Optical potentials derived from microscopic separable interactions including binding and recoil effects*

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We first consider a projectile scattering from a nucleon bound in a fixed potential. A separable Galilean invariant projectile-nucleon interaction is adopted. Without using the fixed scatterer approximation or using closure on the intermediate target nucleon states we obtain various forms for the projectile-bound nucleon t matrix. Effects due to intermediate target excitation and nucleon recoil are discussed. By making the further approximations of closure and fixed scatterers we make connection with the work of previous authors. By generalizing to projectile interaction with several bound nucleons and examining the appropriate multiple scattering series we identify the optical potential for projectile elastic scattering from the many-body system. Different optical potentials are obtained assuming different projectile-bound nucleon t matrices and we study the differences predicted by these dissimilar optical potentials for elastic scattering. In a model problem, we study pion-nucleus elastic scattering and compare the predictions obtained by adopting procedures used by (1) Landau, Phatak, and Tabakin and (2) Piepho-Walker to the predictions obtained in a less restrictive, but computationally difficult treatment.

[NUCLEAR REACTIONS Effects of different approximations on optical potentials and calculated angular distributions. Model problem studies. Nucleon recoil. Momentum distribution. Intermediate nuclear excitation effects.]

I. INTRODUCTION

Separable interactions have a long history of application in the study of scattering of strongly interacting systems. Such interactions are, to be sure, only an approximation to a more realistic energy-dependent nonlocal strong interaction. However, the ease of computation that the separability allows, permits model problems to be studied more completely than would be possible for more realistic interactions.

More recently separable interactions have been frequently adopted in formal studies of medium energy projectile-nucleus scattering. Foldy and Walecka[1] (FW) and others[2,3] have employed separable potentials in their investigation of the optical potential and its "identification" from a many-body multiple scattering series. One clarification from such studies has been the form of the off-energy-shell projectile-nucleon t matrix that enters naturally in the many-body problem and its connection to the free projectile-nucleon t matrix. Of course, the studies to date have made, necessarily, approximations other than the adoption of a separable microscopic potential. The use of fixed scatterers (infinitely heavy target constituents) and closure on the intermediate nuclear target states (subsequent to a suppression of the intermediate nuclear state energy dependence in the Lippmann-Schwinger equation for the system's wave function) are two standard procedures.

One of our objectives in the present work is to consider a model where we do *not* use fixed scatterers or ignore the intermediate state energies, and in an approximate way incorporate some effects due to nucleon binding. In our model problems we use nonrelativistic kinematics and assume a separable Galilean invariant potential which is then iterated in the many-body environment to obtain various forms for the projectile-bound nucleon t matrix. The bound t matrices derived in Sec. II show explicitly how the effects of the target nucleon momentum distribution, binding and recoil, alter in a nontrivial manner the form of the t matrix from that obtained in the free case. Also in Sec. II, adopting standard procedures, we use our results for the projectile-bound nucleon t matrix to derive an optical potential for a projectile scattering from a nucleus. The different forms for the projectile-bound nucleon t matrix lead to diverse forms for the optical potential and, in Sec. III, we compare the results (elastic scattering differential cross sections and total cross sections) obtained using the different forms. We emphasize the role of binding and the finite mass of the target particle in our comparative studies.

Of course, there has been considerable research on the theory of the optical potential and on the appropriateness of the standard approximations adopted in obtaining the "first order optical potential." The early work of Watson[4] and the application to nucleon-nucleus scattering by Kerman, McManus, and Thaler (KMT)[5] are the usual starting points for a discussion of the optical potential via multiple scattering theory. Subsequent studies by Feshbach and co-workers[6] has concentrated on the evaluation of higher order corrections (to the usual lowest order optical potential) that incorporate, for example, effects of two-nucleon correlations in the nucleus. More recently there has been considerable activity on the theory of the optical potential, as applied to pion-nucleus scattering, in anticipation of the forthcoming data from the new meson facilities. In particular, the rapid energy variation of the basic pion-nucleon interaction, associated with the (3, 3) resonance makes more suspect the neglect of binding effects[7-9] and the kinetic energy (and recoil) of the struck nucleon.[7,8,10] The validity of the coherent approximation has also been questioned. Since the ratio of the pion to nucleon mass is $\frac{1}{7}$, it is tempting to treat the pion-nucleon c.m. and lab systems and the pion-nucleus c.m. system as equivalent, thereby avoiding the difficulties associated with the angle transformation from the pion-nucleon c.m. frame to the pion-nucleus c.m. frame. A considerable amount of effort has gone into studies of the pitfalls of this approximation and various prescriptions have been given (and updated) for handling the often needed transformation of the pion-nucleon t matrix from the two-body to the many-body c.m. frame.[11-17]

