Invariant potential for elastic pion-nucleus scattering*

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From the Wick-Dyson expansion of the exact propagator of a pion in the presence of a nucleus, an invariant potential for crossing symmetric elastic pion-nucleus scattering is obtained in terms of a series of pion-nucleon diagrams. The Chew-Low theory is used to develop a model in which the most important class of diagrams is effectively summed. Included in this model is the exclusion principle restriction on the pion-bound nucleon interaction, the effects of the binding of nucleons, a kinematic transformation of energy from the lab to the πN center of mass frame, and the Fermi motion and recoil of the target nucleons. From a numerical study of the effects of these processes on the π-^{12}C total cross section, the relative importance of each is determined. Other processes contributing to the elastic scattering of pions not included in the present model are also discussed.

NUCLEAR REACTIONS Many-body approach to an invariant potential for crossing symmetric, elastic pion-nucleus scattering. $^{12}C(\pi, \pi)$, E_π in (3, 3) resonance region; calculated $\sigma(E)$ showing effects of nuclear corrections to Chew-Low description of πN interaction.

I. INTRODUCTION

In a recent letter[1] we discussed a derivation of the pion-nucleus optical potential which proceeds from the exact propagator of a pion in the presence of a nucleus. From the Wick-Dyson expansion of the propagator and the relation between the potential and the propagator the optical potential was identified with a series of proper self-energy subdiagrams of the pion-nucleon interaction in the nuclear medium. For the subsequent analysis, these πN diagrams were formally summed into a series of direct and crossed pion-nucleus diagrams. It could then be established that the crossed π-nucleus processes are not formally included in the Watson multiple scattering theory[2] even when a crossing symmetric pion-nucleon amplitude is used. It was also shown how the transformation properties of the potential, its off-shell behavior, and the form of the scattering equation determine whether the crossed π-nucleus processes are included in the elastic amplitude.

In the present work we construct an invariant pion-nucleus potential which reflects many of the physical processes so clearly defined by the diagrammatic expansion. After reviewing the propagator approach to the optical potential in the next section, we present in Sec. III a model for the potential to determine crossing symmetric elastic pion-nucleus scattering which effectively sums the most important class of pion-nucleon diagrams, the single-nucleon processes. We include in this model a kinematic transformation of energy from the lab to the πN center of mass frame, and the effects of the binding of nucleons, the exclusion principle restriction on the pion-bound nucleon interaction, and the Fermi motion and recoil of the target nucleons. Though some of these features have been discussed in the literature before,[3] we find that our approach in describing pion-nucleus scattering in terms of a crossing symmetric potential constructed from invariant pion-nucleon amplitudes provides a framework for analyzing these effects which proves convenient and which leads to new dynamical results. In particular, since we identify the pion-nucleus potential as a term appearing in the Klein-Gordon equation analogously to the pion rest mass, the potential necessarily has a Lorentz invariant form which eliminates having to confront the difficult problem[4] of relating off-shell amplitudes in different frames, but, as a consequence of our particular model for this potential, requires that we carry out only a well-defined kinematic transformation of energy. We are also able to show that crossing symmetry has a very important role in determining the contribution of nucleon momenta to the nuclear transition operator for terms through first order in the ratio of pion energy to nucleon mass. The structure of the resulting nucleon motion correction to the pion-nucleus potential significantly differs from that obtained in previous work.[5-7]

From a numerical study of the effects of these processes on the π-^{12}C total cross section, as discussed in Sec. IV, we determine the relative importance of each to elastic π-nucleus scattering. (The specific role played by the crossed π-nucleus processes has been discussed in Ref. 1.) In Sec. V we summarize these results and discuss some further improvements to our potential.

II. REVIEW OF THE PROPAGATOR APPROACH TO THE OPTICAL POTENTIAL

A. Derivation of the scattering equation

Working in the Heisenberg picture, let $|\Psi\rangle$ represent the exact wave function of the nuclear ground state, and let $a^{(H)\dagger}_{\alpha\vec{k}}(t)$ and $a^{(H)}_{\beta\vec{k}'}(t)$ be, respectively, second quantized pion creation and annihilation operators at time t, where α and β denote isospin components and $\vec{k}$ and $\vec{k}'$ are three-momenta. We define the single-pion time-ordered propagator by

$$\langle \beta\vec{k}'|G(\omega)|\alpha\vec{k}\rangle = -i\int dt\, e^{i\omega t}\langle\Psi|T(a^{(H)}_{\beta\vec{k}'}(t), a^{(H)\dagger}_{\alpha\vec{k}}(0))|\Psi\rangle\,, \tag{1}$$

which describes the propagation of a pion with energy parameter ω from the state with momentum $\vec{k}$ and isospin α to the state $(\beta, \vec{k}')$ in the presence of the nucleus in its ground state $|\Psi\rangle$. The symbol T denotes the chronological time ordering of the second quantized operators.

In the absence of interactions the motion of the pion would be described by the free propagator function, given by

$$\langle \beta\vec{k}'|G_0(\omega)|\alpha\vec{k}\rangle = -i\int dt\, e^{i\omega t}\langle 0|T(a^{(H)}_{\beta\vec{k}'}(t), a^{(H)\dagger}_{\alpha\vec{k}}(0))|0\rangle\,, \tag{2}$$

where $|0\rangle$ represents the physical vacuum with $\langle 0|0\rangle = 1$. Let h be the relativistic energy operator for the pion so that, acting on a single-pion state of momentum $\vec{k}$, we have $h|\alpha\vec{k}\rangle = \omega_k|\alpha\vec{k}\rangle$, $\omega_k{}^2 = \vec{k}^2 + m_\pi{}^2$. Then in the absence of interactions the Heisenberg operators satisfy

$$a^{(H)}_{\beta\vec{k}'}(t) = e^{iht}a_{\beta\vec{k}'}e^{-iht}\,, \tag{3}$$

$$a_{\beta\vec{k}'} \equiv a^{(H)}_{\beta\vec{k}'}(0)\,.$$

Inserting a complete set of states in the chronological product of (2), using (3), the canonical commutation relations

$$\begin{aligned}&[a_{\beta\vec{k}'}, a^\dagger_{\alpha\vec{k}}] = (2\pi)^3\delta_{\alpha\beta}\delta(\vec{k}-\vec{k}')\,,\\ &[a_{\beta\vec{k}'}, a_{\alpha\vec{k}}] = [a^\dagger_{\beta\vec{k}'}, a^\dagger_{\alpha\vec{k}}] = 0\,,\end{aligned} \tag{4}$$

and performing the time integration, we have

$$\langle \beta\vec{k}'|G_0(\omega)|\alpha\vec{k}\rangle = \delta_{\alpha\beta}(2\pi)^3\delta(\vec{k}-\vec{k}')(\omega - \omega_{k'} + i\epsilon)^{-1}\,. \tag{5}$$

Expressing the right-hand side as the matrix element of an operator in the single-pion Hilbert space $\mathcal{H}_\pi$, we can write an operator equation defining the free propagator

$$G_0(\omega) = (\omega - h + i\epsilon)^{-1}\,. \tag{6}$$

(Here and in the following an operator will be assumed to act in $\mathcal{H}_\pi$ unless noted otherwise.) If the pion were moving in a potential field which could be represented by an operator v acting in $\mathcal{H}_\pi$, we would have $h \to h' = h + v$, so

$$G_0(\omega) = (\omega - h - v + i\epsilon)^{-1}\,. \tag{7}$$

This suggests that a reasonable ansatz for the pion-nucleus optical potential $U(\omega)$ be given by the operator equation

$$G(\omega) = (\omega - h - U(\omega) + i\epsilon)^{-1}\,. \tag{8}$$

Using (6), this can be written as an implicit equation for $G(\omega)$,

$$G(\omega) = G_0(\omega) + G_0(\omega)U(\omega)G(\omega)\,, \tag{9}$$

which will be recognized as the Dyson equation expressing the exact propagator in terms of the free propagator and "mass operator" $U(\omega)$.

That the operator $U(\omega)$ defined by (8) is an appropriate optical potential, in that it yields the exact many-body T matrix for elastic scattering when used in the Lippmann-Schwinger equation

$$T(z) = U(z) + U(z)(z-h)^{-1}T(z)\,, \tag{10}$$

$$z = \omega + i\epsilon\,,$$

can be established easily. We note that to obtain this result it is necessary to treat the target nucleus as static, thus neglecting recoil.[8]

B. Wick-Dyson expansion

By carrying out the Wick-Dyson expansion of the exact propagator, we obtain the diagrammatic representation of the potential $U(z)$ as the sum of all proper self-energy subdiagrams of the pion-nucleon interaction in the nuclear medium.

To evaluate the perturbation expansion it is necessary to define the Hamiltonian $H = H_0 + H_1$. For the noninteracting part of the Hamiltonian we consider

$$H_0 = H^{(0)}_\pi + H^{(0)}_N + H_B + \text{C.T.} \tag{11}$$

$H^{(0)}_\pi$ and $H^{(0)}_N$ represent free-field Hamiltonians for the pion and nucleon fields, H_B is a nucleon-nucleus potential of the type proposed by Brandow[9] to describe the interaction of any nucleon with the residual core, and C.T. denotes counter terms. For H_1 we consider

$$H_1 = H_{\pi N} + H_{NN} - H_B - \text{C.T.} \tag{12}$$

Here $H_{\pi N}$ describes the πN interaction; we shall assume that it is linear in the pion field, in accordance with most widely used models. H_{NN} describes the nucleon-nucleon interaction, except for pion exchange. By writing the Hamiltonian in this form we shall find a significant simplification in the nature of the diagrams which result from the perturbation expansion. The subdiagrams which represent conventional mass renormalization will be cancelled by the counter terms, and the subdiagrams which represent the renormalization of the nucleon propagator due to the interaction between two or more nucleons produced by H_{NN} or by $H_{\pi N}$ (through second or higher order terms) will be cancelled by the nucleon-nucleus potential, by definition.

