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Relativistic three-body calculation of πd scattering*

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We present a unitary, Lorentz-invariant three-body calculation of pion-deuteron elastic scattering, based upon the idea of quasiparticle-dominated two-body interactions. We make detailed comparisons of these results with those of a conventional fixed-scatterer approach and find that the fixed-nucleon calculation does not adequately reproduce the three-body results, demonstrating the importance of properly treating the three-body kinematics (i.e., of including nucleon recoil and isobar propagation). The multiple scattering expansion converges much more rapidly in the three-body approach than in the fixed-scatterer calculation. Intermediate nucleon-nucleon interactions play an important role, giving contributions to the scattering amplitude of the same order as those given by pion multiple scattering; these effects are especially significant for back-angle scattering. Finally, we compare our results with the available experimental data for the πd total and integrated elastic cross sections and obtain good agreement. Nucleon spin is neglected in all calculations.

[NUCLEAR REACTIONS ^{2}H(π, π), E=80–240 MeV; relativistic three-body calculation of elastic scattering.]

I. INTRODUCTION

Theoretical treatments of hadron-nucleus interactions are almost universally based upon the multiple scattering picture,[1] in which the projectile sequentially scatters from the various nucleons in the nucleus. For light nuclei the multiple scattering series can be explicitly summed, while for heavy nuclei an optical potential is introduced which, when inserted into the projectile equation of motion, is intended to reproduce as closely as possible the original multiple scattering series. However, in either case a fundamental approximation is made to reduce the original many-body problem to a two-body projectile-nucleus problem. For example, the nucleons can be "frozen" through the application of closure on intermediate nuclear states, resulting in a comparatively simple calculational scheme: one computes the projectile scattering amplitude from a set of spatially fixed nucleons and then averages this amplitude over the possible target nucleon configurations (given by the nuclear density as measured in electron scattering). Certain simple corrections can be applied to this picture; for example, the elementary projectile-nucleon scattering amplitude used as input to the calculation can be "Fermi-averaged" and/or evaluated at a shifted energy to account crudely for nucleon binding. Nevertheless, the fact remains that the fixed-scatterer calculations neglect the nuclear dynamics during intermediate stages of the scattering process and that this approximation is extremely difficult to check quantitatively.

The above comments apply to the description of any hadron-nucleus interaction. However, the standard fixed-scatterer approximation may be particularly suspect in the very interesting case of pion scattering at energies near the 3-3 resonance. First, the energy variation of the πN cross section implies that the Fermi motion and binding effects may be especially important here. In addition, this energy variation implies that the πN system has a long interaction time and can be thought to propagate as a quasiparticle or isobar before decaying back into the πN channel. At resonance, this propagation distance is on the order of half the average internucleon separation in nuclei. Given this situation, we can expect that nucleon-nucleon interactions may be important in pion-nucleus reactions, particularly in those such as backward elastic scattering which involve large momentum transfer. A better understanding of the limitations of fixed-scatterer approaches is important in view of the large amount of pion-nucleus reaction data soon to be available.

The simplest case for which we can investigate these questions is pion-deuteron elastic scattering. In this paper, we present a relativistic three-body calculation of this process and make detailed comparisons with a standard fixed-nucleon calculation. The model is that of a pion interacting with an s-state deuteron via a p-wave πN resonant interaction. The calculation respects two-and three-body unitarity and Lorentz invariance and provides an excellent theoretical testing ground for the reliability of the fixed-scatterer approach. The relativistic aspect of the calculation is important both because the pion is highly relativistic and because very large momentum transfers are imparted to the deuteron. In most of our calculations we neglect nucleon spin and isospin, thereby limiting our calculation to one of reasonable size, but at the same time precluding a direct comparison with data; rather, we focus upon the question of the reliability of the fixed-scatterer results in comparison with those of the full three-body treatment. Nevertheless, we finally do include isospin in some calculations and compare these results with πd total and integrated elastic cross section data.

Previous applications of three-body theory have been made to the problem of πd scattering in the resonance region. Myhrer and Koltun[2] performed a nonrelativistic calculation and neglected spin and isospin. Relativistic calculations have been performed by Brayshaw[3] and by Mandelzweig, Garcilazo, and Eisenberg.[4] Brayshaw applied his boundary condition model to only one partial wave influenced by the resonant 3-3 interaction. Mandelzweig *et al.* evaluate all partial waves, including spin and isospin, but neglect the important NN interaction and perform nonrelativistic transformations between the two- and three-body c.m. systems. As mentioned above, our aim is to present a fully unitary, Lorentz invariant calculation, to compare in detail these results with those of a standard fixed-scatterer calculation, and thereby to test the approximations made in conventional multiple scattering approaches.[5]

The relativistic three-body theory which we employ is that developed by Aaron, Amado, and Young (AAY)[6] and is described in Sec. II. Basically, they assume that the two-body interactions are dominated by a bound state or isobar (equivalently, a separable interaction) and write down linear integral equations for the scattering.[7] At this point, the Born terms and propagators which enter the equations are not specified; only Lorentz invariance is imposed. However, the isobar assumption relates the breakup amplitude to that describing elastic scattering from the isobar, and the imposition of unitarity then relates the discontinuities of the Born terms and propagators to the interaction parameters and mass-shell δ functions. Finally, the assumption that these functions have no further discontinuities beyond those required by unitarity allows one to write dispersion relations which fully determine the three-body equations. It is clear that these equations are not unique; rather, they are the simplest which incorporate Lorentz invariance and two- and three-body unitarity. This procedure will be recognized as similar to that introduced by Blankenbecler and Sugar[8] and leads to three-dimensional Lippmann-Schwinger type equations.

The solutions to these equations for πd elastic scattering will provide a theoretical testing ground for the standard fixed-scatterer approach. In Sec. III, we describe briefly our multiple scattering calculation. We compute the single and double scattering terms, including corrections due to Fermi motion, nucleon binding, and frame transformation of the πN amplitude. Section IV contains the numerical results and comparisons (without nucleon spin and isospin). Our over-all conclusion is that the fixed-nucleon calculations are not quantitatively successful in reproducing the elastic scattering cross section in the vicinity of the resonance. In fact, we find that even the single scattering term is in appreciable error for backward scattering, indicating the importance of properly treating the nucleon recoil. Furthermore, effects due to intermediate nucleon-nucleon rescattering are quite important, and, of course, such terms are completely outside the usual multiple scattering framework.

Section V contains the numerical results including isospin and the comparison with data. Intermediate nucleon-nucleon interactions are found to reduce even the forward cross section for πd elastic scattering by about 10%.

The implications of our results will be discussed briefly in a concluding section.

II. RELATIVISTIC THREE-BODY CALCULATION

We start with a review of the three-body theory developed by AAY. The essential ideas are that the elementary two-body interactions are isobar dominated and that the forms of the Born terms and propagators are fixed by unitarity. These ideas will be applied first to the two-body problem, both to review the Blankenbecler-Sugar[8] technique and to define our basic two-body interactions. The reader is referred to AAY for a more detailed discussion of the theory.

