $(\delta k/\kappa)^2$ (which is typically 10^{-4} or less), so that the second iteration is hardly necessary.¹³

Then let $F^C(k)$ be the Coulomb Jost function and k_0 the root of the equation $F^C(k) = 0$, i.e.

$$k_0 = i|\alpha| Z\mu [(\lambda + 1)^2 + \alpha^2 Z^2]^{-1/2}.$$

For $k = k_0$ the function $f^C(k_0, r)$ is proportional¹⁰ to the regular Coulomb solution $\varphi^C(k_0, r)$, and the first order formula (10) for the energy shift simplifies to the form

$$\delta E = \frac{1}{2E_0} \left[\frac{\varphi'(k_0, R)}{\varphi(k_0, R)} - \frac{\varphi^C'(k_0, R)}{\varphi^C(k_0, R)} \right] / \int_0^\infty \left[\frac{\varphi^C(k_0, r)}{\varphi^C(k_0, R)} \right]^2 dr, \quad (11)$$

where $E_0 = (k_0^2 + \mu^2)^{1/2}$. In order to relate (11) to the conventional perturbation formula, let us notice that the Wronskian in formula (10) can be written¹⁰ as

$$W[f^C(k, R), \varphi(k, R)] = F^C(k) + \int_0^\infty f^C(k, r) 2\mu V(r) \varphi(k, r) dr. \quad (12)$$

Remembering that $F^C(k_0) = 0$ and inserting (12) into (10), one finds

$$\delta E = \frac{1}{2E_0} \frac{\varphi^C(k_0, R)}{\varphi(k_0, R)} \frac{\int_0^\infty \varphi^C(k_0, r) 2\mu V(r) \varphi(k_0, r) dr}{\int_0^\infty [\varphi^C(k_0, r)]^2 dr}. \quad (13)$$

Expressions (11) and (13) are equivalent; thus, if one uses the wave Eq. (1) in the integral that enters the numerator of (13), one gets back formula (11). Approximating $\varphi(k_0, r)$ by $\varphi^C(k_0, r)$ in Eq. (13), we recover the conventional (first order in V) perturbation formula. However, this would be a poor approximation unless the potential $V(r)$ were sufficiently weak.

In the hadronic atoms one is interested in the so-called circular orbits for which $n = l + 1$. For this case formula (11) yields the following expression for the total energy shift:

$$\delta E = \frac{\exp(2ik_0 R) (-2ik_0 R)^{2\lambda+3}}{2E_0 R^2 \Gamma(2\lambda+3)} \times \left[R \frac{\varphi'(k_0, R)}{\varphi(k_0, R)} - (\lambda+1) - ik_0 R \right]. \quad (14)$$

Our final formulas (11) and (14) have two rather advantageous features: (i) they are valid to all orders in the perturbing potential $V(r)$, and (ii) the shift δE is expressed only in terms of the logarithmic derivative of the regular wave function at the nuclear boundary. These two features allow for a fast and accurate evaluation of the shift.

The accuracy of formula (14) has been tested for a model kaonic atom calculation where $V(r)$ is of a Woods-Saxon shape¹⁴

$$V(r) = V_0 \{1 + \exp[(r-c)/a]\}^{-1}, \quad (15)$$

with $a = t/(4\ln 3)$, where t and c are the usual density parameters and where the complex depth V_0 is given by

$$V_0 = (\bar{a}/2\mu) (1 + m_K/m_N) (3A/c^3) [1 + (\pi a/c)^2]^{-1}. \quad (16)$$

Here m_K and m_N are the kaon and the nucleon masses, respectively, A is the mass number, and the complex parameter \bar{a} is the measure of the strength of the potential.

To provide a numerical illustration we have compared three methods of calculation of δE : (i) conventional perturbation theory, (ii) the S -matrix approach developed in this work, and (iii) the method of solving numerically the complex eigenvalue problem for the Klein-Gordon equation. In Table I we present the results of the computations performed for the $K^- - {}^{32}\text{S}$ atom, for several levels and for various values of the strength parameter \bar{a} [the corresponding depth (16) is rather big, V_0 is of the order of 100 MeV]. As seen from Table I

TABLE I. The values of ϵ and Γ for the K^- - ^{32}S atom in a model calculation. The perturbing potential was of Woods-Saxon shape (15) with density parameters $c=2.93$ fm, $t=2.45$ fm, the same as used in Ref. 14. The exact values have been obtained (Ref. 14) by a numerical solution of the complex eigenvalue problem for the Klein-Gordon equation [we have verified those entries with our code (Ref. 15) and corrected some typographical errors of Ref. 14]. The perturbation theory values have been obtained from the conventional first order perturbation formula with relativistic Coulomb wave functions. In Ref. 14 nonrelativistic perturbation theory was used and therefore slightly different values were obtained. The S-matrix theory values have been obtained from formula (14) with $R=30$ fm.