In translating the results of the perturbation analysis of the propagator into diagrams, we shall use Feynman-Goldstone diagrams as they yield the form (8) readily in addition to conveniently exhibiting the roles of nuclear excitation, pion-nucleus crossing, and other features we wish to consider. It should be emphasized that we fully include the antinucleon degrees of freedom in our analysis, though for our purposes it will not be necessary to explicitly indicate the antinucleon states in the diagrams.

A few of the lowest order diagrams for the πN interaction in the nuclear medium are shown in Fig. 1. For simplicity, diagrams involving exchange of bosons between the pion and a nucleon are not exhibited. We note that for every diagram for which the absorption vertex for the incoming pion precedes the emission vertex for the outgoing pion there is another diagram which is identical, except that the absorption and emission vertices are reversed. This is the expression of crossing in the πN interaction in terms of Feynman-Goldstone diagrams.

From the relation (9) it follows by well-known arguments[10] that the optical potential $U(z)$ has the diagrammatic interpretation as the sum of the proper self-energy subdiagrams of the Wick-Dyson expansion of the propagator. Having thus identified the series of subdiagrams which constitute the potential $U(z)$, we can exhibit them as a sum of pion-nucleus diagrams after summing over all possible generalized time ordering. This process may be viewed as simply a regrouping of the πN diagrams. The resulting series of proper π-nucleus diagrams are shown schematically in Fig. 2, where the ground state nucleus is taken as the vacuum. The open circle of Fig. 2 represents the basic set of proper diagrams in which the nucleus in excited by the first interaction and remains excited until the last interaction. For every diagram in the set there is another which

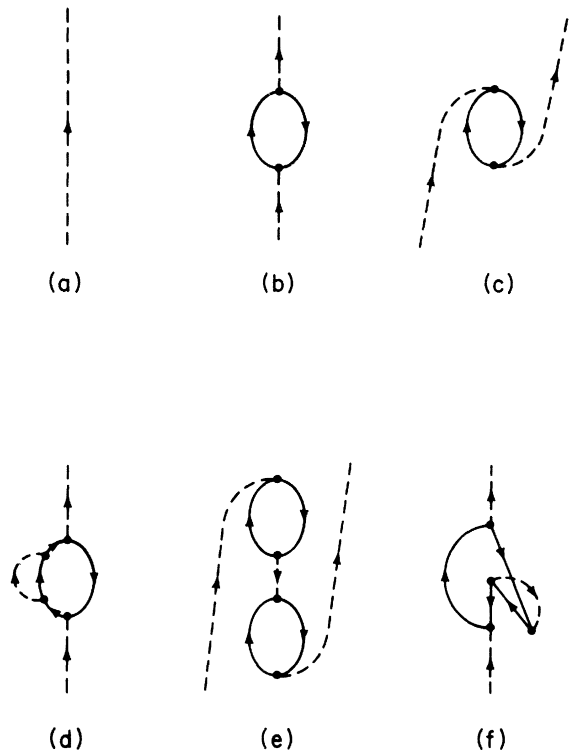

FIG. 1. Several low order Feynman-Goldstone diagrams for the πN interaction in the nuclear medium which are generated from the Wick-Dyson expansion of the single-pion propagator are shown. The dotted lines represent pions; the solid lines which are directed upward represent nucleons, while those directed downward represent holes.

is obtained by crossing the incoming and the outgoing pion lines; thus the set is crossing symmetric. Its value will be represented by $\langle \beta \vec{k}' | D(z) | \alpha \vec{k} \rangle$. The diagrams in D can be broadly grouped into two subsets: one containing the diagrams where the incoming and the outgoing pions both interact with the same nucleon (single-nucleon processes), and the other containing all remaining diagrams (multinucleon processes). An example of a single-nucleon process is shown in the diagram of Fig. 3(a); a multinucleon process is shown in Fig. 3(b). The single-nucleon processes represent the elementary π-nucleon scattering in the nuclear medium. The resulting amplitude differs from the free πN scattering amplitude in three important aspects: (i) the exclusion principle restricts the available phase space for the interaction, (ii) the nucleons are bound in a potential and possess a definite distribution of momenta so there exists a certain threshold of energy to produce a particle-hole pair, and (iii) the intermediate state pions can

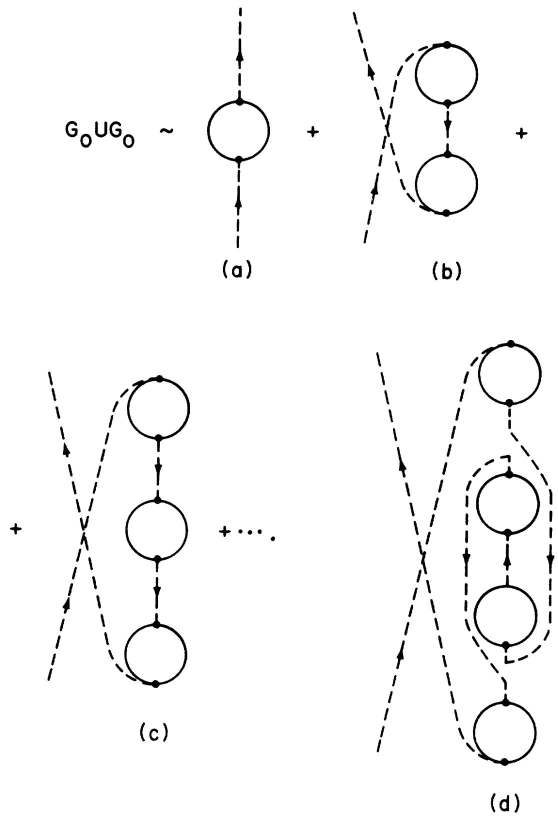

FIG. 2. Diagrams (a), (b), and (c) represent the first three terms of the series of pion-nucleus diagrams which constitute the pion-nucleus optical potential $U(z)$. Diagram (d) is an example of a diagram exhibiting crossing within crossing, and is therefore not part of the series represented by $U(z)$.

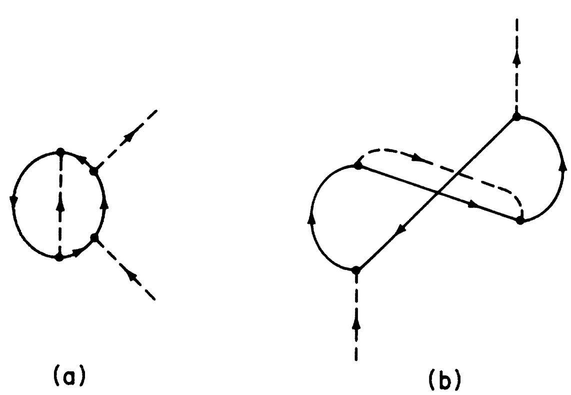

FIG. 3. An example of a single-nucleon process (a) and a multinucleon process (b).

rescatter off the entire nucleus. These effects must be considered in the formulation of any model description of the pion-bound nucleon interaction. The multinucleon processes describe inherently many-nucleon aspects of the interaction which are not represented in free scattering.

Figure 2(b) represents the simplest crossed (proper) diagram, and Fig. 2(c) represents the second term of a series of crossed diagrams. These diagrams describe the process where the nucleus emits two pions and returns to its ground state. One of the pions comes out while the other suffers a series of D interactions and is eventually absorbed, as is the incoming pion. We note that diagrams such as the one shown in Fig. 2(d), where we have crossing within crossing, are not to be included in U. These diagrams are iterations [in the context of (10)] of the series of crossed diagrams, and would therefore be counted twice in the elastic amplitude if also included in U. We further note that since each of the crossed π-nucleus diagrams has a ground state nucleus in an intermediate state, they are not formally represented in optical potentials derived from the Watson multiple scattering theory.[2] The principle reason for this difference is that, unlike the Watson formalism, here we explicitly keep track of the facts that (i) the pion is a boson and (ii) a nucleus can emit or absorb two pions (one real, one virtual) and still remain in its ground state. In Ref. 1 it was demonstrated that the crossed π-nucleus processes play an important role in the nuclear elastic scattering of low energy pions.

From the rules for evaluating π-nucleus diagrams given in Appendix A, and after carrying out all possible generalized time orderings, the optical potential $U(z)$ is analytically expressed in terms of the series of proper diagrams by

$$\begin{aligned} U(z) &= D(z) + D(z)G_0(-z)D(z) \\ &\quad + D(z)G_0(-z)D(z)G_0(-z)D(z) + \cdots \\ &= D(z) - D(z)[z + h + D(z)]^{-1}D(z) \end{aligned} \tag{13}$$

or, equivalently,

$$U(z) = D(z) - D(z)(z + h)^{-1}U(z). \tag{14}$$

The first term on the right of these equations represents the contribution of the direct diagram of Fig. 2, while the remaining terms describe the series of crossed π-nucleus diagrams. Using (14) in (10) yields

$$T(z) = D(z) + D(z)(2h)^{1/2}(z^2 - h^2)^{-1}(2h)^{1/2}T(z). \tag{15}$$

We note that h is a positive definite operator, so that $(2h)^{1/2}$ is well defined. By multiplying (15) on the right and on the left by $(2h)^{1/2}$ and defining the operators

$$\hat{T}(z) = (2h)^{1/2} T(z) (2h)^{1/2} ,$$
$$V(z) = (2h)^{1/2} D(z) (2h)^{1/2} , \quad (16)$$

we have

$$\hat{T}(z) = V(z) + V(z)(z^2 - h^2)^{-1} \hat{T}(z) . \quad (17)$$

Following the usual prescription, (17) will be recognized as the time-independent Klein-Gordon equation

$$\langle \vec{k}' | \hat{T}(\omega_k + i\epsilon) | \vec{k} \rangle = \langle \vec{k}' | V(\omega_k) | \varphi_{\vec{k}}^{(+)} \rangle , \quad (18a)$$

where

$$| \varphi_{\vec{k}}^{(+)} \rangle = | \vec{k} \rangle + (\omega_k^2 - h^2 + i\epsilon)^{-1} V(\omega_k) | \varphi_{\vec{k}}^{(+)} \rangle \quad (18b)$$

or

$$[h^2 + V(\omega_k)] | \varphi_{\vec{k}}^{(+)} \rangle = \omega_k^2 | \varphi_{\vec{k}}^{(+)} \rangle . \quad (18c)$$

Thus, to determine the amplitude for elastic π-nucleus scattering which includes both the direct and crossed π-nucleus processes, we need to develop a model which effectively sums the diagrams represented by $V(z)$, and then solve the Klein-Gordon equation (17). We observe that the inclusion of the crossed diagrams has led to a potential which is Lorentz invariant. This is not surprising when we consider that by including the crossed diagrams we have included the negative energy states of the pions, and that relativity requires that we handle the negative and positive energy states symmetrically.