A. Two-body interaction

The πN and NN interactions in our model are represented by Fig. 1; namely, the πN interaction proceeds by p-wave coupling to the Δ isobar, while the NN interaction is restricted to the bound state, s-wave deuteron channel. The vertex functions $v(k)$ and $u(k)$, where k is the relative momentum, describe dissociation of the quasiparticle. Therefore, πN scattering is shown schematically in Fig. 2, with the heavy circle appearing on the propagator in the last diagram corresponding to the insertion of all πN "bubbles," producing the renormalized or dressed Δ propagator. Since the πN channel is open, this Δ self energy produces both a mass shift and a width corresponding roughly to a Breit-Wigner shape for the dressed propagator.

In order to specify the two-body relativistic equation corresponding to Fig. 2, we resort to the Blankenbecler-Sugar[8] prescription. Assume a linear integral equation of the Bethe-Salpeter type

$$T_{pq}(s) = V_{pq} + \int \frac{d^4k}{(2\pi)^4} V_{pk} G_k(s^+) T_{kq}(s), \quad (1)$$

where k is the relative momentum, V_{pq} is a Born term (the first diagram on the right-hand side in Fig. 2), and $G_k(s)$ is an as yet unspecified propagator for the πN system. However, two-body unitarity requires that

$$G_k(s^+) - G_k(s^-) = (2\pi)^2 i \delta^+(k_1^2 - m_1^2)\delta^+(k_2^2 - m_2^2), \quad (2)$$

where the subscripts 1 and 2 label the particles. The assumption that Eq. (2) represents the only discontinuity of the propagator in the complex s plane then yields, through a simple dispersion relation,

$$G_k(s) = \frac{\pi}{\omega_1\omega_2}\, \delta\!\left(k_0 - \frac{\omega_1 - \omega_2}{2}\right) \frac{\omega_1 + \omega_2}{(\omega_1 + \omega_2)^2 - s}, \quad (3)$$

with $\omega_i \equiv (m_i^2 + k^2)^{1/2}$. This is the Blankenbecler-Sugar[8] result, and reduces Eq. (1) to an integral equation in one vector variable,

$$T_{pq}(s) = V_{pq}(s) + \int \frac{d\vec{k}}{(2\pi)^3 2\omega_1\omega_2} V_{pk}(s) \frac{\omega_1 + \omega_2}{(\omega_1 + \omega_2)^2 - s^+} T_{kq}(s). \quad (4)$$

Finally, with the interaction proceeding as in Fig. 2, i.e.,

$$V_{pq}(s) = \frac{\vec{p}\cdot\vec{q}\, v(p^2) v(q^2)}{s - m_\Delta^2}, \quad (5)$$

we have, in the two-body c.m. system,

$$T_{pq}(s) = \frac{\vec{p}\cdot\vec{q}\, v(p^2) v(q^2)}{D(s)}, \quad (6a)$$

where $D(s)^{-1}$ is the dressed isobar propagator, given by

$$D(s) = s - m_\Delta^2 + \frac{1}{3}\int_0^\infty \frac{dq\, q^2}{(2\pi)^2} \frac{\omega_q + E_q}{\omega_q E_q} \frac{q^2 v^2(q^2)}{(\omega_q + E_q)^2 - s^+}. \quad (6b)$$

Here, m_Δ is the bare mass of the isobar, and we use ω and E for the pion and nucleon energies, respectively. For our calculations, we shall employ the simple form

$$v(q^2) = \frac{g}{\alpha^2 + q^2}, \quad (7)$$

meaning that we have three parameters (g, m_Δ, α) at our disposal for describing the πN interaction.[9] We choose these parameters by fitting the phase of $D(s)$ to the experimental πN scattering phase shifts in the 3-3 partial wave. The fit obtained with the parameters $m_\Delta = 6.83\ \mathrm{fm}^{-1}$, $\alpha = 1.8\ \mathrm{fm}^{-1}$, and $g^2/m_\Delta^2 = 3.14$ is shown in Fig. 3.

A similar procedure is followed in describing the NN interaction. However, in this case we have the additional information that the two-nucleon propagator must have the deuteron pole at $s = M_d^2$. This allows us to perform a subtraction at that point, yielding the propagator

$$d(s) = (s - M_d^2)\int_0^\infty \frac{dq\, q^2}{(2\pi)^2} \frac{u^2(q^2)}{E_q(4E_q^2 - s^+)(4E_q^2 - M_d^2)}. \quad (8)$$

If we now assume that

$$u(q^2) = \frac{G}{\beta^2 + q^2}, \quad (9)$$

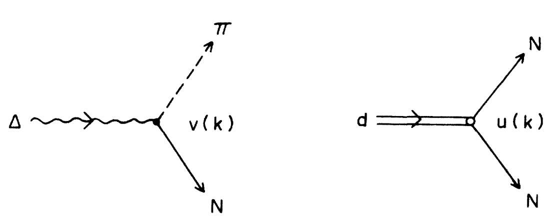

FIG. 1. Diagrammatic representation of the quasiparticle (isobar) dominated two-body interactions. The vertex functions v and u are functions of the relative momentum k in the two-body channel.

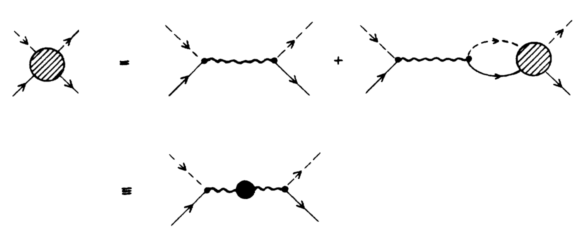

FIG. 2. Diagrammatic representation of the πN scattering amplitude. The circle on the isobar propagator represents the insertion of all πN bubble diagrams.

then a choice of M_d and β determines the NN scattering phase shift. With a deuteron binding energy of 2.225 MeV, the value $\beta = 1.3\ \mathrm{fm}^{-1}$ leads to the results shown in Fig. 4; this represents a rough fit to the low energy ${}^3S_1$ phase shift. The coupling constant G is determined from the normalization condition on the deuteron wave function

$$G^2 = \left[\int \frac{d\vec{p}}{(2\pi)^3 E_p} \frac{1}{(\beta^2 + p^2)^2 (4E_p{}^2 - M_d{}^2)^2} \right]^{-1} \tag{10}$$

and has the value $G = 64.6\ \mathrm{fm}^{-3}$.