Since the three features ("binding," Fermi motion, and approximate treatment of kinematics) on which we concentrate have been studied previously, we make connection with previous research where appropriate. We note the investigations of Schmidt[7] and of Kujawski and Aitken[8] which contain some features similar to the present research. These authors have adopted explicit models for the pion-nucleon interaction which allow one to obtain, in the model, an expression for the pion-nucleus optical potential that is of the form of an off-shell pion-nucleon t matrix containing the nucleon momentum variables. The off-shell π-n t matrix is integrated over the range of nucleon momentum variables weighted by the nuclear ground state momentum distribution. Free particle kinematics are not adopted and some effects due to nuclear binding are included. Based on this previous work[7,8] which found that the combined effects of binding, Fermi motion, and a correct treatment of the kinematics were important to include in the optical potential for pion-nucleus scattering even below the 3,3 resonance, we have undertaken the research reported in this paper.

One of our goals is to compare results of models which have been used in the past to make a broad spectrum of predictions for elastic scattering with the results obtained in a more complete but computational difficult approach. In our discussion below we make a detailed model and derivation of the bound pion-nucleon t matrix. Of course, one could obtain the result by simply stating that a particular approximation to the bound t matrix of Watson or KMT was being adopted. Hopefully, however, the detailed derivation gives the reader a better feeling for the approximations that were required to obtain the final pion-bound nucleon t matrix and *why* the approximations were necessary in order to proceed.

As mentioned above, practical applications of separable interactions have been made in medium energy pion-nucleus scattering.[8,11-13] An objective of the present work is to compare, in a model problem, "model correct" results for pion-nucleus elastic scattering with results obtained using the different approximations adopted by those groups using a separable interaction to study medium energy pion-nucleus scattering. In particular, here we are interested in the effects of using an approximate angle transformation and using the $t\rho$ factorization (with and without subsequent Fermi-folding) as in the original treatment by Landau, Phatak, and Tabakin,[11] and the effects of using the fixed scatterer approximation and closure as in the work of Piepho and Walker.[13] The results of this comparison and discussion are presented below in Sec. III.

We are motivated to present the kind of study given here in hope that such an investigation can aid in clarifying the validity (or range and type of error included) of some of the standard approximations used to obtain the form of the medium energy optical potential. If one does not have confidence in the procedures used to derive the optical potential from microscopic considerations, then deviation of the theory from experiment may be interpretable as due to a host of sins of approximation and the underlying physics may be difficult or impossible to distill.

In the next section we illustrate the assumptions needed to obtain the standard results and in Sec. III we compare predictions of the standard forms for the optical potential with those predictions of elastic scattering obtained adopting other less restrictive approximations. We illustrate the kind of results obtained when the projectile is light compared to the target constituents (nucleons) as in pion-nucleus scattering and also when the constituent nucleon is light compared with the projectile as in α-nucleus scattering. Because of the details involved in the derivations presented in Sec. II, we have included a comprehensive summary at the end of the paper for those not interested in the details of the formalisms involved.