The evaluation of the scattering amplitude from (17) has the obvious advantage over the corresponding calculation based on the Lippmann-Schwinger equation (10) that the potential $V(z)$ that we must determine represents a smaller and simpler set of πN subdiagrams than does $U(z)$. We now consider a model for the potential V.

III. MODEL FOR THE INVARIANT PION-NUCLEUS POTENTIAL

As the potential $V(z)$ appears in the Klein-Gordon equation analogously to the pion rest mass, we shall refer to it as the invariant pion-nucleus potential. From the Dyson-Wick expansion V, by its relation to D through (16), is identified as a set of crossing symmetric proper self-energy sub diagrams describing the pion-nucleon interaction in the nuclear medium. Our goal is to sum this series of diagrams.

Our previous analysis has shown that the diagrams represented by the invariant potential can be grouped into two classes: the single-nucleon and the multinucleon processes. The first class represents pion-single nucleon scattering in the nuclear medium. Except for modifying terms arising from the presence of other nucleons, this subset of diagrams is identical with the set one obtains from a perturbation expansion of pion-free nucleon scattering. This class of diagrams may thus be considered analogous to the description of the pion-nucleus interaction which is provided by a first-order optical potential. The multinucleon processes may correspondingly be viewed as being analogous to the various higher-order corrections to the potential. Consequently, we shall develop a model for V in which a large class of diagrams describing free πN scattering is effectively summed; and with corrections to account for the presence of other nucleons, we will take this result as an approximation to the single-nucleon processes described by V. We will not consider the more difficult problem of summing the diagrams for the multinucleon processes, though our numerical studies will indicate that they are not entirely negligible.

A. Impulse approximation with Chew-Low amplitude

We first discuss a very simple model for $V(z)$ based on the impulse approximation. Thus we not only neglect the multinucleon diagrams, we also neglect the effects of the other nucleons on π-single nucleon scattering. As a result, the diagrams considered are those which represent the free πN scattering amplitude. In the simple model we make the further approximation of using the Chew-Low[11] theory to describe the πN scattering amplitude. This theory is approximate in that it only considers the p-wave interaction with the nucleon regarded as static; i.e., the ratio ω/M_N of the pion energy to the nucleon mass is considered negligibly small. However, it is crossing symmetric, and has the virtue that it is simple and gives a fairly good description of the p-wave phase shifts in the resonance region.

Thus, in our model

$$\langle \beta\vec{k}' | V(z) | \alpha\vec{k} \rangle = \langle \beta\vec{k}' | (2\omega_{k'})^{1/2} D(z) (2\omega_k)^{1/2} | \alpha\vec{k} \rangle = \langle \Psi | -4\pi v(\vec{k}^2) v(\vec{k}'^2) \sum_{n=1}^{A} \sum_{\mu=1}^{4} P_\mu^{(n)}(\beta\vec{k}', \alpha\vec{k}) h_\mu(z) | \Psi \rangle , \quad (19)$$

where $|\Psi\rangle$ denotes the exact ground state nuclear wave function, $v(k^2)$ is the form factor (cutoff) of the πN interaction, and the sum over n represents the sum over the A nucleons of the nucleus. [With $A=1$, and $|\Psi\rangle$ replaced by a single (static) nucleon state, (19) becomes (except for energy factors) the πN T-matrix element of the Chew-Low theory.] For the nth (static) nucleon, the complete projection operators are given by

$$P_1^{(n)}(\beta\vec{k}', \alpha\vec{k}) = \mathcal{T}_{1/2}^{(n)}(\beta,\alpha)\mathcal{J}_{1/2}^{(n)}(\vec{k}',\vec{k})e^{i(\vec{k}-\vec{k}')\cdot\vec{r}_n},$$
$$P_2^{(n)}(\beta\vec{k}', \alpha\vec{k}) = \mathcal{T}_{1/2}^{(n)}(\beta,\alpha)\mathcal{J}_{3/2}^{(n)}(\vec{k}',\vec{k})e^{i(\vec{k}-\vec{k}')\cdot\vec{r}_n},$$
$$P_3^{(n)}(\beta\vec{k}', \alpha\vec{k}) = \mathcal{T}_{3/2}^{(n)}(\beta,\alpha)\mathcal{J}_{1/2}^{(n)}(\vec{k}',\vec{k})e^{i(\vec{k}-\vec{k}')\cdot\vec{r}_n},$$
$$P_4^{(n)}(\beta\vec{k}', \alpha\vec{k}) = \mathcal{T}_{3/2}^{(n)}(\beta,\alpha)\mathcal{J}_{3/2}^{(n)}(\vec{k}',\vec{k})e^{i(\vec{k}-\vec{k}')\cdot\vec{r}_n},$$

where (20)

$$\mathcal{T}_{1/2}^{(n)}(\beta,\alpha) = \tfrac{1}{3}\tau_\beta^{(n)}\tau_\alpha^{(n)},$$
$$\mathcal{T}_{3/2}^{(n)}(\beta,\alpha) = \delta_{\alpha\beta}^{(n)} - \mathcal{T}_{1/2}^{(n)}(\beta,\alpha),$$
$$\mathcal{J}_{1/2}^{(n)}(\vec{k}',\vec{k}) = \vec{\sigma}_n\cdot\vec{k}'\vec{\sigma}_n\cdot\vec{k},$$
$$\mathcal{J}_{3/2}^{(n)}(\vec{k}',\vec{k}) = 3\vec{k}'\cdot\vec{k} - \mathcal{J}_{1/2}^{(n)}(\vec{k}',\vec{k}),$$

where $\mathcal{T}_I$ is the projection operator onto states of total isospin I, $\mathcal{J}_J$ is the projection operator onto states of total angular momentum J, $\vec{\sigma}_n$ is the spin vector of the nth nucleon, and $\tau_\alpha^{(n)}$ is the αth component of the nucleon isospin vector.

As ^{12}C has received the most extensive experimental study as the target for elastic pion scattering, we shall confine our calculations to this nucleus. The application of our procedure to other nuclei is straightforward. The ground state of ^{12}C has $(J^P, I) = (0^+, 0)$, and thus in the nuclear expectation value of the projection operators in (19) only the scalar, isoscalar, even parity part of these operators will contribute. The three nucleon variables $\vec{\tau}_n$, $\vec{r}_n$, and $\vec{\sigma}_n$ present in the $P_\mu^{(n)}$ can be coupled to obtain the relevant part using standard techniques; this is described in detail in Appendix B. We find that

$$\langle\beta\vec{k}'|V(\omega)|\alpha\vec{k}\rangle = -16\pi^2\delta_{\alpha\beta}v(\vec{k}^2)v(\vec{k}'^2)\vec{k}'\cdot\vec{k}\tilde{\rho}(q)H^{(+)}(\omega), \quad (21)$$

$$H^{(+)}(\omega) = \tfrac{1}{3}[h_1(\omega) + 2h_2(\omega) + 2h_3(\omega) + 4h_4(\omega)],$$

$$\tilde{\rho}(q) = \int r^2\,dr\,j_0(qr)\rho(r), \quad q = |\vec{k}' - \vec{k}|,$$

where $\rho(r)$ is the nuclear density, which is normalized to the number of nucleons. This is the result of our simple model. It is a straightforward exercise to show that this invariant potential generates a crossing symmetric pion-nucleus amplitude from (17). For completeness we include this calculation in Appendix C.

B. Corrections to the impulse approximation

To make the model for the invariant potential more realistic we consider four corrections to the impulse approximation.

1. Pauli exclusion principle

The Pauli principle restricts the states into which a nucleon can scatter after interaction with a pion to be those which are not occupied by the other nucleons of the nucleus. The effects of this blocking of the πN interaction in the nuclear medium have been extensively studied,[12–18] and this effect is generally regarded as a necessary (though not dominant) correction to the impulse approximation treatment of pion-nucleus scattering. We incorporate this restriction into our model for the invariant potential by replacing the πN form factor $v(k^2)$ by $v(k^2)\beta(k)$, where $\beta(k)$ is the probability amplitude that a nucleon can change its momentum state by $\vec{k}$ and scatter into an unoccupied state. This change can be viewed as a modification of the vertex function of the πN interaction. The effect of this prescription is that the *entire* πN amplitude appearing in the invariant potential will now have $\beta(k)\beta(k')$ as a multiplicative factor. Since there are higher-order diagrams contributing to the effective pion-nucleon scattering amplitude in the nucleus which are not proportional to this factor,[15] our procedure will necessarily overestimate the role of Pauli blocking unless $\beta(k)$ is constructed as an "effective" (perhaps phenomenological) function.