B. Three-body equations

Having defined the two-body interactions we now construct the equations for the πd scattering. The easiest approach is to consider the possible diagrams that can contribute to the scattering process. Given the assumption of isobar dominance, the interaction of the pion with the deuteron must lead to a ΔN intermediate state which subsequently rescatters back into the πd channel. This is shown schematically in Fig. 5(a). The $\Delta N \to \pi d$ has a Born term (single nucleon exchange) and higher order terms corresponding to multiple pion and nucleon rescatterings. The equation for this amplitude is shown in Fig. 5(b). The expression for the πd scattering equations is

$$\begin{aligned} T_{11} &= B_{12} G_2 T_{21}, \\ T_{21} &= B_{21} + B_{22} G_2 T_{21} + B_{21} G_1 T_{11} \\ &= B_{21} + (B_{22} + B_{21} G_1 B_{12}) G_2 T_{21}, \end{aligned} \tag{11}$$

where the subscripts 1 and 2 denote the πd and ΔN channels, respectively. G_2 is the propagator for the particle-isobar intermediate state and B is a Born term {nucleon exchange for B_{12} [Fig. 6(a)], pion exchange for B_{21} [Fig. 6(b)]}. At this point the forms of neither the Born terms nor the propagators have been specified. In analogy with the two-body case, B and G are chosen to have the simplest form consistent with Lorentz invariance and two- and three-body unitarity.

Unitarity relates the elastic scattering amplitude to that for breakup which in our model proceeds via the dissociation of an isobar (or quasiparticle). Consequently the unitarity relation can be expressed solely in terms of two-body amplitudes and isobar propagators. The procedure for choosing the Born terms and isobar propagators so that the proper two- and three-body discontinuities

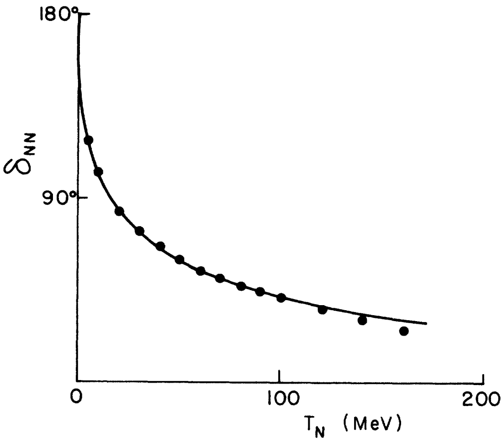

FIG. 4. The NN s-wave scattering phase shift as a function of nucleon laboratory kinetic energy T_N (data from Ref. 11).

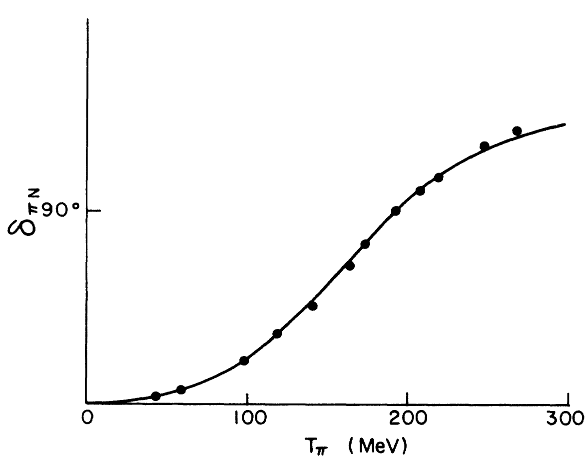

FIG. 3. The πN p-wave scattering phase shift as a function of pion laboratory kinetic energy T_π (data from Ref. 10).

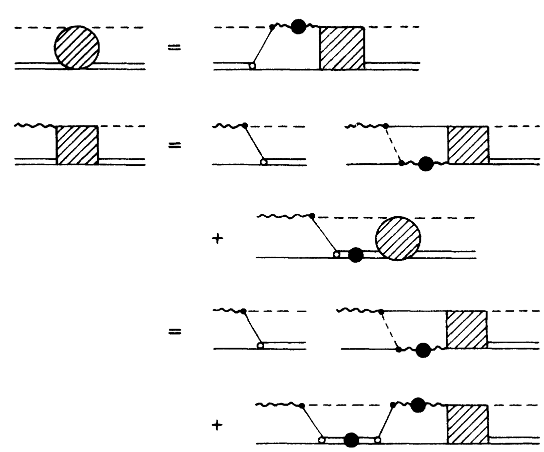

FIG. 5. Schematic representation of the πd scattering equations. The shaded circle represents the πd elastic amplitude and the shaded square represents the transition amplitude for $\Delta N \to \pi d$.

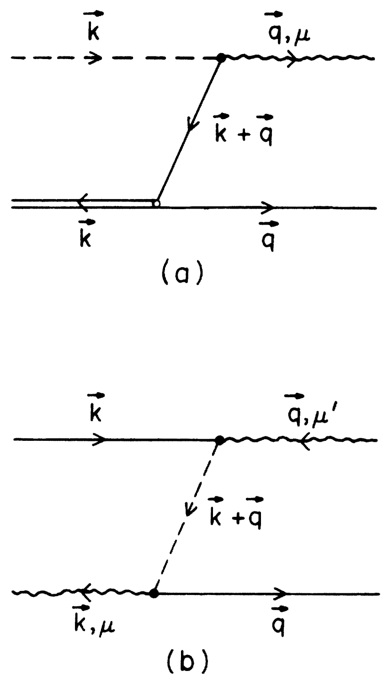

FIG. 6. The Born terms appearing in the three-body equations: (a) nucleon exchange, (b) pion exchange.

are obtained is discussed explicitly in AAY and we simply quote the results here.

The propagators G_1 (G_2) describe intermediate propagation of the "spectator" plus quasiparticle system $\pi + d(N+\Delta)$, with total c.m. four-momentum P and spectator four-momentum k. In the AAY approach, the spectator is kept on the mass shell and we have[6]

$$G_1 = -2\pi\,\delta^+(k^2 - M^2)\,d(\sigma_k)^{-1}$$

and

$$G_2 = -2\pi\,\delta^+(k^2 - \mu^2)D(\sigma_k)^{-1}, \tag{12}$$

where μ and M are the pion and nucleon masses, respectively, and $\sigma_k \equiv (P-k)^2$ is the invariant mass squared of the quasiparticle. The clustering property is enforced in Eq. (12); namely, the propagation of the isobar is a function only of its invariant mass.[12]

In writing down the Born terms, we must remember that the Δ isobar is a spin-one particle in this model (i.e., $\pi N\, p$ wave). This means, of course, that the πN t matrix, in the two-body c.m. system, has the form given by Eq. (6), and the problem is to find the four-vector dot product that reduces to $\vec{k}\cdot\vec{k}'$ in the c.m. system. Labeling the pion and nucleon four-momenta as k_1 and k_2, respectively, the four-vector

$$\kappa = \frac{k_1 k_2 \cdot K - k_2 k_1 \cdot K}{K^2}, \tag{13}$$

with $K \equiv k_1 + k_2$, clearly reduces to the c.m. momentum in that frame, Therefore, $\vec{k}\cdot\vec{k}'$ can be replaced by the Lorentz scalar $-\kappa\cdot\kappa'$ in all frames. However, this is now not very convenient for performing a partial wave decomposition and it is easily shown[6] that $\kappa\cdot\kappa'$ can be written as a three-dimensional dot product in all frames:

$$\vec{k}\cdot\vec{k}' \to -\kappa\cdot\kappa' = \vec{M}\cdot\vec{M}', \tag{14}$$

where the vector $\vec{M}$ is

$$\vec{M} = \vec{\kappa} - \vec{K}\,\frac{\vec{\kappa}\cdot\vec{K}}{K_0(K_0 + \sqrt{K^2})}. \tag{15}$$