Level	\bar{a} (fm)	ϵ (keV)			$\frac{1}{2} \Gamma$ (keV)		
		Perturbation	S matrix	Exact	Perturbation	S matrix	Exact
3d	-0.6-i0.8	+0.972	-0.357	-0.355	1.295	1.011	1.012
	0 -i0.8	0	-0.478	-0.477	1.295	0.674	0.674
	+0.6-i0.8	-0.972	-0.721	-0.721	1.295	0.439	0.440
4f	-0.6-i0.8	+1.125 $\times 10^{-3}$	+0.389 $\times 10^{-3}$	+0.389 $\times 10^{-3}$	1.450 $\times 10^{-3}$	1.798 $\times 10^{-3}$	1.798 $\times 10^{-3}$
	0 -i0.8	0	-0.343 $\times 10^{-3}$	-0.343 $\times 10^{-3}$	1.450 $\times 10^{-3}$	1.293 $\times 10^{-3}$	1.293 $\times 10^{-3}$
	+0.6-i0.8	-1.125 $\times 10^{-3}$	-1.084 $\times 10^{-3}$	-1.084 $\times 10^{-3}$	1.450 $\times 10^{-3}$	1.016 $\times 10^{-3}$	1.016 $\times 10^{-3}$
5g	-0.6-i0.8	+0.706 $\times 10^{-6}$	+0.582 $\times 10^{-6}$	+0.594 $\times 10^{-6}$	0.942 $\times 10^{-6}$	1.071 $\times 10^{-6}$	1.071 $\times 10^{-6}$
	0 -i0.8	0	-0.087 $\times 10^{-6}$	-0.081 $\times 10^{-6}$	0.942 $\times 10^{-6}$	0.914 $\times 10^{-6}$	0.914 $\times 10^{-6}$
	+0.6-i0.8	-0.706 $\times 10^{-6}$	-0.713 $\times 10^{-6}$	-0.707 $\times 10^{-6}$	0.942 $\times 10^{-6}$	0.822 $\times 10^{-6}$	0.822 $\times 10^{-6}$

the S-matrix method is sufficiently accurate to be used in all realistic calculations of the hadronic atom level shifts. It should also be added that since formula (14) involves no iterative scheme this method of computation is extremely fast.

III. RELATION BETWEEN THE COMPLEX LEVEL SHIFT AND THE ZERO ENERGY PHASE SHIFT

Many years ago Deser *et al.*² derived a simple linear relation between the s-wave scattering length and the energy shift for the 1s level of a pionic atom. Their relation, accurate to all orders in the strong interaction and obtained under fairly general assumptions, was essentially a direct consequence of the fact that the strong interaction is of a short range relative to the atomic scale. Since the work of Deser *et al.* several papers have appeared³⁻⁷ trying to generalize their formula and through a more quantitative treatment to find correction terms that would make the formula more accurate. All these analyses, however, are not satisfactory because the authors, more or less explicitly, make the rather unrealistic assumption that the hadron-nucleus potential is weak (e.g., the assumptions that both the hadron-nucleus scattering length and the range of the potential are much smaller than the corresponding Bohr radius puts an upper limit on the potential strength). Consequently, it was not at all clear whether the derived relationship would still hold for arbitrarily strong potentials. Furthermore, since in general neither the scattering length nor the energy shift could be calculated perturbatively, it was far from obvious whether these two quantities could be connected by a formula whose validity rested heavily upon perturbative arguments. To prove

that this is indeed the case a nonperturbative calculational scheme for δE is necessary. Having developed such scheme in the preceding section we shall now derive the relation between the level shift δE and the complex hadron-nucleus scattering length.