A simple model for the Pauli blocking function was first constructed by Bethe[13] by considering a zero-temperature Fermi gas with equal densities of protons and neutrons. For the scattering of negative pions by nuclei, the diagram shown in Fig. 4(a) (a part of the Born term of the Chew-Low theory) will contribute to the scattering only if the intermediate proton state of momentum $(\vec{q} - \vec{k}')$ is above the Fermi sea [specified by the Fermi momentum (p_F)]. As shown by Bethe using the diagram of Fig. 4(b), the fraction of initial neutron states for which the intermediate state $(\vec{q} - \vec{k}')$ is empty is

$$F(k') = \frac{3}{4}\left(\frac{k'}{p_F}\right)\left(1 - \frac{1}{12}\frac{k'^2}{p_F^2}\right) \quad \text{for } k' \leq 2p_F$$
$$= 1 \quad \text{for } k' > 2p_F. \quad (22)$$

We note that if $\beta(k) \equiv [F(k)]^{1/2}$, then $\beta(k) \to 0$ as $k \to 0$. This, however, is an unacceptable threshold behavior for a blocking function,[15] since the theoretical analysis of pionic atom data suggests relatively small blocking as $k \to 0$.

To obtain a more realistic blocking function and, in particular, to improve its threshold behavior, we extend Bethe's analysis to include the renormalization of the occupation numbers for nucleon states in the Fermi sea. For simplicity we consider only an average occupation number n for such states. There are then three classes of πN interactions which lead to intermediate states consistent with Fermi statistics:

(a) A nucleon is initially with the Fermi sea; at the πN vertex the momentum state of the nucleon changes by $\vec{k}$ such that the nucleon enters a state above the sea.

(b) A nucleon is initially in a state above the sea; at the πN vertex the momentum state of the nucleon changes by $\vec{k}$ such that the nucleon remains above the sea.

(c) A nucleon is initially within the sea; at the πN vertex the momentum state of the nucleon changes by $\vec{k}$ such that the nucleon enters some unoccupied state within the sea.

The probability for process (a) is $nF(k)$. For (b) it is $(1-n)P_b(k)$, where $P_b(k)$ is the probability that the nucleon remains above the sea after changing its momentum state by $\vec{k}$; a reasonable approximation is to take $P_b(k)\simeq 1$. Finally, the probability for process (c) is $n[1-F(k)](1-n)$. Combining yields the probability amplitude that a nucleon can change its momentum state by $\vec{k}$ and scatter into an unoccupied state,

$$\beta(k)=\{nF(k)+(1-n)+n[1-F(k)](1-n)\}^{1/2}. \qquad (23)$$

It can be seen that $\beta(k)\geq [F(k)]^{1/2}$ for all k, and as $k\to 0$, $\beta(k)\to(1-n^2)^{1/2}$. Thus our blocking function is weaker than the expression proposed by Bethe,[13] and it has a form which provides a reasonable threshold behavior.

The average occupation probability can be determined from the results of Brueckner-Hartree-Fock calculations,[19] though in light of our previous remarks it may ultimately be better to view n as an effective parameter, thus making (23) a convenient parametrization of the blocking function. When one considers that in the nuclear interior the attenuation of the pion wave is very strong near resonance,[18] which implies that the elastic scattering becomes a peripheral process for which Pauli blocking plays a negligible role,[12] it is then clear that n must be considered an energy-dependent parameter.

2. Nuclear binding effects

As the nucleons of the target nucleus are bound in an effective potential due to their mutual interactions, we expect that this will result in a shift from the value of the impulse approximation in the energy at which the πN interaction takes place. This point was discussed on physical grounds in Ref. 1. Subsequently, in a study of pion-deuteron scattering near the πN resonance energy using a three-body model based on Faddeev's equations, Myhrer and Koltun[20] found clear numerical evidence that the potential which binds the nucleon to the target increases the pion energy at which the πN amplitudes resonate. Thus, we relate the energy of the π-nucleus interaction ω to the energy of the π-nucleon interaction $\hat{\omega}$ by

$$\hat{\omega}=\omega+E_B, \qquad (24)$$

where E_B is the average nucleon single-particle energy ($E_B<0$). This represents the simplest approximation to a more careful kinematical[21] and dynamical[12] treatment of nucleon binding. A similar approach to simulate the effects of nucleon binding has also been considered by Schmit[22] and by Kujawski and Aitkin.[21] In the present usage, (24) should be considered more an intuitive prescription rather than a systematic correction to the impulse approximation.[23]

3. Nucleon recoil and Fermi motion

In the simple model for the invariant potential there were no terms in the nuclear transition operator involving nucleon momenta as a consequence of our use of the static approximation for nucleons. As a result the invariant potential is insensitive to the nucleon momentum distribution (except as this is reflected in the nuclear form factor) as well as the recoil effects of the bound nucleons in their interaction with the scattering pion. Previous theoretical work[7,24,25] provides some indication that nucleon motion can have a strong influence on pion-nucleus scattering. We shall discuss the analysis of this effect in some detail.

It is tempting to think that nucleon motion can be taken into account by simply rewriting the projection operators of (20) in terms of the π-nucleon relative momenta.[26] Unfortunately, this is not a complete solution. The dynamics of absorption and emission produce a purely p-wave interaction only in the static limit. In reality the crossed processes produce interactions in other partial waves as well. (We hasten to add that exchange of bosons between the pion and nucleon produces interaction in all partial waves also. But here we are interested in an energy region where the absorption-emission mechanism dominates.) These effects are of first order in ω/M_N. Thus, to take the effects of nucleon motion on the off-shell πN scattering amplitude into account fully, one must improve upon the Chew-Low theory by abandoning the static approximation. Fortunately, a limited program where we take into account only correction terms linear in ω/M_N can be carried out without having to redo the Chew-Low theory. This is possible because the recoil induced admixture of the other partial waves to the dominant p-wave amplitude is fixed in the lowest order by crossing symmetry.

Let us consider the scattering event shown in Fig. 4(a), where $\vec{k}$ and $\vec{k}'$ are the initial and final pion momenta and $\vec{q}$ and $\vec{q}'(=\vec{q}-\vec{k}'+\vec{k})$ are the corresponding nucleon momenta. Since we are interested in effects of first order in ω/M_N, we neglect the change in the pion energy due to recoil. The projection operators, correct up to ω/M_N, are easily obtained by the replacements

$$\vec{k}'\to\vec{K}'=\frac{M_N\vec{k}'-\omega\vec{q}'}{M_N+\omega},\qquad \vec{k}\to\vec{K}=\frac{M_N\vec{k}-\omega\vec{q}}{M_N+\omega}. \tag{25}$$

For simplicity we confine our discussion to the isosymmetric part of the πN scattering amplitude. For the pion optical potential due to ^{12}C, the target under study in this paper, this is the quantity of interest. Retaining the same dynamical assumption that the absorption-emission process dominates over boson exchange processes, one may be tempted to write for the isosymmetric part of the dominant p-wave πN amplitude

$$\begin{aligned}\langle\vec{q}',\vec{k}'|t(\omega)|\vec{q},\vec{k}\rangle &= -4\pi v(\vec{K}^2)v(\vec{K}'^2)\langle s'|[A^{(+)}(\omega)\vec{K}\cdot\vec{K}'+A^{(-)}(\omega)i\vec{\sigma}\cdot\vec{K}'\times\vec{K}]|s\rangle\\ &= -4\pi v(\vec{k}'^2)v(\vec{k}^2)\langle s'|[A^{(+)}(\omega)\vec{k}'\cdot\vec{k}+A^{(-)}(\omega)i\vec{\sigma}\cdot\vec{k}'\times\vec{k}-(\omega/M_N)\{A^{(+)}(\omega)\tfrac{1}{2}[(\vec{k}+\vec{k}')^2+(\vec{k}+\vec{k}')\cdot(\vec{q}+\vec{q}')]\\ &\qquad +A^{(-)}(\omega)i\vec{\sigma}\cdot[\vec{k}'\times\vec{k}-\vec{q}'\times\vec{q}]\}+\cdots]|s\rangle,\end{aligned} \tag{26}$$

where s represents the nucleon spin, and for simplicity we set the nucleon at the origin of coordinates. In the second line we have retained only terms through first order in ω/M_N. Furthermore, terms linear in ω/M_N arising from the form factors have been ignored. [This is because the form factor in the Chew-Low theory is very well represented by e^{-k^2/Λ^2} and the terms arising from it are of the order $(\omega/M_N)k^2/\Lambda^2$. Since $\Lambda\sim M_N$ these terms are negligible.]

The reason (26) is not quite correct is that it cannot satisfy the crossing property of the isosymmetric part of the πN amplitude which requires that

$$\langle\vec{q}',\vec{k}'|t(\omega)|\vec{q},\vec{k}\rangle=\langle\vec{q}',-\vec{k}|t(-\omega)|\vec{q},-\vec{k}'\rangle. \tag{27}$$

Note that the two terms inside each of the two square brackets behave differently under crossing.

An obvious solution to this problem is to symmetrize (26) with the amplitude describing the crossed process. The expression resulting from this procedure (which we shall follow) has the virtues that (i) it exhibits the dominance of the p-wave interaction, (ii) it reduces to the Chew-Low result in the static limit, (iii) it is (necessarily) crossing-symmetric, and (iv) because of the last feature it includes an s-wave πN interaction term due to nucleon recoil. To understand this last feature we recall that the crossing matrix is not diagonal in orbital angular momentum, but has off-diagonal terms of order ω/M_N. The s-wave term, which is explicitly of this order, is thus a consequence of crossing.