Finally, we can write the Born term[6] in the three-body c.m. frame as (see Fig. 6)

$$B_{12} = \frac{u(\omega_{\vec{k}} + E_{\vec{q}} + E_{\vec{k}+\vec{q}})\, v M_\mu}{E_{\vec{k}+\vec{q}}[s - (\omega_{\vec{k}} + E_{\vec{q}} + E_{\vec{k}+\vec{q}})^2]} \tag{16}$$

and

$$B_{22} = \frac{M_\mu\, v\,(E_{\vec{k}} + E_{\vec{q}} + \omega_{\vec{k}+\vec{q}})\, v M_{\mu'}}{\omega_{\vec{k}+\vec{q}}[s - (E_{\vec{k}} + E_{\vec{q}} + \omega_{\vec{k}+\vec{q}})^2]}. \tag{17}$$

Using these Born terms [Eqs. (16) and (17)] and propagators [Eq. (12)], we write down explicitly all the terms in Eq. (11). For example, the single scattering approximation is just

$$T_{11}(\vec{k},\vec{k}',s) = \int \frac{d\vec{q}}{(2\pi)^3 2E_q} B_{12} G_2 B_{21}$$

$$= \int \frac{d\vec{q}}{(2\pi)^3 2E_q}\left[\frac{u(Q^2)(\omega_{\vec{k}} + E_{\vec{q}} + E_{\vec{k}+\vec{q}})}{E_{\vec{k}+\vec{q}}(s - [\omega_{\vec{k}} + E_{\vec{q}} + E_{\vec{k}+\vec{q}}]^2)}\right]\left[\frac{v(M^2)\vec{M}\cdot\vec{M}'\,v(M'^2)}{D(\sigma_q)}\right]\left[\frac{(\omega_{\vec{k}'} + E_{\vec{q}} + E_{\vec{k}'+\vec{q}})\,u(Q'^2)}{E_{\vec{k}'+\vec{q}}(s - [\omega_{\vec{k}'} + E_{\vec{q}} + E_{\vec{k}'+\vec{q}}]^2)}\right]$$

where

$$\sigma_q = (\sqrt{s} - E_q)^2 - \vec{q}^{\,2}, \qquad \vec{M} = \vec{\kappa} - \vec{q}\,\frac{\vec{q}\cdot\vec{\kappa}}{K_0(K_0 + \sqrt{K^2})}, \qquad \vec{\kappa} = \vec{k} + \frac{\vec{q}}{2}\left[1 + \frac{\mu^2 - M^2}{K^2}\right], \tag{18}$$

$$K_0 = \omega_{\vec{k}} + E_{\vec{k}+\vec{q}}, \qquad K^2 = K_0^{\,2} - \vec{q}^{\,2}, \tag{19}$$

and

$$\vec{Q} = (\vec{q} + \tfrac{1}{2}\vec{k}) - \vec{k}\,\frac{\vec{k}\cdot(\vec{q}+\tfrac{1}{2}\vec{k})}{K_0'(K_0' + \sqrt{K'^2})}, \tag{20}$$

with similar definitions for $\vec{M}'$ and $\vec{Q}'$. Up to some kinematic factors, $u(Q)[s-(\omega_{\vec{k}} + E_{\vec{q}} + E_{\vec{k}+\vec{q}})^2]^{-1}$ is the deuteron wave function ϕ transformed into the πd c.m. frame and, using Eq. (6), it is clear that the single scattering approximation gives the expected form, namely $\int \phi T_{\vec{M}\vec{M}'}(\sigma_q)\phi$. The πN t matrix, $T_{\vec{M},\vec{M}'}$ is fully off shell and evaluated at an energy consistent with energy conservation and with the fact that the spectator nucleon is on mass shell.

The solution of Eq. (11) is obtained by standard techniques. After partial wave analysis, we have remaining a linear integral equation in one scalar variable q, the magnitude of the three-momentum of the intermediate spectator particle. Using the contour rotation method of Hetherington and Schick,[11] this equation is then solved along a ray in the complex q plane. We have performed all the requisite checks on the numerical accuracy of our calculations and are convinced that the results are reliable. Before presenting these results, however, we go on to discuss the fixed-scatterer calculation which will be used for comparison.

III. FIXED-SCATTERER CALCULATION

In this section, we describe a standard fixed-nucleon calculation which we have performed with the same input parameters as described above. There is an extensive literature[1] on such calculations to which the reader is referred for more detailed discussions. The basic idea is that the pion scattering amplitude is computed for a fixed configuration of the target nucleons and then averaged over all possible configurations. Keeping only the single and double scattering contributions, we have[14]

$$F_{\pi d}(\vec{k},\vec{k}') = \int d\vec{r}\,\rho(r) F_{\pi d}(\vec{k},\vec{k}';\vec{r}),$$

$$-4\pi F_{\pi d}(\vec{k},\vec{k}';\vec{r}) = -4\pi f(\vec{k},\vec{k}')\left[e^{i(\vec{k}-\vec{k}')\cdot\vec{r}/2} + e^{-i(\vec{k}-\vec{k}')\cdot\vec{r}/2}\right]$$
$$+(-4\pi)^2 \int \frac{d\vec{p}}{(2\pi)^3}\,\frac{f(\vec{k},\vec{p})f(\vec{p},\vec{k}')}{k^{2+}-p^2}\left[e^{i(\vec{k}-\vec{p})\cdot\vec{r}/2 - i(\vec{p}-\vec{k}')\cdot\vec{r}/2} + (\vec{r}\to-\vec{r})\right]. \tag{21}$$

Here, $\vec{r}$ is the relative coordinate between the two nucleons and $\rho(r)$ is the target ground state density; this controls the relative probability for finding various target configurations and is determined by the Yamaguchi-type interaction specified by Eq. (9).