II. DERIVATION OF FORMULAS

A. Scattering from a nucleon bound in a fixed potential

The basic problem we consider is that of a distinguishable projectile scattering from an infinitely heavy nucleus composed of finite mass nucleons. At this stage we assume the projectile interacts with only one of the target nucleons. The other nucleons only participate indirectly in that they are the source of binding for the interacting nucleon and they may give rise to exclusion principle effects. The basic projectile-nucleon interaction is assumed to be a spin and isospin independent Galilean invariant nonlocal separable potential of the form

$$V(\vec{r}_p,\vec{r}_t,\vec{r}_p',\vec{r}_t')=\frac{\hbar^2}{2\mu}\sum_{l,m}4\pi\lambda_l v_l(r)v_l(r')Y_{lm}(\Omega_{\vec{r}})Y^*_{lm}(\Omega_{\vec{r}'})\delta\left(\frac{1}{m_t+m_p}(m_t\vec{r}_t+m_p\vec{r}_p-m_t\vec{r}_t'-m_p\vec{r}_p')\right), \tag{1}$$

$$\vec{r}=\vec{r}_p-\vec{r}_t,\quad \vec{r}'=\vec{r}_p'-\vec{r}_t',\quad \mu=\frac{m_tm_p}{m_t+m_p},$$

where the subscript $t(p)$ refers to the target nucleon (projectile).

It is well known that the Fourier transform of the separable form factor, defined by

$$v_l(k)=4\pi\int_0^\infty v_l(r)j_l(kr)r^2dr \tag{2}$$

can be obtained from the projectile-free nucleon phase shifts under suitable conditions by solving the inverse scattering problem.[18] In the simple situation where there are no bound states and $\delta(0)-\delta(\infty)=0$ the explicit relationship between the phase shift $\delta_l(k)$ and $v_l(k)$ is given by

$$\lambda_l[v_l(k)]^2=-4\pi\frac{\sin\delta_l(k)}{k}\exp\left[-\frac{2P}{\pi}\int_0^\infty\frac{\delta_l(k')k'dk'}{(k')^2-k^2}\right]. \tag{3}$$

For simplicity we shall adopt nonrelativistic kinematics for the particles involved and shall assume a Lippmann-Schwinger equation is appropriate for describing the various systems under consideration.

First let us consider, under the assumptions above, the scattering of a projectile from a single nucleon bound in a *fixed* potential. The appropriate Lippmann-Schwinger equation for describing the system is given by

$$\Psi^{(+)}_{\vec{k}}(\vec{r}_p,\vec{r}_t)=e^{i\vec{k}\cdot\vec{r}_p}\varphi_0(\vec{r}_t)-\sum_n\int\frac{d\vec{t}}{(2\pi)^3}e^{i\vec{t}\cdot(\vec{r}_p-\vec{r}_p')}\frac{\varphi_n(\vec{r}_t)\varphi_n^*(\vec{r}_t')}{E(t)-E(k)+E_n-E_0-i\epsilon}$$
$$\times V(\vec{r}_p',\vec{r}_t',\vec{r}_p'',\vec{r}_t'')\Psi^{(+)}_{\vec{k}}(\vec{r}_p'',\vec{r}_t'')d\vec{r}_p'd\vec{r}_p''d\vec{r}_t'd\vec{r}_t'', \tag{4}$$

where $\vec{k}$ is the incident momentum of the projectile, $E(t)$ and $E(k)$ are, respectively the intermediate state energy and initial energy of the projectile, and E_n and E_0 are the bound particle intermediate state energy and initial state energy corresponding to the nucleon wave functions φ_n and φ_0, respectively. The differential cross section and scattering amplitude (t matrix) for elastic scattering may be determined from

$$\frac{d\sigma}{d\Omega}=|f(\vec{k}',\vec{k})|^2, \tag{5}$$

where

$$\frac{\hbar^2}{2m_p}f(\vec{k}',\vec{k})\equiv t(\vec{k}',\vec{k})=-\frac{1}{4\pi}\int e^{-i\vec{k}'\cdot\vec{r}_p}\varphi_0^*(\vec{r}_t)V(\vec{r}_p,\vec{r}_t,\vec{r}_p',\vec{r}_t')\psi^{(+)}_{\vec{k}}(\vec{r}_p',\vec{r}_t')d\vec{r}_pd\vec{r}_p'd\vec{r}_td\vec{r}_t'. \tag{6}$$