The prescription of crossing symmetrization gives

$$\begin{aligned}\langle\vec{q}',\vec{k}'|t(\omega)|\vec{q},\vec{k}\rangle &= -4\pi v(\vec{k}^2)v(\vec{k}'^2)\langle s'|\{\tfrac{1}{2}[A^{(+)}(\omega)+A^{(+)}(-\omega)][\vec{k}'\cdot\vec{k}-(\omega/2M_N)(\vec{k}+\vec{k}')\cdot(\vec{q}+\vec{q}')]\\ &\qquad+\tfrac{1}{2}[A^{(-)}(\omega)-A^{(-)}(-\omega)]i\vec{\sigma}\cdot[\vec{k}'\times\vec{k}+(\omega/M_N)\vec{q}'\times\vec{q}]\\ &\qquad-(\omega/2M_N)\tfrac{1}{2}[A^{(+)}(\omega)-A^{(+)}(-\omega)](\vec{k}+\vec{k}')^2-(\omega/M_N)\tfrac{1}{2}[A^{(-)}(\omega)+A^{(-)}(-\omega)]i\vec{\sigma}\cdot\vec{k}'\times\vec{k}\}|s\rangle.\end{aligned} \tag{28}$$

In the static limit, $(\omega/M_N)\to 0$, $A^{(\pm)}(\omega)$ are linear combinations of the Chew-Low amplitudes, namely,

$$A^{(+)}(\omega) \xrightarrow[\omega/M_N\to 0]{} H^{(+)}(\omega)=\tfrac{1}{3}[h_1(\omega)+2h_2(\omega)+2h_3(\omega)+4h_4(\omega)]=H^{(+)}(-\omega),$$

and

$$A^{(-)}(\omega) \xrightarrow[\omega/M_N\to 0]{} H^{(-)}(\omega)=\tfrac{1}{3}[h_1(\omega)-h_2(\omega)+2h_3(\omega)-2h_4(\omega)]=-H^{(-)}(-\omega). \tag{29}$$

It then follows that the combinations

$A^{(+)}(\omega)-A^{(+)}(-\omega)$ and $A^{(-)}(\omega)+A^{(-)}(-\omega)$

must contain as a factor an odd power (≥ 1) of ω/M_N. Therefore, as long as our aim is to obtain the πN amplitude correct up to the first power of ω/M_N, we can neglect the last two terms in (28) and write

$$\langle \vec{q}', \vec{k}'|t(\omega)|\vec{q}, \vec{k}\rangle$$
$$=-4\pi v(\vec{k}^2)v(\vec{k}'^2)\langle s'|\{\tilde{H}^{(+)}(\omega)[\vec{k}'\cdot\vec{k}-(\omega/2M_N)(\vec{k}+\vec{k}')\cdot(\vec{q}+\vec{q}')]+\tilde{H}^{(-)}(\omega)i\vec{\sigma}\cdot[\vec{k}'\times\vec{k}+(\omega/M_N)\vec{q}'\times\vec{q}]\}|s\rangle, \tag{30}$$

where

$$\tilde{H}^{(+)}(\omega)=\tfrac{1}{2}[A^{(+)}(\omega)+A^{(+)}(-\omega)], \quad \tilde{H}^{(-)}(\omega)=\tfrac{1}{2}[A^{(-)}(\omega)-A^{(-)}(-\omega)]. \tag{31}$$

To interpret the new amplitude and, in particular, the quantities $\tilde{H}^{(+)}(\omega)$ and $\tilde{H}^{(-)}(\omega)$, let us go to the center of mass (c.m.) frame where $\vec{q}=-\vec{k}$ and $\vec{q}'=-\vec{k}'$. We obtain

$$\langle -\vec{k}', \vec{k}'|t(\omega)|-\vec{k}, \vec{k}\rangle$$
$$=-4\pi v(\vec{k}^2)v(\vec{k}'^2)\langle s'|\{\tilde{H}^{(+)}(\omega)(\omega/2M_N)(\vec{k}^2+\vec{k}'^2)+[1+(\omega/M_N)][\tilde{H}^{(+)}(\omega)\vec{k}'\cdot\vec{k}+\tilde{H}^{(-)}(\omega)i\vec{\sigma}\cdot\vec{k}'\times\vec{k}]\}|s\rangle. \tag{32}$$

We examine the p-wave term first. If we write the full scattering amplitude, without the static approximation, in the form

$$\langle -\vec{k}', \vec{k}'\beta|t(\omega)|-\vec{k}, \vec{k}\alpha\rangle=\langle s'\tau'|-4\pi v(\vec{k}^2)v(\vec{k}'^2)\sum_{\mu=1}^{4}h'_\mu(\omega)P_\mu(\beta\vec{k}', \alpha\vec{k})|s\tau\rangle, \tag{33}$$

with τ the nucleon isospin ($\tau=\tau'$ for the isosymmetric amplitude), then the unitarity relation requires

$$\operatorname{Im} h'_\mu(\omega_k)=[v(\vec{k}^2)]^2k^3|h'_\mu(\omega_k)|^2\frac{M_N}{M_N+\omega_k}. \tag{34}$$

The last factor is not present in the static approximation. From (34) it follows that $h'_\mu(\omega_k)$ is related to the phase shifts δ_μ by

$$h'_\mu(\omega_k)=\frac{M_N+\omega_k}{M_N}\frac{e^{i\delta_\mu(\omega_k)}\sin\delta_\mu(\omega_k)}{k^3[v(\vec{k}^2)]^2}, \tag{35}$$

while in the static approximation

$$h_\mu(\omega_k)=\frac{e^{i\delta_\mu(\omega_k)}\sin\delta_\mu(\omega_k)}{k^3[v(\vec{k}^2)]^2}. \tag{36}$$

Noting that

$$[1+(\omega_k/M_N)]\tilde{H}^{(+)}(\omega_k)$$
$$=\tfrac{1}{3}[h'_1(\omega_k)+2h'_2(\omega_k)+2h'_3(\omega_k)+4h'_4(\omega_k)]$$

and

$$[1+(\omega_k/M_N)]\tilde{H}^{(-)}(\omega_k)$$
$$=\tfrac{1}{3}[h'_1(\omega_k)-h'_2(\omega_k)+2h'_3(\omega_k)-2h'_4(\omega_k)],$$

it then follows that we can approximately equate $\tilde{H}^{(\pm)}(\omega)$ to the Chew-Low amplitudes $H^{(\pm)}(\omega)$, defined in (29), as long as the constituent amplitudes $h_\mu(\omega)$ produce phase shifts in reasonable agreement with experimental data.

The s-wave term which appears in (32) is by no means the full s-wave amplitude which has contributions from the boson exchange process. Even if the boson exchange contributions are insignificant the s-wave term is still not quite right as it cannot be unitary when the p-wave term is unitary. But the lack of unitarity is of order (ω/M_N). It should also be stressed that the terms which occur due to boson exchange are likely to have a different functional dependence on $\vec{k}$, $\vec{k}'$, and ω. Even though we do not have a satisfactory theory of the s-wave πN off-shell scattering amplitude, (30) is an improvement over the static πN amplitude.

It is straightforward to generate the π-nucleus optical potential using (30) and techniques similar to those used in Appendix B. The operator Λ in the nuclear space associated with the $(\vec{k}+\vec{k}')\cdot(\vec{q}+\vec{q}')$ term is defined by the relation

$$\langle\vec{q}'|\Lambda|\vec{q}\rangle=(2\pi)^3\delta(\vec{q}'+\vec{k}'-\vec{q}-\vec{k})(\vec{k}+\vec{k}')\cdot(\vec{q}+\vec{q}'),$$

and so it has the form

$$\Lambda = (\vec{k}' + \vec{k})\cdot\{-i\vec{\nabla}e^{-i(\vec{k}'-\vec{k})\cdot\vec{r}} - i e^{-i(\vec{k}'-\vec{k})\cdot\vec{r}}\overleftarrow{\nabla}\}\,.$$

Since this operator has the property that its time-reversed form is the negative of its adjoint, it does not contribute to the optical potential when the target nucleus wave function is unchanged under time reversal. This is the situation with a zero angular momentum target.

Finally, the optical potential for a zero angular momentum, zero isospin target nucleus is given by

$$\langle \beta\vec{k}'|V(\omega)|\alpha\vec{k}\rangle = -16\pi^2\delta_{\alpha\beta}v(\vec{k}^2)v(\vec{k}'^2)\beta(\vec{k})\beta(\vec{k}') \times[\vec{k}'\cdot\vec{k}H^{(+)}(\omega)\tilde{\rho}(|\vec{k}'-\vec{k}|) - (\omega/M_N)H^{(-)}(\omega)\rho'(|\vec{k}'-\vec{k}|)]\,, \tag{37}$$

where we have equated each $\tilde{H}$ with the corresponding H. The quantities $\tilde{\rho}(q)$ and q have been defined in (21). The quantity $\rho'(q)$ is given by the expression

$$\rho'(q) = \int dr\, j_0(qr)\frac{d}{dr}r\sum_{nlj}\langle\vec{\sigma}\cdot\vec{L}\rangle_{nlj}\rho_{nlj}(r)\,. \tag{38}$$

The sum runs over the occupied states, $\langle\vec{\sigma}\cdot\vec{L}\rangle_{nlj} = j(j+1) - l(l+1) - \frac{3}{4}$ and $\rho_{nlj}(r) = (2j+1)|u_{nlj}(r)|^2$. The result (38) is specialized to the case of closed shell nuclei. From the above it can be seen that the linear combination of Chew-Low amplitudes $H^{(-)}$ results from a spin-orbit coupling of the nucleons, an effect which has previously been considered in elastic electron scattering from nuclei,[27] but has not previously been considered in pion-nucleus scattering.[28] Our result for the nucleon motion correction to the impulse approximation differs from other work[7,25] primarily in that we do not retain terms which are not crossing symmetric.