The familiar expression for the effective range expansion in the presence of the Coulomb field takes the following form⁷:

$$\frac{p_l(\eta)}{2\eta(2l+1)} k^{2l+1} [C_0^2(\eta) \cot \delta_l + 2\eta h(\eta)] = -\frac{1}{A_l} + \frac{1}{2} R_l k^2 + O(k^4), \quad (17)$$

where δ_l is the l th partial wave phase shift, A_l is the generalized scattering length of the dimension $(\text{length})^{2l+1}$, R_l is the effective range of the dimension $(\text{length})^{1-2l}$. The other quantities in formula (17) are defined as follows:

$$\eta = \alpha Z\mu/k,$$

$$h(\eta) = \text{Re } \psi(i\eta) - \ln(|\alpha| Z\mu/k),$$

$$C_0^2(\eta) = 2\pi\eta/(e^{2\pi\eta} - 1),$$

$$p_l(\eta) = \frac{2\eta 2^{2l}}{(2l)!(2l+1)!} \prod_{s=1}^l (s^2 + \eta^2),$$

where $\psi(x)$ is the logarithmic derivative of the Γ function. The phase shift δ_l , and subsequently the parameter A_l defined by formula (17), is obtained by matching at $r=R$ the logarithmic derivative of the regular wave function $\varphi(k, r)$ with the logarithmic derivative of the appropriate linear combination of the regular and irregular wave functions. The final expression for A_l can be written as⁷

$$A_l = [\varphi^C(0, \mathbf{R})]^2 \chi(R) [1 + \theta(R) \varphi^C(0, R) \chi(R)]^{-1}, \quad (18)$$

where

$$\chi(R) = \frac{\varphi'(0, R)}{\varphi(0, R)} - \frac{\varphi^{C'}(0, R)}{\varphi^C(0, R)}, \quad (19a)$$

$$\varphi^C(0, R) = \frac{\Gamma(2l+2)}{|2\mu\alpha Z|^{l+1}} |2\mu\alpha ZR|^{1/2} \times J_{2l+1}(2|2\mu\alpha ZR|^{1/2}), \quad (19b)$$

$$\theta(R) = -\pi \frac{|2\mu\alpha Z|^l}{\Gamma(2l+2)} |2\mu\alpha ZR|^{1/2} Y_{2l+1}(2|2\mu\alpha ZR|^{1/2}), \quad (19c)$$

and J_{2l+1} and Y_{2l+1} are Bessel functions of the first and second kind, respectively.

Thus far no approximation has been made in the derivation of formula (18). The dependence on the interaction $V(r)$ enters, as in the preceding section, through the difference of the logarithmic derivatives (19a). This time, however, this difference is taken at zero momentum, rather than at $k = k_0$ as in formula (11). On the other hand, for values of R such that $2\mu|V_{\text{cul}}(R)| \gg |k_0^2|$ and $|k_0^2| \gg 2\mu|V(R)|$ these quantities will not differ much from each other, i.e.,

$$\left[\frac{\varphi'(k_0, R)}{\varphi(k_0, R)} - \frac{\varphi^{C'}(k_0, R)}{\varphi^C(k_0, R)} \right] \simeq \left[\frac{\varphi'(0, R)}{\varphi(0, R)} - \frac{\varphi^{C'}(0, R)}{\varphi^C(0, R)} \right] + O(k_0^2). \quad (20)$$

Eliminating the above difference of logarithmic derivatives between Eqs. (11) and (18) one arrives at the desired relation between δE and A_l :

$$A_l \simeq \frac{\delta E \xi}{1 + \delta E [\xi \theta(R) / \varphi^C(0, R)]}, \quad (21)$$

where

$$\xi = 2E_0 [\varphi^C(0, R) / \varphi^C(k_0, R)]^2 \int_0^\infty [\varphi^C(k_0, r)]^2 dr.$$

Neglecting δE in the denominator of (21) and setting $\varphi^C(0, R) \approx \varphi^C(k_0, R)$, an extremely simple relation follows:

$$A_l = \delta E 2E_0 \int_0^\infty [\varphi^C(k_0, r)]^2 dr. \quad (22)$$

For circular orbits, introducing the binding energy $B = E - \mu$, the Bohr radius $r_B = (|\alpha|Z\mu)^{-1}$ and the Coulomb energy $B_0 = k_0^2/2\mu$, formula (22) takes a more familiar form

$$\frac{\delta B}{B_0} = -\frac{1}{(2l+1)!} \left(\frac{2}{l+1} \right)^{2l+2} \frac{A_l}{r_B^{2l+1}}, \quad (23)$$

which is nothing but the well-known relationship obtained by many authors for weak potentials.⁵⁻⁷