To apply Eq. (21), we must still specify the πN scattering amplitude $f(\vec{k},\vec{k}')$. First of all, we shall use only the on-shell scattering amplitudes; this is an approximation first employed by Brueckner[15] and corresponds to there being little "overlap" between the scatterers. This is certainly a reasonable approximation for a loosely bound target such as the deuteron. However, there is still a problem, since the amplitudes in Eq. (21) must be specified in the πd c.m. frame. We follow the usual procedure and first generate the amplitude in the two-body c.m. system, using the p-wave phase shift shown in Fig. 3. We can now relate the pion scattering angle in the two-body c.m. system to that in the πd c.m. system and write[16]

$$\frac{f(\vec{k},\vec{k}')}{k} = \frac{3e^{i\delta}\sin\delta}{\kappa^2}\left[\frac{\kappa^2-k^2}{\kappa^2} + \frac{k^2}{\kappa^2}\cos\theta\right], \tag{22}$$

where $\kappa(k)$ is the pion momentum in the two- (three-) body c.m. system and θ is the pion scattering angle in the πd c.m. system (i.e., $\cos\theta = \hat{k}\cdot\hat{k}'$). In effect, an s-wave term is generated by the transformation to the three-body rest system. While this is not the most sophisticated treatment of the frame transformation problem, it is both simple and commonly used.[16]

Strictly speaking, this completes the specification of the fixed-scatterer calculation. Nevertheless, two additional corrections, the binding and Fermi-averaging corrections, are often applied to the elementary amplitude in order to include some effect of the nuclear dynamics. The binding correction takes into account the fact that the target nucleons are bound, and the resulting prescription is that we evaluate the projectile-nucleon amplitude at a pion energy shifted from the free value by the average NN potential energy[2]

$$\langle V\rangle = -[E_B + \beta(E_B/M)^{1/2}]. \tag{23}$$

For a binding energy $E_B = 2.225$ MeV and a range parameter $\beta = 1.3\ \mathrm{fm}^{-1}$, this effectively "shifts the resonance" up by about 15 MeV. The Fermi-averaging correction recognizes that the target nucleons are not stationary but have momentum distributions given by the ground state wave function. The correction is obtained by numerically averaging the πN amplitude with the ground state momentum distribution. A far more detailed discussion of such multiple scattering theories can be found, for example, in Ref. 1. We shall present results both with and without the binding and Fermi-averaging corrections.

IV. RESULTS AND DISCUSSION

We shall now present the results of the three-body and fixed-scatterer calculations, performed with exactly the same input (i.e., πN phase shift and deuteron wave function). The calculations discussed in this section contain neither spin nor isospin and so cannot be compared to any data. Rather, our goal here is to learn something about the convergence of the multiple scattering series, the importance of recoil effects, and the role of intermediate NN interactions.

First, we shall consider only single scattering, that is, the approximation in which the pion interacts with only one nucleon. This is represented by Fig. 7 and, in fixed-scatterer approximation, is given simply as the product of the πN scattering amplitude and twice the deuteron form factor. In the three-body calculation, the single scattering term is explicitly given by Eqs. (18)–(20). The results for the total and integrated elastic cross sections as a function of pion lab kinetic energy T_π are shown in Fig. 8, while Fig. 9 shows the differential cross section for $T_\pi = 180$ MeV. The fixed-scatterer result for the total cross section is just twice the elementary πN total cross section, and we note that the single scattering approximation obviously violates unitarity. It is clear that the simple fixed-nucleon calculation is substantially off even in the single-scattering approximation. The total cross section curves will come into somewhat better agreement with the inclusion of the binding energy shift [Eq. (23)], since this would shift the fixed-scatterer curve up by about 15 MeV (this effect has been discussed in some detail by Myhrer and Koltun[2]); therefore, the remainder of our fixed-nucleon calculations will include the binding and Fermi averaging corrections to the elementary amplitude. Nevertheless, examination of Fig. 9 shows that, even if we "renormalize" the elementary amplitude to fit the total cross section (this is the aim of the binding

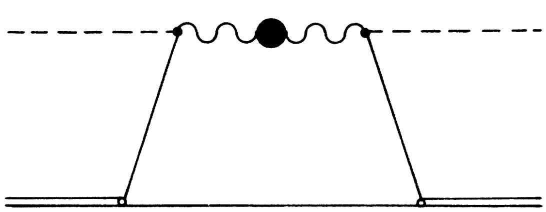

FIG. 7. Single scattering approximation to the πd scattering amplitude.

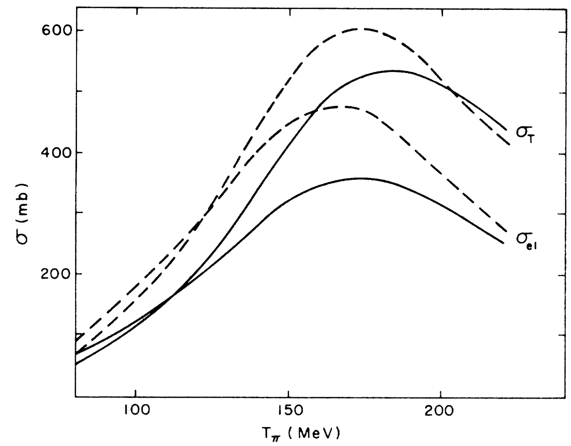

FIG. 8. Total (σ_T) and integrated elastic (σ_{el}) πd cross sections in single scattering approximation as a function of pion kinetic energy. The full curve corresponds to the single scattering approximation to the three-body equations [Eq. (18)]. The dashed curve corresponds to the fixed-scatterer approximation.

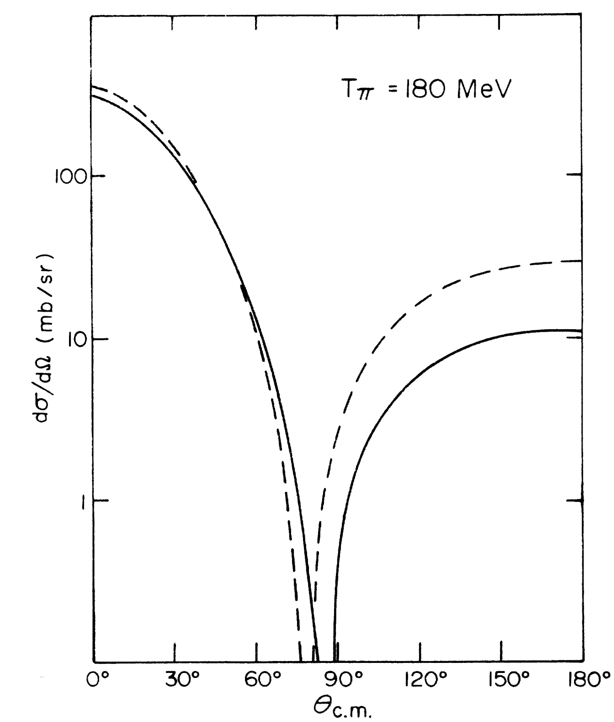

FIG. 9. The c.m. differential cross sections as a function of the c.m. scattering angle in single scattering approximation. The full curve corresponds to the single scattering approximation to the three-body equation [Eq. (18)]. The dashed curve corresponds to the fixed-scatterer approximation.

and Fermi motion corrections), the differential cross section would still be in appreciable error.[17] In fact, the backward cross sections differ by a factor of 3, reflecting the importance of properly treating the three-body kinematics (i.e., nuclear recoil and isobar propagation) in large momentum-transfer scattering. The elastic cross sections are considerably smaller in the three-body calculations, implying that, in the language of the fixed-scatterer approach, the nucleon recoil and isobar propagation effectively reduce the overlap with the final state deuteron wave function. The important result here is that, in a resonance-dominated situation, it is important to retain the full (off-shell) momentum dependence of the elementary resonant amplitude and to retain the nucleon density matrix[18] rather than simply the form factor; neither of these is done in conventional fixed-scatterer calculations.