4. *Energy transformation*

As the potential which we use to describe pion-nucleus scattering is a Lorentz scalar we may directly evaluate the Klein-Gordon scattering equation (17) for the elastic amplitude in any frame which proves convenient. For the numerical analysis presented in the following section we choose to solve this equation with variables referred to the pion-nucleus center of mass frame. (Our previous remarks[8] imply that to a good approximation this frame is equivalent to the lab frame for the range of pion energies we are considering.) However, since the argument of the Chew-Low amplitudes used in the invariant potential refers to the pion energy in the π-nucleon center of mass frame,[29] we must carry out the well-defined kinematic transformation of energy between these two frames. It should be stressed that in this formalism we need not confront the problem of relating off-shell amplitudes in different frames. This problem does occur in the treatment of pion-nucleus scattering based on the Lippmann-Schwinger equation (10). This is critically discussed by Heller, Bohannon, and Tabakin,[4] where a prescription for carrying out a transformation appropriate to their potential formalism is presented and where references to other work can be found.

The necessary energy transformation is effected using the Lorentz invariant $s = (p_N + p_\pi)^2$, where $p_N(p_\pi)$ is the nucleon (pion) energy-momentum four-vector. Taking the pion and nucleon to be on their mass shells and the pion on its energy shell, we have in the lab frame (ignoring nucleon Fermi motion)

$$s = M_N^2 + m_\pi^2 + 2M_N\omega_k\,, \tag{39}$$

while in the πN center of mass

$$s \simeq M_N^2 + m_\pi^2 + 2(\hat{\omega}_k^2 - m_\pi^2) + 2\hat{\omega}_k M_N\,. \tag{40}$$

Thus,

$$\hat{\omega}_k = -\tfrac{1}{2}M_N + \tfrac{1}{2}[M_N^2 + 4(M_N\omega_k + m_\pi^2)]^{1/2}\,.$$

It is apparent that we have neglected terms in nucleon momentum in (40). Though this is somewhat inconsistent with the analysis of the previous section, it is a necessary approximation in view of the interpretation of the Chew-Low amplitudes.[29] This therefore defines a limitation of our present model. Though the improvement of these amplitudes to include nucleon motion terms (of order M_N^{-1}) is possible, this will not be considered further here.

If we now include the effects of binding of the nucleons we necessarily relinquish the previous condition that the nucleons are on the mass shell. Including the shift in energy due to nucleon binding, we have

$$\hat{\omega}_k = -\tfrac{1}{2}M'_N + \tfrac{1}{2}[M'^2_N + 4(M'_N\omega_k + m_\pi^2)]^{1/2} + E_B\,, \tag{41}$$

where $M'_N \cong M_N + E_B$ for $|E_B/M_N| \ll 1$.

IV. NUMERICAL RESULTS

Given a model for the invariant potential V, the elastic π-nucleus scattering amplitude $\hat{T}$ is obtained as the solution of (17). This equation can be conveniently solved in momentum space by expanding in partial waves and using the methods

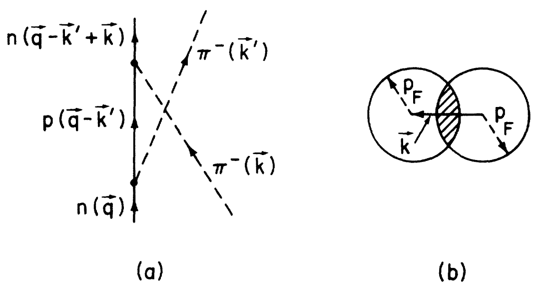

FIG. 4. (a) Interpreted as a Feynman diagram, this is the main graph for the formation of the (3,3) resonance in the scattering of negative pions by nuclei. (b) From these two intersecting spheres of radius p_F (the Fermi momentum), the fraction F of initial neutron states in (a) for which the intermediate proton state is empty can be determined from $F=1-$(shaded volume)/$(\frac{4}{3}\pi p_F^3)$.

of Noyes[30] and Kowalski.[31]

To carry out the numerical solution of (17) we must first complete our definition of the invariant potential by choosing expressions for the πN form factor $v(k^2)$ and the nuclear ground state density $\rho(r)$. For the form factor we have used a Gaussian function with a range of the order of the nucleon Compton wavelength

$$v(\vec{k}^2)=e^{-\vec{k}^2/49m_\pi^2}\,. \qquad (42)$$

This form is suggested by the Chew-Low theory, as there the form factor is given as simply the Fourier transform of the nucleon's density. We have found that our optical potential pion-nucleus calculations are rather insensitive to the range of this form factor; e.g., doubling the range affects the calculations by an average of only a few percent.

For the full nuclear density we have used

$$\rho(r)=12[a^3\pi^{3/2}(1+2a^2/b^2)]^{-1}(1+\tfrac{4}{3}r^2/b^2)e^{-r^2/a^2}\,, \qquad (43)$$

which is a modification of the density predicted by the harmonic oscillator shell model (which has $a=b$). This form has been used to parametrize electron scattering from ^{12}C.[32] The values of a and b, with corrections for the nucleon's spatial extension,[24] have been determined as $a=1.59$ fm, $b=1.66$ fm.

When considering the spin-orbit correction to the potential resulting from nucleon Fermi motion and recoil, we also need an expression for the density of the eight $p_{3/2}$ nucleons in ^{12}C. In accordance with our use of the modified shell model density (43), we take

$$\sum_{nlj}\langle\vec{\sigma}\cdot\vec{L}\rangle_{lj}\,\rho_{nlj}(r)=\frac{4b^2}{\pi^{3/2}a^5}\,\frac{4}{3}\,\frac{r^2}{b^2}e^{-r^2/a^2}\,. \qquad (44)$$

We now examine the results of our numerical study of (17) for $\pi-{}^{12}$C elastic scattering.

The dash-dotted curve in Fig. 5 shows the results of calculations for the total $\pi-{}^{12}$C cross section with the potential of (21), which represents the impulse approximation. The data points shown here and in subsequent figures are from the work of Binon *et al.*[33] The most striking feature of this calculation is that the peak in the cross section occurs at the pion energy of only 85 MeV. This should be compared with the energy of the peak in the experimental $\pi-{}^{12}$C cross section at 145 MeV, and the energy of the peak in the πN cross section at 180 MeV.

We have found that the reason for this downward shift in the calculated cross section lies in the energy dependence of the linear combination of Chew-Low amplitudes $H^{(+)}$ which appears in V and, in particular, the energy broadening of the π-nucleus amplitude. In Fig. 6 we show the imaginary parts of $H^{(+)}$ (scaled by a factor of 3) and the (3,3) Chew-Low amplitude h_4 (scaled by a factor of 4). We see that the peak in Im$H^{(+)}$ occurs approximately 17 MeV below the peak in the πN amplitude. When we consider the effects of nuclear binding it will be seen that the downward shift in the peak of Im$H^{(+)}$ compared to Imh_4 is directly translated into a downward shift in the $\pi-{}^{12}$C cross section compared to the πN cross section. Furthermore, as pointed out by Landau, Phatak, and Tabakin,[24] the presence of the nuclear form factor in V and the higher order multiple scattering obtained when solving the scattering

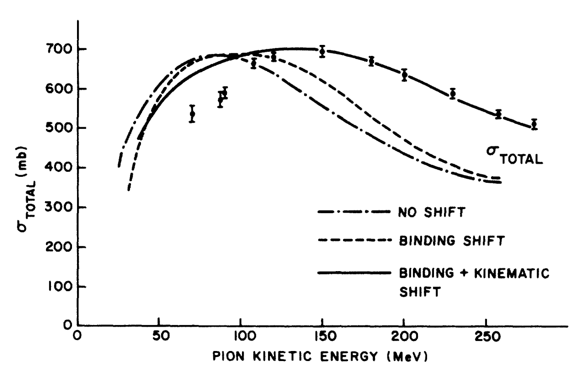

FIG. 5. $\pi-{}^{12}$C total cross sections as a function of the pion lab kinetic energy showing the effects of including corrections for nucleon binding and a kinematic transformation applied to the energy at which the πN interaction in the nucleus is evaluated.

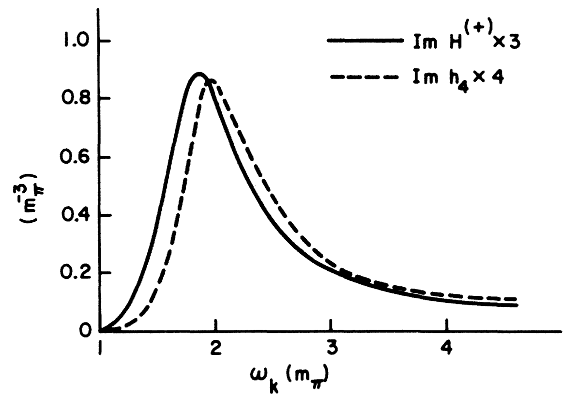

FIG. 6. The imaginary parts of $H^{(+)}$ (scaled by a factor of 3) and the (3, 3) Chew-Low amplitude h_4 (scaled by a factor of 4) are shown as functions of the pion energy in the πN c.m.

equation for the π-nucleus amplitude both tend to broaden the πN amplitude; and this results in the π-nucleus amplitude $\hat{T}$ having a much slower energy variation than the πN amplitude. Then, since the total cross section is obtained from the scattering amplitude by

$$\sigma_{\rm tot}(\vec{k}) = -\frac{4\pi}{k}\sum_{l}(2l+1)\,{\rm Im}\,\hat{T}_l(k,k), \tag{45}$$

the presence of the k^{-1} factor in conjunction with the broadened π-nucleus amplitude can be seen to result in a substantial shift in the π-nucleus cross section peak.