As seen from (22), the dependence on the potential $V(r)$ that enters (18) through the logarithmic derivative (19a), has been eliminated in (22) in favor of δE . Thus, formula (22) constitutes the most general relationship between the shift δE and the scattering length A_l that may be formulated in a model independent way. The approximate equality (20), so crucial for the derivation of (21), holds if $2\mu|\alpha|Z/R \gg |k_0^2|$, or if

$$(R/2r_n) \ll 1, \quad (24)$$

that is, if the range of the hadron-nucleus potential is much smaller than the classical radius, $r_n = (l+1)/|k_0|$. This condition seems to be quite well fulfilled in all practical applications. The derivation of formula (23) implies a number of corollaries:

(1) Formula (23) is valid to all orders in the perturbing potential. Even though, for strong potentials, both A_l and δE are nonlinear functions of the potential strength, they are still correlated by a linear relationship.¹⁶

(2) Formula (23) is model independent; notice that even the dependence on the matching radius does not appear.

(3) Finite size of the nuclear charge distribution and vacuum polarization effects can be completely disregarded. They can always be calculated perturbatively and enter as additive corrections to the binding energy. Since δE is a difference of two binding energies these corrections will eventually cancel out and need not be considered.¹⁵

(4) The scattering length A_l is defined in the presence of the Coulomb field.

(5) A_l need not necessarily be less than r_B^{2l+1} ; actually, formula (23) still holds even if $|A_l| > r_B^{2l+1}$ provided l is large enough so that $|\delta B/B_0| \ll 1$.

The accuracy of formula (23) has been tested on a model calculation for the kaonic atom case considered in the preceding section. We used the same Wood-Saxon potential (15) and compared the scattering length calculated from the exact formula (18) with the approximate values obtained by means of formula (23); the results of the computations are presented in Table II. As seen from this table the error, being always less than 4%, is getting smaller for higher orbital momenta, because then the ratio (R/r_n) decreases in accordance with the requirement (24). Notice that since $r_B \approx 3.5$ fm, for $l = 4$ one has $|A_l| > r_B^{2l+1}$; nevertheless, the accuracy of formula (23) is better than 1%.

IV. CONCLUSIONS

In the nonperturbative S -matrix approach developed in this paper we have derived two important relations. The first one expresses the com-

TABLE II. The scattering length A_l in $(\text{fm})^{2l+1}$ for K^- - ^{32}S scattering in a model calculation. The scattering lengths A_l have been calculated for the Woods-Saxon potential (15) with $c=2.93$ fm, $t=2.45$ fm. The exact values have been obtained from formula (18) for $R=30$ fm, the corresponding S -matrix values have been calculated from the approximate relation (23).

State	\bar{a} (fm)	$\text{Re}A_l (\text{fm})^{2l+1}$		$\text{Im}A_l (\text{fm})^{2l+1}$	
		Exact	S matrix	Exact	S matrix
$l=2$	$-0.6-i0.8$	0.6804×10^3	0.6621×10^3	-0.1780×10^4	-0.1875×10^4
	$0-i0.8$	0.8753×10^3	0.8861×10^3	-0.1197×10^4	-0.1249×10^4
	$+0.6-i0.8$	0.1294×10^4	0.1338×10^4	-0.7753×10^3	-0.8151×10^3
$l=3$	$-0.6-i0.8$	-0.1421×10^5	-0.1473×10^5	-0.6737×10^5	-0.6802×10^5
	$0-i0.8$	0.1300×10^5	0.1300×10^5	-0.4828×10^5	-0.4889×10^5
	$+0.6-i0.8$	0.4056×10^5	0.4102×10^5	-0.3783×10^5	-0.3843×10^5
$l=4$	$-0.6-i0.8$	-0.1118×10^7	-0.1129×10^7	-0.2048×10^7	-0.2060×10^7
	$0-i0.8$	0.1695×10^6	0.1680×10^6	-0.1746×10^7	-0.1758×10^7
	$0.6-i0.8$	0.1364×10^7	0.1372×10^7	-0.1569×10^7	-0.1581×10^7