Another question of some interest regards the convergence of the multiple scattering series, considered as an expansion in the number of πN scattering amplitudes. Our fixed-scatterer calculations include only single and double scattering terms.[19] However, Koch and Walecka[20] have examined this question in great detail for fixed-nucleon calculations of πd elastic scattering (no spin or isospin), using finite range separable πN interactions. They find (see Fig. 6 of Ref. 20) that, with a reasonable deuteron wave function and short-range πN interaction, the multiple scattering expansion diverges. More specifically, although single plus double scattering is a reasonable approximation to the exact result in the forward direction, inclusion of the triple scattering term increases the cross section by orders of magnitude.[20] While those results depend strongly upon the particular parameters used, rapid convergence of the fixed-nucleon multiple scattering series is achieved only with extreme choices of the interaction parameters.[21] The three-body calculations lead to entirely different conclusions. Figure 10 shows the 180 MeV differential cross section (in the backward hemisphere) as an expansion in the number of πN scatterings; note that both the pion exchange diagram, Fig.(11a), and the NN rescattering diagram, Fig. (11b), correspond to double scattering terms in our expansion. If we first examine the results without the intermediate NN scattering (presumably this corresponds more closely to the conventional approach), we see that the convergence to the full result is extremely rapid; in fact, triple scattering is already fairly small and brings us essentially into agreement with the full calculations. This again points out the importance of retaining the three-body kinematics, especially in situations involving large momentum transfer. In triple scattering, the pion must "backscatter" from one of the nucleons, giving it a very large recoil momentum.

When the intermediate NN interactions are retained, the results are somewhat changed. First, the backward cross sections are reduced. This means that, for example, since the double scattering term has the opposite phase composed to single scattering, its strength is increased with the inclusion of NN scattering. Basically, this is because the momentum transfer to the nucleons can be more easily shared. By the same token, the higher order scatterings are relatively more important and the multiple scattering series converges more slowly. Nevertheless, considering that the backward cross section in the single scattering approximation is 14 times that given by the

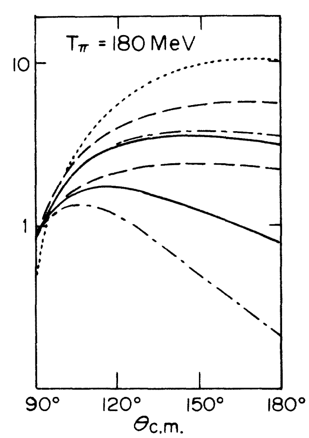

FIG. 10. Comparison of the large-angle differential cross section including single (• • •), double (---), and triple (-•-•-) scattering with the exact solution of the three-body equations. The middle three curves are calculated without nucleon-nucleon rescattering. In the lower three curves nucleon-nucleon rescattering is included.

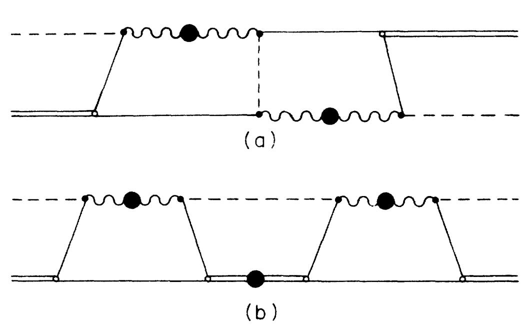

FIG. 11. Double scattering contribution to the πd elastic scattering amplitude: (a) pion exchange diagram, (b) nucleon-nucleon rescattering diagram.

full calculation, the convergence is still quite good. This convergence is guaranteed because all integrals over intermediate momenta are suppressed at large momentum by inclusion of nucleon recoil and isobar propagation. This is not the case in fixed scatterer calculations.[5,20]

Another indication of the importance of intermediate NN interactions is given in Fig. 12. There, we show the total and integrated elastic cross sections as a function of energy in the single and double scattering approximations. Roughly speaking, the double scattering term with intermediate NN scattering [Fig. 11(b)] is as important as the usual double scattering term.

Figures (13) and (14) contain more detailed comparisons between the fixed-scatterer and three-body calculations. The total and integrated elastic cross sections are shown in Figs. 13(a) and 13(b), respectively. It is clear that the binding and Fermi motion corrections act to shift the peak towards higher energies, more nearly in agreement with the three-body calculation, but that serious differences remain with regard to normalization and detailed shape of the curves. The normalization is particularly bad for the integrated elastic cross section, where the NN interaction reduces the cross section by almost 30% at the peak.[22] Similarly, the differential cross section in fixed-scatter approximation is substantially different from that resulting from the three-body calculation. The two fixed-nucleon curves appear quite different, but this is only because the double scattering amplitude in the backward direction has opposite phase to, and is larger than, the single-scattering amplitude, causing the extra "dip" at 140°; this is no longer the case when the πN amplitude includes the binding and Fermi-averaging correction. Comparing the Fermi-averaged result with that from the full three-body calculation, we see that the backward cross sections differ by a factor of 7. Obviously, the nucleon-nucleon dynamics and three-body kinematics are essential for a quantitative evaluation of the process.

Finally, we show in Fig. (15) the s-, p-, and d-wave Argand plots for $kf_l(k) = (\eta_l e^{2i\delta_l} - 1)/2i$. It is striking that higher order multiple scatterings and intermediate nucleon-nucleon scatterings affect strongly only the πd p-wave amplitude. In our model, this is the only partial wave which contains the Δ isobar and "spectator" nucleon in a relative s wave. The NN interaction makes the p wave considerably more absorptive and is necessary to satisfy unitarity at the lower energies. Phase shift analyses of πd elastic scattering data would be ex-

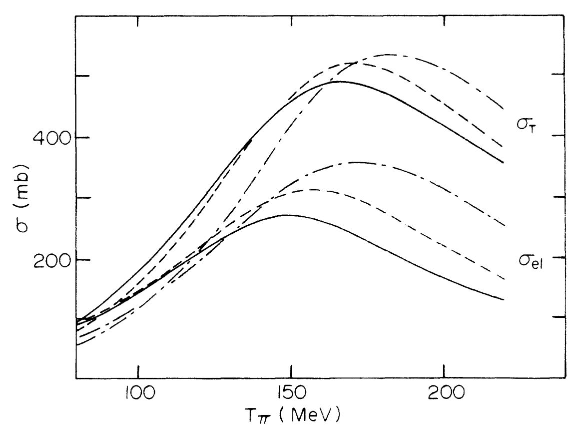

FIG. 12. Comparison of total and integrated elastic cross sections including single scattering (-·-·-), double scattering with pion exchange only (---), and double scattering with both pion exchange and nucleon rescattering (——).