Turning now to the corrections to the impulse approximation, we first consider the effects of nuclear binding. As discussed in Sec. III B, the binding of the target nucleon to the residual nucleus suggests that we evaluate the πN amplitudes appearing in the potential of (21) at an energy $\hat{\omega}_k$ which is shifted from the energy of the π-nucleus interaction ω_k according to $\hat{\omega}_k = \omega_k + E_B$, for E_B the nucleon binding energy. In addition, there is also a kinematic shift which results from the fact that in the present model the energy of the πN interaction must be referred to the πN c.m. Including this transformation results in the relation between ω_k and $\hat{\omega}_k$ which is given by (41).

In Fig. 5 we show the results of the calculated π-nucleus total cross sections in which the energy of the πN amplitudes is the same as the energy of the π-nucleus interaction (dash-dotted curve), is shifted by the nucleon binding energy (dashed curve), and is shifted by binding and includes the kinematic transformation (solid curve). For the binding energy we have used the value $E_B = -0.1 m_\pi$. It can be seen that by including the effects of the nuclear binding and a kinematic shift in the energy at which we evaluate the πN amplitudes, a significant improvement results in the calculated cross section. Consequently, in all subsequent numerical results we relate the energy of the πN interaction to the energy of the π-nucleus interaction in accordance with (41).

The restriction on the pion-bound nucleon interaction imposed by the exclusion principle ("Pauli blocking") is incorporated into our calculation by replacing the πN form factor $v(k^2)$ appearing in the invariant potential by $v(k^2)\beta(k)$, where $\beta(k)$ is the probability amplitude that a nucleon can change its momentum state by $\vec{k}$ and scatter into an unoccupied state. For ^{12}C the Fermi momentum is $p_F = 221$ MeV/c, so from its defining equation (23), $\beta(k)$ is completely determined once we specify the average occupation probability of the states in the Fermi sea, n. Because of the singular nature of the nucleon-nucleon interaction, the ground state of a nucleus has a complicated short-range correlation structure and, therefore, the normally occupied single-particle orbitals are expected to be depleted with a significant probability and, correspondingly, high-momentum continuum orbitals become occupied. Thus, n is expected to differ from one by a significant amount. In Fig. 7 we display the form of $\beta(k)$ for several values of n. For use in our study of the role of Pauli blocking in $\pi - {}^{12}$C scattering, we have taken $n = 0.8$, as this value is in approximate agreement

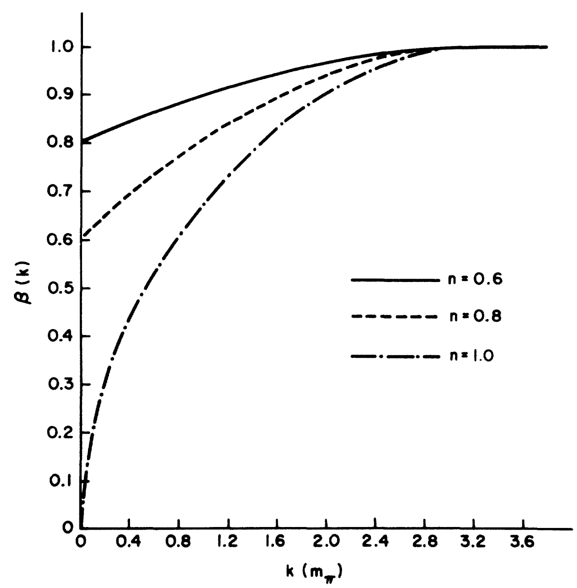

FIG. 7. $\beta(k)$, the probability amplitude that a nucleon can change its momentum state by $\vec{k}$ and scatter into an unoccupied state, is shown as a function of k for various values of n, the average occupation probability of states in the Fermi sea.

with the results of various Brueckner-Hartree-Fock calculations.[19] However, from our previous discussion it is clear that this value will result in an overestimate of the magnitude of the blocking correction.

In Fig. 8 we show the effects of including the Pauli blocking correction in the calculation of the $\pi - {}^{12}C$ total cross section. As expected, the greatest change occurs at lower pion energies where the probability of the target nucleon absorbing enough momentum to enter an unoccupied state is smallest.

(37) exhibits the invariant potential which includes a correction for the recoil and Fermi motion of the target nucleons, in addition to the above corrections. Our procedure for introducing this correction is based on the elimination of the static constraint on the target nucleons in the context of the Chew-Low theory. In Fig. 9 we show as the dash-dotted curve the results of the calculation of the total cross section using the invariant potential given in (37). The other two curves shown here are the same as those of Fig. 8 and are included for comparison. We find that for energies smaller than 75 MeV the term which is of first order in ω/M_N adds in phase to the main term of Eq. (37) and thus enhances the total cross section. At 50 MeV, the cross section is increased by nearly 30%. For energies larger than 75 MeV the correction term has the opposite sign and tends to reduce the cross section. Over the energy range 75-200 MeV the reduction is less than 1%. Since the Chew-Low theory provides a rather poor description of amplitudes for channels other than (3, 3), we expect that a more accurate description of these πN amplitudes could quantitatively but not qualitatively modify the results of the nucleon motion correction.

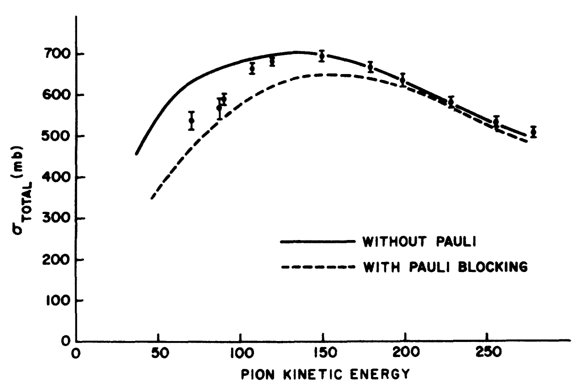

FIG. 8. π-${}^{12}C$ total cross sections showing the effects of including a correction for Pauli blocking in the model for the invariant potential.

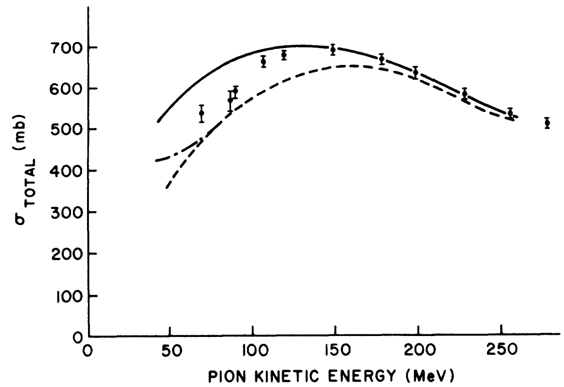

FIG. 9. π-${}^{12}C$ total cross sections showing the effects of including a correction for Fermi motion in the model for the π-nucleus invariant potential. The solid and dashed curves are the same as the corresponding curves in Fig. 8.

V. SUMMARY AND DISCUSSION

In this work we have presented a theoretical model for the analysis of elastic pion-nucleus scattering which has been developed from the exact propagator of a pion in the presence of a nucleus. We have demonstrated how a crossing symmetric elastic amplitude can be generated from a potential constructed from invariant pion-nucleon amplitudes. This potential has been evaluated in the impulse approximation retaining only single-nucleon processes. Four important corrections to this treatment of the potential have been discussed in detail and compared with previous work. Considering scattering from ${}^{12}C$ for the illustration of our theoretical results, we have shown that corrections to the impulse approximation treatment of the invariant potential for the binding and kinematic shift in the energy of the πN interaction have a large effect on calculated cross sections. We have presented an estimate of the effects of Pauli blocking in the π-nucleus interaction, but more importantly, we have obtained an expression for the blocking function which possesses a reasonable low momentum behavior and which can be used as a convenient parametrization of the blocking phenomenon. And finally, we have demonstrated that crossing symmetry has a very important role in determining the contribution of nucleon momenta to the nuclear transition operator for terms through first order in the ratio of pion energy to nucleon mass. The resulting correction to the invariant potential accounting for the Fermi motion and recoil of the target nucleons was found to be important only for pion energies below about 75 MeV.

It is clear that still more work needs to be done before one can claim to have a theoretically complete description of pion-nucleus scattering. In relation to the present work we note in particular that a better treatment of Pauli blocking is most definitely needed. The further analysis of this process is expected to be quite difficult since one must include details of the single-particle structure of the nucleus as well as the energy dependence of the πN interaction to obtain a better model of the blocking phenomenon. It is also clear that there are several additional physical effects which have not been included in our present model but which can be important in understanding the process of elastic pion-nucleus scattering. These include the multinucleon processes and the elastic nuclear rescattering of pions.

An estimate of the relative importance of the multinucleon processes has been obtained by Eisenberg[34] using a multiple scattering formulation of the optical potential. He finds that at the (3, 3) resonance energy the two-particle correlation diagrams (the simplest multinucleon process) contribute about 30% (in magnitude) of the single-nucleon processes to the optical potential, with higher particle correlation terms less important. We also know from the analysis of intermediate and high-energy elastic nucleon-nucleus scattering[35] that the higher order terms become increasingly important at angles away from the diffraction region. This should also be true for the pion-nucleus problem.