plex energy shift in a hadronic atom in terms of the logarithmic derivative of the regular wave function at the nuclear boundary. The obtained formula allows for a fast and accurate evaluation of the complex level shift once the equivalent nuclear potential is given. The second relation correlates the complex level shift with the zero-energy scattering phase shift. The relation obtained is model independent, that is, it does not depend on the form of the potential, nor on its range, as long as the latter is small in comparison with the classical Coulomb radius. Thus, the low energy

data, viz., the hadronic atoms binding energies, and the corresponding zero-energy cross sections have been correlated by means of a simple linear formula. On the theoretical side, this means that multiple scattering theory may be used to explain the observed hadronic atom level shifts; there is no necessity to invoke the notion of the equivalent nuclear potential.

We wish to thank Dr. J. Law for a helpful correspondence concerning the derivation of the energy shift formula in Sec. II.

[†]Present address: Department of Physics, University of Guelph, Guelph, Ontario, Canada.

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⁶P. O. Mazur, M. M. Block, J. Keren, and J. L. Meyer, *Phys. Rev. D* **1**, 20 (1970).

⁷E. Lambert, *Helv. Phys. Acta* **42**, 667 (1969).

⁸J. Uretsky, in *Proceedings of the Second International Conference on High-Energy Physics and Nuclear Structure, Rehovoth, Israel, 1967*, edited by G. Alexander (North-Holland, Amsterdam, 1967).

⁹Since for practical purposes nonrelativistic quantum mechanics is used to construct the potential, two types of nonlocality may be of interest: the interaction nonlocal in time where the potential is energy dependent, and the interaction nonlocal in space where $V\psi$ should be understood as $\int V(\mathbf{r}, \mathbf{r}') \psi(\mathbf{r}') d\mathbf{r}'$. Notice that, in general, $V(\mathbf{r}, \mathbf{r}') \neq V(\mathbf{r}', \mathbf{r})$ because the potential is absorptive and thus manifestly not time reversal in-

variant. However, nonlocality of the potential adds nothing new to our discussion and to simplify the notation we write $V(\mathbf{r})$ as if it were a local potential.

¹⁰R. G. Newton, *Scattering Theory of Waves and Particles* (McGraw-Hill, New York, 1966).

¹¹Usually the derivative $\dot{H}(\kappa)$ is a slowly varying function, so that it is sufficient to calculate $\dot{H}(\kappa)$ only in the first step and keep this value unchanged throughout the iterative procedure. This simplification speeds up the computations considerably.

¹²For a potential nonlocal in time (i.e., V depends on κ) the right-hand side of (9) contains the term $\int_0^R 2\mu \times [\partial V(\kappa, r)/\partial \kappa] [\varphi(\kappa, r)]^2 dr$. Similarly, for a potential nonlocal in space a term $\int_0^R 2\mu [V(\mathbf{r}, \mathbf{r}') - V(\mathbf{r}', \mathbf{r})] \varphi(\kappa, \mathbf{r}') \times \dot{\varphi}(\kappa, \mathbf{r}) d\mathbf{r} d\mathbf{r}'$ should be added. In either case these extra terms are small and can be neglected.

¹³Of course we have tacitly assumed that the potential is adequate to explain the experimental shifts, whereas for a quite arbitrary potential the first order correction δk does not necessarily need to be small. Such a situation may, for instance, occur if one wants to adjust certain parameters of the potential to fit the data. Nevertheless, the method is still very efficient even if more than one iteration has to be performed.

¹⁴R. Seki, *Phys. Rev. C* **5**, 1196 (1972).

¹⁵A. Deloff and J. Law, *Phys. Rev. C* **10**, 1688 (1974).

¹⁶Strictly speaking, the relation between δE and A_l is nonlinear, for solving eq. (21) for δE one ends up with formula (23) multiplied on the right-hand side by the factor $[1 + A_l \theta(R)/\varphi^C(0,R)]^{-1}$. In realistic situations this factor is close to one, but has to be included if the

potential is strong enough to yield binding. If this is the case, A_l jumps from $-\infty$ to $+\infty$ when the depth of the potential increases and reaches sufficient strength to produce a zero-energy bound state. However, due to the extra factor δE remains finite, as it should.