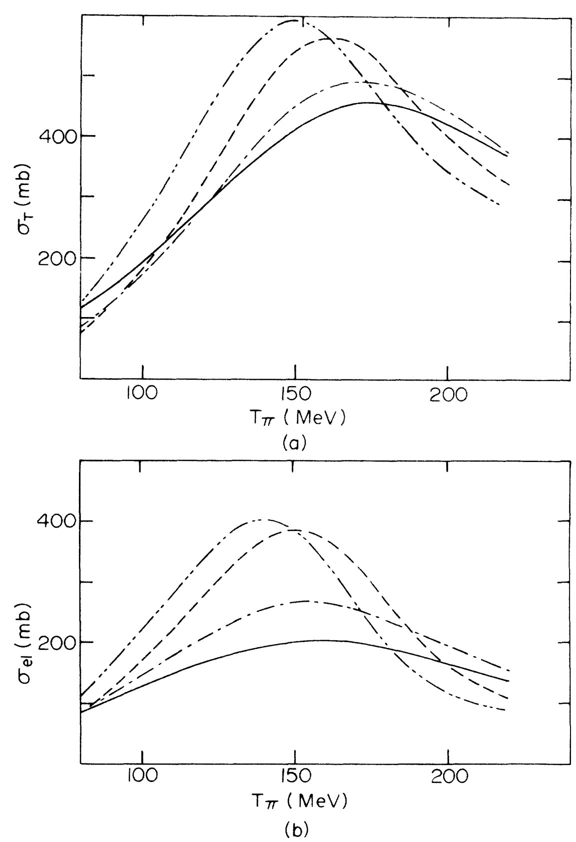

FIG. 13. Comparison of (a) total and (b) integrated elastic cross section in fixed-scatter approximation with the three-body results. The curves --- and -··-·· show the fixed-scatterer results with and without Fermi averaging. The curves —— and -·-· gives the result of the three-body calculation with and without nucleon rescattering.

tremely helpful in furthering our understanding of multiple scattering theories.

V. COMPARISON WITH DATA

The results presented so far have included neither nucleon spin nor isospin, and we concentrated upon questions involving the quantitative reliability of multiple scattering calculations. Unfortunately, incorporation of spin and isospin into the full three-body calculations would tremendously increase the magnitude of the computations.[4] Therefore, in order to compare with data and to examine the role of intermediate nucleon-nucleon interactions in "true" πd elastic scattering, we shall follow a much less ambitious course and do only the minimum required for a reasonable comparison.

Spin effects are known to be very important for describing large angle elastic scattering.[23] However, we shall compare only to total cross section and to integrated elastic cross section data. The first of these is, of course, given by the forward elastic amplitude, while most of the contribution to σ_{el} comes from fairly small angles. Therefore, we shall continue to neglect spin effects in this section. On the other hand, isospin must be taken into account at least approximately. That is, charge exchange processes will not be included, but we do include the proper statistical weighing factors to give the correct pion-proton and pion-neutron cross section (i.e., at resonance, $\sigma_{\pi+p} = \sigma_{\pi-n} \approx 210$ mb and $\sigma_{\pi+n} = \sigma_{\pi-p} \approx 70$ mb). Note that the charged pion has an especially strong interaction with only one of the nucleons, so that the importance of multiple scattering will be much less now than in the model problem discussed so far. This is also why any reasonable calculation should be in at least semiquantitative agreement with πd elastic data. Finally, we will not solve

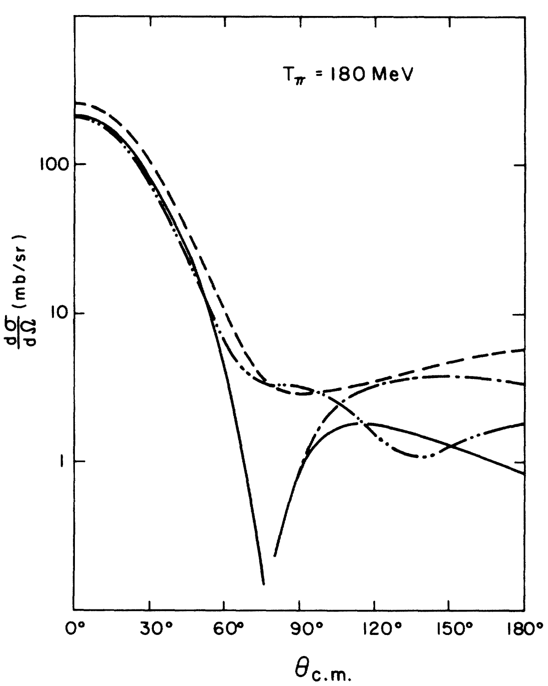

FIG. 14. Comparison of the c.m. differential cross section in fixed-scatterer approximation with the three-body results. Different approximations are indicated as in Fig. 13.

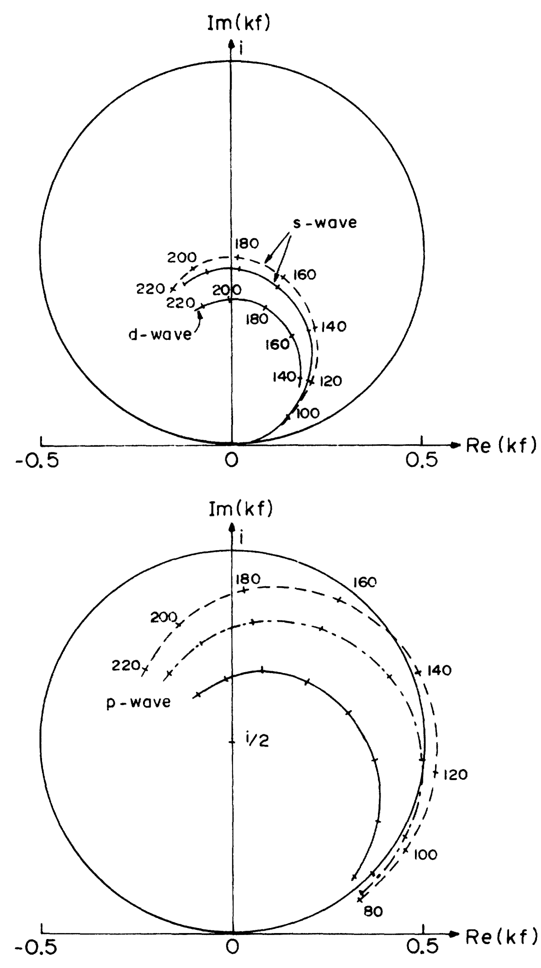

FIG. 15. Argand plot of s-, p-, and d-wave πd partial wave amplitudes. The solid curve gives the three-body result including nucleon rescattering while the dashed-dot curve gives the three-body result without nucleon rescattering. The dashed curve shows the partial wave amplitude including single and double scattering without nucleon rescattering. For s and d waves, the dot-dash and solid curves are indistinguishable. The numbers along the trajectories label the pion kinetic energy.

the full set of three-body equations [analogous to Eq. (11)], but instead evaluate only the first three orders of multiple scattering (including NN interactions and the full three-body kinematics). It is expected that the convergence of the series will be extremely rapid in this case, and this is in fact borne out by the calculations.