The strong attractiveness of the pion-nucleus interaction near the resonance further suggests that the nuclear elastic rescattering of pions also may be an important process. As the field-theoretic equation of Low[36] provides a natural way (i.e., through off-shell unitarity) of determining an elastic scattering amplitude which includes the rescattering of intermediate state pions, we have examined the numerical solution of this equation for $\pi - {}^{12}C$ scattering. Though the details of this work will be reported elsewhere,[37] we note here that our results indicate that rescattering enhances the strength of the pion-nucleus interaction. Thus it appears that both the multinucleon processes and the nuclear rescattering of pions are necessary ingredients for a theoretically complete description of pion-nucleus elastic scattering.

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APPENDIX A

Using the Hamiltonian defined in (11) and (12), the set of rules which allow the expression of the single-pion propagator $\langle \beta \vec{k}' | G(z) | \alpha \vec{k} \rangle$ in a series of pion-nucleon diagrams are obtained using the standard analysis[10] based on Wick's theorem for time-ordered products and the basic contractions for the pion, nucleon, and antinucleon (interaction picture) operators. From the explicit evaluation of a series of terms of the Wick-Dyson expansion of the propagator we have derived, by induction, the following rules:

1. At each vertex (πNN, $\pi \overline{N}\overline{N}$, or $\pi N\overline{N}$) associate the vertex function defined by the π-nucleon interaction Hamiltonian $H_{\pi N}$.
2. Nucleon lines directed upward stand for particle states (above the Fermi sea) and those directed downward stand for hole states (below the Fermi sea). The rule for antinucleon lines is not stated as we will not refer to such lines.
3. All internal pion lines are directed upwards.
4. The rule for associating energy with the external pion line requires that we establish a rule for its direction. The required rule is as follows. Replace the two external line segments by a single directed line from the exit point of the external line to the entry point. The associated energy is z.
5. Between successive interactions there is an energy denominator equal to the sum of energies of all downward lines (hole lines and possibly the external line) minus the sum of energies of all upward lines.
6. Sum over all internal line variables.

As the pion-nucleus diagrams represent a formal summation of the πN diagrams, these rules essentially determine the analytic expression corresponding to any pion-nucleus diagram. We need to add only that the open circles of Fig. 2, which constitute the basic set of πN diagrams, are represented by the operator D.

APPENDIX B

In relation to (21) we wish to evaluate the nuclear expectation value

$$a_\mu = \langle \Psi | \sum_{n=1}^{A} P_\mu^{(n)}(\beta \vec{k}', \alpha \vec{k}) | \Psi \rangle \ , \qquad \text{(B1)}$$

for $|\Psi\rangle$ having the quantum numbers $(J^P, I) = (0^+, 0)$, so that only the part of the operator shown which is an even parity scalar and isoscalar will contribute. We illustrate the technique by considering a_1,

$$a_1 = \tfrac{1}{3} \sum_{n=1}^{A} \langle \Psi | e^{i(\vec{k} - \vec{k}')\cdot \vec{r}_n} \tau_\beta^{(n)} \tau_\alpha^{(n)} \vec{\sigma}_n \cdot \vec{k}' \vec{\sigma}_n \cdot \vec{k} | \Psi \rangle \ , \qquad \text{(B2)}$$

where (20) has been used to determine $P_1^{(n)}$.

As the coupling of the nucleon isospin operator $\vec{\tau}$ to the position or spin vectors is not physically defined, the only contribution from the isospin factor will occur when $\alpha = \beta$ so that $\tau_\beta^{(n)} \tau_{\alpha=\beta}^{(n)} = 1$.

Thus,

$$a_1 = \tfrac{1}{3}\delta_{\alpha\beta}\vec{k}'\cdot\vec{k}\sum_{n=1}^{A}\langle\Psi|e^{i(\vec{k}-\vec{k}')\cdot\vec{r}_n}|\Psi\rangle + \tfrac{i}{3}\delta_{\alpha\beta}\sum_{n=1}^{A}\langle\Psi|e^{i(\vec{k}-\vec{k}')\cdot\vec{r}_n}\vec{\sigma}_n\cdot\vec{k}'\times\vec{k}|\Psi\rangle\,, \quad \text{(B3)}$$

where $\vec{\sigma}_n\cdot\vec{k}'\vec{\sigma}_n\cdot\vec{k} = \vec{k}'\cdot\vec{k} + i\vec{\sigma}_n\cdot(\vec{k}'\times\vec{k})$ has been used. This expression is evaluated using the expansion

$$e^{i(\vec{k}-\vec{k}')\cdot\vec{r}_n} = 4\pi\sum_{lm} i^l j_l(qr_n)Y^*_{lm}(\hat{q})Y_{lm}(\hat{r}_n)\,, \quad \text{(B4)}$$

where $q=|\vec{k}'-\vec{k}|$ and $\hat{q}=(\vec{k}'-\vec{k})/q$. In the first term of (B3) the only contribution will result from the scalar term in (B4) involving $Y_{00}(\hat{r}_n)$. As there is no way to couple $Y_{lM}(\hat{r}_n)$ and $\vec{\sigma}_n$ to form an even parity configuration space scalar, the second term in (B3) vanishes. Therefore

$$a_1 = \tfrac{1}{3}\delta_{\alpha\beta}\vec{k}'\cdot\vec{k}\sum_{n=1}^{A}\langle\Psi|j_0(qr_n)|\Psi\rangle = \tfrac{4}{3}\pi\delta_{\alpha\beta}\vec{k}'\cdot\vec{k}\,\tilde{\rho}(q)\,, \quad \text{(B5)}$$

with

$$\tilde{\rho}(q) = \int r^2 dr\, j_0(qr)\rho(r)\,,$$

and where the nuclear density is normalized to the nucleon number.

Proceeding with a similar analysis for the remaining terms, we find

$$a_2 = a_3 = \tfrac{8}{3}\pi\delta_{\alpha\beta}\vec{k}'\cdot\vec{k}\tilde{\rho}(q)\,,\qquad a_4 = \tfrac{16}{3}\pi\delta_{\alpha\beta}\vec{k}'\cdot\vec{k}\tilde{\rho}(q)\,. \quad \text{(B6)}$$

So,

$$\langle\beta\vec{k}'|V(z)|\alpha\vec{k}\rangle = -4\pi v(\vec{k}^2)v(\vec{k}'^2)\sum_{\mu=1}^{4}a_\mu h_\mu(z) = -16\pi^2 v(\vec{k}^2)v(\vec{k}'^2)\delta_{\alpha\beta}\vec{k}'\cdot\vec{k}\times\tilde{\rho}(q)H^{(+)}(z)\,, \quad \text{(B7)}$$

$$H^{(+)}(z) = \tfrac{1}{3}[h_1(z)+2h_2(z)+2h_3(z)+4h_4(z)]\,.$$

APPENDIX C

Using the model of (21) for the invariant potential, the crossing relation for the pion-nucleus elastic amplitude can be derived. First we verify the crossing relation for this model of the invariant potential. For the crossed process we have

$$\langle\bar{\alpha}, -\vec{k}|V(-\omega)|\bar{\beta}, -\vec{k}'\rangle = -16\pi^2\delta_{\bar{\alpha}\bar{\beta}}v(\vec{k}^2)v(\vec{k}'^2)\vec{k}'\cdot\vec{k}\tilde{\rho}(q)H^{(+)}(-\omega)\,, \quad \text{(C1)}$$

where $|\bar{\beta}, -\vec{k}'\rangle$ is defined as a pion state of isospin $\bar{\beta}$ and three-momentum $-\vec{k}'$. Working with the Cartesian representation of the pion isospin states so that $\alpha, \beta=(1,2,3)$, we have $\bar{\alpha}=\alpha$, $\bar{\beta}=\beta$.[38] It is easily shown, using the crossing relation for the Chew-Low amplitudes, that $H^{(+)}(\omega)$ is even under crossing[37]; i.e., $H^{(+)}(\omega)=H^{(+)}(-\omega)$. From (21) and (C1), the crossing relation for matrix elements of the invariant potential is therefore

$$\langle\alpha, -\vec{k}|V(-\omega)|\beta, -\vec{k}'\rangle = \langle\beta\vec{k}'|V(\omega)|\alpha\vec{k}\rangle\,. \quad \text{(C2)}$$

This is a necessary condition which follows from the nature of the diagrams represented by this potential.

We can obtain a useful operator statement of crossing for the invariant potential using the antiunitary time-reversal operator θ. Defining $\theta^{-1}V(-\omega)\theta \equiv V^{(T)}(-\omega)$, we readily obtain for the matrix elements of the invariant potential

$$\langle\alpha, -\vec{k}|V(-\omega)|\beta, -\vec{k}'\rangle = \langle\alpha\vec{k}|V^{(T)}(-\omega)|\beta\vec{k}'\rangle^*\,. \quad \text{(C3)}$$

Comparing this with (C2) gives

$$V^{(T)}(-\omega) = V^\dagger(\omega) \quad \text{(C4)}$$

as the operator statement of crossing for the invariant potential. The elastic amplitude $\hat{T}$ can be shown to satisfy an identical relation by first re-expressing (17) in a closed form in V

$$\hat{T}(-\omega) = V(-\omega) + V(-\omega)[\omega^2 - h^2 - V(-\omega)]^{-1}V(-\omega)\,.$$

Using θ and (C4) gives

$$\hat{T}^{(T)}(-\omega) = \hat{T}^\dagger(\omega)\,. \quad \text{(C5)}$$

Repeating the analysis which led to (C3) for matrix elements of $\hat{T}$ and using (C5) then yields

$$\langle\alpha, -\vec{k}|\hat{T}(-\omega)|\beta, -\vec{k}'\rangle = \langle\beta\vec{k}'|\hat{T}(\omega)|\alpha\vec{k}\rangle\,. \quad \text{(C6)}$$

This is the expression of crossing symmetry in pion-nucleus scattering.

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