By introducing the Fourier transform of the bound nucleon wave function, adopting the form for the potential given by Eq. (1), and making the variable substitution

$$\vec{r}=\vec{r}_p-\vec{r}_t,\quad \alpha\equiv\frac{m_p}{m_t+m_p},\quad \vec{R}=\alpha\vec{r}_p+\beta\vec{r}_t,\quad \beta\equiv\frac{m_t}{m_t+m_p}, \tag{7}$$

we can rewrite Eq. (6) in the form

$$t(\vec{k}', \vec{k}) = -\frac{1}{4\pi} \int \cdots \int d\vec{r}\, d\vec{r}'\, d\vec{R}\, d\vec{R}'\, d\vec{p}'\, e^{-i(\beta\vec{k} - \alpha\vec{p}')\cdot\vec{r}}\, e^{-i(\vec{k}' + \vec{p}')\cdot\vec{R}}\, \frac{\varphi_0^*(\vec{p}')}{(2\pi)^{3/2}}$$

$$\times \sum_{lm} 4\pi \frac{\hbar^2}{2\mu} \lambda_l v_l(r) v_l(r') Y_{lm}(\Omega_{\vec{r}}) Y_{lm}^*(\Omega_{\vec{r}'}) \delta(\vec{R} - \vec{R}') \Psi_{\vec{k}}^{(+)}(\vec{r}', \vec{R}') . \tag{8}$$

Now by defining $\Psi_{lm}(\vec{k}, \vec{k}' + \vec{p}')$ and $v_{lm}(\vec{p})$ according to

$$\Psi_{lm}^{(+)}(\vec{k}, \vec{k}' + \vec{p}') = \sqrt{4\pi} \int e^{-i(\vec{k}' + \vec{p}')\cdot\vec{R}}\, v_l(r) Y_{lm}^*(\Omega_{\vec{r}}) \Psi_{\vec{k}}^{(+)}(\vec{r}, \vec{R})\, d\vec{r}\, d\vec{R} , \tag{9a}$$

$$v_{lm}(\vec{p}) \equiv \sqrt{4\pi}\, i^l Y_{lm}^*(\Omega_{\vec{p}}) v_l(p) ,$$

$$v_l(p) \equiv 4\pi \int_0^\infty j_l(pr) v_l(r) r^2 dr \quad [\text{see Eq. (2)}] , \tag{9b}$$

making a partial wave decomposition of the plane waves appearing in Eq. (8), and making use of the two-particle center-of-mass δ function, permits Eq. (8) to be written as

$$t(\vec{k}', \vec{k}) = -\frac{1}{4\pi} \sum_{lm} \int d\vec{p}'\, \frac{\hbar^2}{2\mu} \lambda_l v_{lm}^*(\beta\vec{k}' - \vec{\alpha}\vec{p}')\, \frac{\varphi_0^*(\vec{p}')}{(2\pi)^{3/2}}\, \Psi_{lm}^{(+)}(\vec{k}, \vec{k}' + \vec{p}') . \tag{10}$$

We obtain the following expression for $\Psi_{lm}(\vec{k}, \vec{k}' + \vec{p}')$ by projecting expression (9a) out of both sides of Eq. (4) and using δ functions to eliminate integrations where possible

$$\Psi_{lm}^{(+)}(\vec{k}, \vec{k}' + \vec{p}') = v_{lm}(\vec{k} - \alpha(\vec{k}' + \vec{p}'))\, \varphi_0(\vec{p}' + \vec{k}' - \vec{k})(2\pi)^{3/2}$$

$$- \sum_n \int \frac{d\vec{t}}{(2\pi)^3}\, d\vec{p}_i\, \frac{\hbar^2}{2\mu} v_{lm}(\vec{t} - \alpha(\vec{k}' + \vec{p}'))\, \frac{\varphi_n(\vec{k}' + \vec{p}' - \vec{t})\varphi_n^*(\vec{p}_i)}{E(t) - E(k) + E_n - E_0 - i\epsilon}$$

$$\times \sum_{l'm'} \lambda_{l'} v_{l'm'}^*(\beta\vec{t} - \alpha\vec{p}_i) \psi_{l'm'}^{(+)}(\vec{k}, \vec{t} + \vec{p}_i) . \tag{11}$$