Figure (16) shows the data[24-26] and the results of the calculations both with and without NN scattering. Also shown are the fixed-nucleon results both with and without the binding and Fermi-averaging corrections. The rapid convergence of the multiple scattering expansion for the three-body calculation with intermediate NN interaction can be seen by examining Table I (recall from the discussion of Sec. IV that the convergence is slower in this case), and we are confident that higher order terms are essentially negligible. The data are of generally poor quality and we note that the 142 MeV π^- data of Pewitt *et al.*[25] is low with respect to both the calculations and the total cross section data of Ashkin *et al.*[24] (the "crosses"). In any case, several conclusions can be drawn from Fig. (16) (besides the obvious one that more and better data are needed). First, inclusion of the binding and Fermi-averaging corrections in the fixed-nucleon calculations is essential for getting rough agreement with the data. Nevertheless, this calculation is still about 20–25 mb larger at the peak than the three-body results, with approximately 10 mb of the difference coming from the intermediate NN scattering. The three-body calculation itself is in fairly good agreement with the data, but it must be remembered that a more complete calculation would include several improvements (spin- and isospin-flip, absorption, a better deuteron wave function, and additional πN partial waves). Nevertheless, it is clear that the correct three-body kinematics and NN scattering must be taken into account for a quantitatively accurate description of πd elastic scattering.

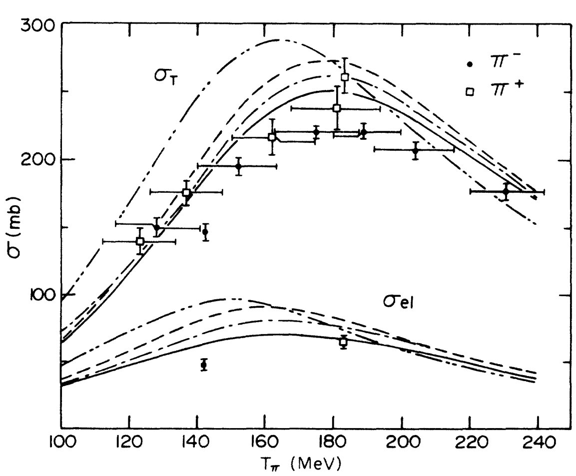

FIG. 16. Total and integrated elastic cross sections with isospin. Curves are the same as those in Fig. 13.

VI. SUMMARY AND CONCLUDING REMARKS

We have performed a unitary, Lorentz-invariant three-body calculation of πd elastic scattering and have presented detailed comparisons of these results with those of a standard fixed-scatterer calculation. Application of the binding and Fermi motion corrections in the fixed-scatterer calculation did improve agreement with the shape of the total and integrated elastic cross sections but an appreciable quantitative difference remained. This difference became especially serious for large angle elastic scattering. In fact, even the single scattering approximations were significantly different in the two calculations, demonstrating the importance of properly treating nucleon recoil and isobar propagation (both of which are neglected in the usual fixed-nucleon approach). One implication of this result for the treatment of pion scattering from heavier nuclei is that the factorized form of the lowest order optical potential (i.e., nuclear form factor times πN t matrix) may not be sufficiently accurate to allow meaningful statements about higher order terms; rather, this simple optical potential should be replaced by a folding of the t matrix with the nuclear single particle density matrix [similar to Eq. (18)].

When the multiple scattering expansion is evaluated using the full three-body kinematics, the series converges much more rapidly than in the fixed-nucleon case. This may be important for computing pion scattering from few-nucleon systems. That is, it may be better to evaluate only the first few terms in the multiple scattering expansion, retaining the correct kinematics, spin

TABLE I. Multiple scattering expansion (including NN rescattering) of the total and integrated elastic cross sections. SS, DS, and TS include terms through single, double, and triple scattering, respectively.

T_π	σ_T (mb)			σ_{el} (mb)		
(MeV)	SS	DS	TS	SS	DS	TS
100	55.3	69.5	71.2	30.0	31.6	31.3
120	103.7	121.0	121.6	48.5	48.6	47.1
140	171.7	185.0	182.5	70.5	65.3	62.1
100	237.6	237.6	233.0	87.1	73.9	70.2
180	267.3	253.2	250.0	89.2	70.5	68.5
200	254.9	235.3	234.7	78.5	60.2	59.8
220	221.0	202.8	203.5	63.7	48.8	49.1
240	182.3	168.3	169.6	49.6	38.6	39.2

and isospin effects, and nucleon-nucleon interaction, than to sum the entire series in fixed-scatterer approximation.

The effects of intermediate NN scattering were extremely important. A rough characterization of these effects is that they are as important as the multiple scattering contributions, significantly decreasing the total and integrated elastic cross sections at the resonances and even more strongly affecting the back-angle scattering. Of course, the NN interactions must be included to guarantee a unitary calculation, as demonstrated by the p wave Argand diagram (Fig. 15). This figure also shows that the πd p wave is the only partial wave strongly affected by the NN rescattering and points out the importance of a πd phase shift analysis for our detailed understanding of this process; hopefully, high resolution pion beams will soon make this possible. In any case, theoretical descriptions of pion-nucleus interactions obviously must treat the intermediate nuclear dynamics in order to achieve quantitative success.

Finally, with the inclusion of isospin, the three-body calculation is in reasonable agreement with the available, rather poor quality data on the total and integrated elastic cross sections. Inclusion both of the proper three-body kinematics and of the NN interaction serve to decrease the cross section appreciably (20% in σ_{el}) from the fixed-scatterer values. Comparison with large angle data is precluded primarily because of our neglect of nucleon spin. Including spin in our calculations presents no problems in principle but increases enormously the computational difficulties. Nevertheless, it is clear that, when high quality πd scattering data become available, such calculations will be essential for our understanding this basic pion-nucleus interaction.

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[12]Mandelzweig *et al.* (Ref. 4) violate the cluster property which we feel is an essential ingredient of any many-body theory. Of course a full crossing-symmetric relativistic theory, which would contain analyticity, unitarity, and the cluster property, is not available. We enforce the clustering property, as well as two- and three-body unitarity, which results in the introduction of a spurious square root branch point very far from the physical region ($s=0$).

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[17]We remind the reader that the zero in the differential cross section originates in the two-body p-wave amplitude and not from the deuteron form factor.

[18]The first and last factors in Eq. (18) correspond to the density matrix.

[19]In our fixed-scatterer calculation we have used only the on-shell πN amplitudes, implicitly corresponding to a zero-range interaction. However, in this case the intermediate pion propagator is the free propagator and contains an r^{-1} singularity, where r is the separation between the "fixed" nucleons. Consequently, higher order terms in the multiple scattering expansion will diverge in the fixed-scatterer approximation with zero-range interactions; finite-range interactions cut off the propagator but results in more complicated numerical calculations.

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[21]For example, Koch and Walecka (Ref. 20) find that an NN hard core radius greater than 1.8 fm guarantees convergence of the series.

[22]In a calculation performed with a "deuteron" of binding energy $E_B=8$ MeV, this ratio of the integrated elas-

tic cross sections without and with *NN* rescattering was a factor of 2.

[23]See the papers of Gabathuler and Wilken (Ref. 5) and of Mandelzweig, Garcilazo, and Eisenberg (Ref. 4).

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