Now in order to proceed further we apparently need to make a model for the intermediate nuclear states φ_n and the associated energies, the E_n. If we assume simple product wave functions for the many-body nucleus composed of A nucleons, then the intermediate states of all the nucleons noninteracting with the projectile must be the same as their initial states. If we assume that the energy of the nucleus can be written in terms of the single particle energies of the individual nucleons, then since only the interacting nucleon can change its state during the scattering, we can reduce $E_n - E_0$ to E_n (interacting nucleon) $- E_0$ (interacting nucleon). Instead of simply assuming that one can ignore the dependence on n of the difference $E_n - E_0$ in the denominator and then use closure on the intermediate states, we shall make a nuclear matter type approximation for the intermediate interacting target nucleon states. More specifically, we shall assume the intermediate target nucleon states are plane waves. Thus $\sum_n \to \int d\vec{p}_n (2\pi)^{-3}$ and the intermediate single state energy, E_n is written as $E(p_n)$. We shall consider different forms for the expression $E(p_n)$ in our discussion. There is an interesting effect due to an approximate treatment of the exclusion principle which puts a restriction on the intermediate state integrations. This topic is still under investigation, and while of considerable interest, we do not attempt to treat it here. The basic idea is that for multistep processes where one uses a Fermi gas model, the momentum transferred in a particular step must be zero for intermediate nucleon states below the Fermi momentum. For intermediate state densities characterized by a momentum above the Fermi sea no such restriction exists. We note that it is *not* proper to attempt to treat exclusion effects by simply limiting the intermediate state momentum integral to momenta above the Fermi sea. Such a procedure neglects important forward scattering contributions occurring while the nucleus (in a Fermi gas model) stays in its ground state in the intermediate states.

Substituting plane wave states for the $\varphi_n(\vec{r}_t)$ in Eq. (4) allows Eq. (11) to be written in the considerably simpler form

$$\psi_{lm}^{(+)}(\vec{k},\vec{k}'+\vec{p}')=v_{lm}(\vec{k}-\alpha(\vec{k}'+\vec{p}'))\varphi_0(\vec{p}'+\vec{k}'-\vec{k})(2\pi)^{3/2}$$
$$-\frac{\hbar^2}{2\mu}\int\frac{d\vec{t}}{(2\pi)^3}\frac{v_{lm}(\vec{t}-\alpha(\vec{k}'+\vec{p}'))\sum_{l'm'}\lambda_{l'}v_{l'm'}^*(\vec{t}-\alpha(\vec{k}'+\vec{p}'))}{E(t)-E(k)+E(\vec{p}_n=\vec{k}'+\vec{p}'-\vec{t})-E_0-i\epsilon}\psi_{l'm'}^{(+)}(\vec{k},\vec{k}'+\vec{p}')\ . \quad (12)$$

In order to obtain a simple closed form expression for ψ_{lm} without using determinants it is necessary that $l=l'$ and $m=m'$. This can be accomplished in a variety of situations (see later discussion) but certainly not in the general case. We first consider the general solution using determinants. We can rewrite Eq. (12) in the form

$$\psi_{lm}^{(+)}(\vec{k},\vec{k}'+\vec{p}')=\chi_{lm}(\vec{k},\vec{k}'+\vec{p}')-\sum_{l'm'}D_{lm,l'm'}(\vec{k},\vec{k}'+\vec{p}')\psi_{l'm'}^{(+)}(\vec{k},\vec{k}'+\vec{p}')\ , \quad (13a)$$

where

$$\chi_{lm}(\vec{k},\vec{k}'+\vec{p}')\equiv(2\pi)^{3/2}v_{lm}(\vec{k}-\alpha(\vec{k}'+\vec{p}'))\varphi_0(\vec{p}'+\vec{k}'-\vec{k})\ , \quad (13b)$$

$$D_{lm,l'm'}(\vec{k},\vec{k}'+\vec{p}')\equiv\frac{\hbar^2}{2\mu}\int\frac{d\vec{t}}{(2\pi)^3}\lambda_{l'}\frac{v_{lm}(\vec{t}-\alpha(\vec{p}'+\vec{k}'))v_{l'm'}^*(\vec{t}-\alpha(\vec{p}'+\vec{k}'))}{E(t)-E(k)+E(\vec{p}_n=\vec{k}'+\vec{p}'-\vec{t})-E_0-i\epsilon}\ . \quad (13c)$$

This set of inhomogeneous linear equations (13a) can be written as a matrix equation ($l,m\equiv i$)

$$\psi_i^{(+)}=\chi_i-\sum_{i'}D_{ii'}\psi_{i'}^{(+)} \quad (14a)$$

or

$$\psi^{(+)}=\chi-\underline{D}\psi^{(+)}\ . \quad (14b)$$

Equation (14b) has the solution

$$\psi^{(+)}=\underline{R}\chi\ , \quad (15a)$$

where

$$\underline{R}(\underline{1}+\underline{D})=\underline{1}\ . \quad (15b)$$

For actual numerical calculation of $\psi^{(+)}$, it is our experience that it is much quicker (computer time) to solve the matrix equation $(1+\underline{D})\psi^{(+)}=\chi$ by Gaussian elimination than to actually find the inverse, $\underline{R}$. The dimension of $\psi^{(+)}$ depends on the maximum value of l used in the interaction, Eq. (1). If the maximum value of l is L, then $\psi^{(+)}$ has $(L+1)^2$ dimensions and $\underline{D}$ is an $(L+1)^2$ by $(L+1)^2$ matrix. Once we have $\psi^{(+)}$, we can solve for $t(\vec{k}',\vec{k})$ since from Eq. (10)

$$t(\vec{k}',\vec{k})=\sum_i A_i\psi_i^{(+)}(\vec{k},\vec{k}'+\vec{p}')\ , \quad (16)$$

where A_i is the integral operator

$$-\frac{1}{4\pi}\frac{\hbar^2}{2\mu}\int d\vec{p}'\lambda_i v_i^*(\beta\vec{k}'-\alpha\vec{p}')\frac{\varphi_0^*(\vec{p}')}{(2\pi)^{3/2}}\ . \quad (17)$$

We now investigate the conditions under which $l,m=l',m'$ in Eq. (11) so that the form of $t(\vec{k}',\vec{k})$ given by Eq. (16) greatly simplifies. The procedure is to carry out the angular integrations over $\vec{t}$ and, for several situations, this allows a contraction of the l',m' sum because of the orthogonality of the spherical harmonics that are factors in the v_{lm}'s. For our study it is sufficient to consider the integral

$$\int\frac{d\vec{t}}{(2\pi)^3}\frac{v_{lm}(\vec{t}-\alpha(\vec{p}'+\vec{k}'))v_{l'm'}^*(\vec{t}-\alpha(\vec{p}'+\vec{k}'))}{E(t)-E(k)+E(\vec{p}_n=\vec{k}+\vec{p}_0-\vec{t})-E_0-i\epsilon}\ . \quad (18)$$

The basic models that allow one to obtain various simple closed form expressions for $t(\vec{k}',\vec{k})$ incorporate different assumed forms for the energy denominator and drop terms of order $\alpha=[m_p/(m_p+m_t)]$ (α, for example, is $\approx\frac{1}{7}$ for a pion projectile, $\frac{1}{2}$ for a nucleon projectile, and $\frac{4}{5}$ for a ^{4}He projectile).

First we consider the special model used by Foldy and Walecka[1] which leads to a particularly simple form for $t(\vec{k}',\vec{k})$. If we assume $\alpha=[m_p/(m_t+m_p)]=0$ and argue that the term $E(\vec{p}_t=\vec{k}'+\vec{p}'-\vec{t})-E_0$ in the denominator can be ignored, then Eq. (18) becomes

$$4\pi\int_0^\infty\frac{t^2dt\,v_{lm}{}^2(t)}{E(t)-E(k)-i\epsilon}\delta_{l,l'}\delta_{m,m'}\ . \quad (19)$$

This result when inserted into Eq. (17) allows one to obtain the following expression for $t(\vec{k}',\vec{k})$,

$$t(\vec{k}',\vec{k})=-\frac{1}{4\pi}\int\varphi_0^*(\vec{p}')\varphi_0(\vec{p}'+\vec{k}'-\vec{k})d\vec{p}'\frac{\hbar^2}{2\mu}\frac{\lambda_l v_l(k)v_l(k')(2l+1)}{1+\frac{\lambda_l}{(2\pi)^3}\frac{\hbar^2}{2\mu}\int\frac{d\vec{t}\,v_l{}^2(t)}{E(t)-E(k)-i\epsilon}}P_l(\cos\theta_{k'k})\ . \quad (20)$$

Ignoring the nuclear energy differences allows FW to use closure on the intermediate target states and this means one does not have to assume a Fermi-gas model for these states. Of course, we obtain the same result as FW because once the nuclear energies in the denominator are ignored, any complete set of states assumed for the intermediate states will, if treated exactly, lead to the closure result. The energy denominator in Eq. (20), $E(t)-E(k)$, may be taken as

$$E(t)-E(k)=\frac{\hbar^2t^2}{2m_p}-\frac{\hbar^2k^2}{2m_p}\ . \tag{21}$$

Of course, for an infinitely heavy target constituent

$$\mu=\frac{m_t m_p}{m_t+m_p}\to m_p\ .$$

Also, as $\alpha\to 0$ the δ function appearing in the separable nonlocal interaction given in Eq. (1) simply makes the potential nonlocal in the projectile coordinate only. Equation (20) will be referred to in the following as the FW result.

Another variation that has been previously discussed[2] is that obtained when one assumes finite mass scatterers $\alpha\neq 0$ but still ignores the nuclear excitation energies (so that, in earlier treatments closure could be used on the intermediate nuclear states). In this case, by making the substitution $\vec{t}\,'=\vec{t}-\alpha(\vec{p}\,'+\vec{k}\,')$, and noting

$$E(t)\to E(|\vec{t}\,'+\alpha(\vec{p}\,'+\vec{k}\,')|)$$

$$=\frac{\hbar^2}{2m_p}\left[t'^2+2\alpha(\vec{p}\,'+\vec{k}\,')\cdot\vec{t}\,'+\alpha^2(\vec{p}\,'+\vec{k}\,')^2\right]\ , \tag{22}$$

one obtains for Eq. (16)

$$\frac{2m_p}{\hbar^2}\int\frac{d\vec{t}\,'\,4\pi v_l(t')v_{l'}(t')\,i^{(l-l')}Y^*_{lm}(\Omega_{t'})Y_{l'm'}(\Omega_{t'})}{t'^2+2\alpha(\vec{p}\,'+\vec{k}\,')\cdot\vec{t}\,'+\alpha^2(\vec{p}\,'+\vec{k}\,')^2-k^2-i\epsilon}\ . \tag{23}$$

There is an angle dependence in the denominator of expression (23) that precludes the angle integration over $\vec{t}\,'$ from yielding $l=l'$, $m=m'$ trivially because of the orthogonality of the Y_{lm}'s in the numerator. Of course, the denominator can be shown to be independent of the (azimuthal) φ integration so that one obtains $m=m'$ from integration over the azimuthal angle. Thus if only one l is present (a pure s wave or p wave interaction for example) the sum can be contracted down to a single lm and a simple closed form expression for $t(\vec{k}\,',\vec{k})$ can be obtained [which, of course, in the limit $\alpha\to 0$ becomes identical with Eq. (20)]. In the more general case where several l's are present, one can adopt an additional "angle average" approximation which allows one to still obtain a closed form expression for $t(\vec{k}\,',\vec{k})$ without using determinants. This particular set of approximations is discussed in more detail in Ref. 2 and will not be pursued further here.

Now we wish to present a new result for a simple closed form expression for $t(\vec{k}\,',\vec{k})$ which *neither* uses the assumption that $\alpha=0$ (fixed scatterers) or that the intermediate nuclear state excitation energies can be ignored (but treated only in some "average" manner). Consider a Fermi-gas model for the intermediate nuclear states where it is assumed

$$E_n(p_n)=\frac{\hbar^2p_n^2}{2m_t}+\langle V\rangle\ \ \text{(independent of } p_n) \tag{24a}$$

and where one writes the initial nuclear energy E_0 as

$$E_0=\langle T\rangle_0+\langle V\rangle_0\ . \tag{24b}$$

Now the difference E_n-E_0 will reflect effects due to nucleon recoil (in the kinetic energy term) and differences in the binding potential felt by a particle in its ground state compared to some intermediate excited state. The binding potential is approximately 40 MeV deep for a particle below the Fermi sea, above the Fermi sea it should be considerably less. Substituting the expressions given by Eqs. (24a) and (24b) into Eq. (18) and making the substitution $\vec{t}\,'=\vec{t}-\alpha(\vec{p}\,'+\vec{k}\,')$ yields the expression

$$\int\frac{d\vec{t}\,'}{(2\pi)^3}\,\frac{v_{lm}(\vec{t}\,')v^*_{l'm'}(\vec{t}\,')}{\dfrac{\hbar^2}{2m_p}[\vec{t}\,'+\alpha(\vec{p}\,'+\vec{k}\,')]^2-\dfrac{\hbar^2k^2}{2m_p}+\dfrac{\hbar^2(\vec{k}\,'+\vec{p}\,'-\vec{t})^2}{2m_t}-\langle T\rangle_0-\langle V\rangle_0+\langle V\rangle-i\epsilon} \tag{25}$$

which after some manipulation may be rewritten as

$$\int\frac{d\vec{t}\,'}{(2\pi)^3}\,\frac{v_{lm}(\vec{t}\,')v^*_{l'm'}(\vec{t}\,')}{\dfrac{\hbar^2t'^2}{2\mu}-\dfrac{(\beta\vec{k}\,'-\alpha\vec{p}\,')^2}{2\mu}-\langle T\rangle_0+\dfrac{p'^2}{2m_t}+\langle V\rangle-\langle V\rangle_0-i\epsilon}\ , \tag{26}$$

where $\mu = m_t m_p / m_t + m_p$.

There is no dependence on the angles of $\vec{t}'$ in the denominator of expression (26) so when the angular integration is carried out one obtains $l = l'$, $m = m'$ from the orthogonality of the spherical harmonics in the numerator. Thus, the lm sums contract to a single sum and the following closed form expression is obtained for $t(\vec{k}', \vec{k})$,

$$t(\vec{k}', \vec{k}) = -\frac{1}{4\pi} \int \varphi_0^*(\vec{p}')\varphi_0(\vec{p}' + \vec{k}' - \vec{k})d\vec{p}' \, \frac{\hbar^2}{2\mu} \sum_{l,m} \lambda_l \frac{v_{lm}^*(\beta\vec{k}' - \alpha\vec{p}')v_{lm}(\vec{k} - \alpha(\vec{k}' + \vec{p}'))}{1 + \frac{\lambda_l}{(2\pi)^3}\frac{\hbar^2}{2\mu}\int \frac{d\vec{t}'\,|v_l(t')|^2}{\hbar^2 t'^2/2\mu - (\beta\vec{k}' - \alpha\vec{p}')^2/2\mu - B - i\epsilon}}, \tag{27}$$

where

$$B \equiv \langle T \rangle_0 - (p')^2/2m_t + \langle V \rangle_0 - \langle V \rangle. \tag{28}$$

If we ignore the term B in the denominator (i.e., the excitation effect) then Eq. (27) simply represents what should probably be referred to as the free two-body t matrix expressed in a coordinate system with origin at the origin of the fixed bound nucleon potential and Fermi averaged over (p') the bound nucleon's final momentum. There are often adopted approximations for evaluating this expression and we compare, in the next section, results obtained from exactly evaluating expression (27) in a model problem, with results obtained by following approximate methods. The reason that, in general, approximate methods have been adopted for evaluating expression (27) is that the three dimensional $\vec{p}'$ integration is very time consuming to perform [notice $\vec{p}'$ appears in the nuclear wave functions, in the v_{lm} in the numerator and in the integral over $\vec{t}'$ in expression (27)].

Finally and briefly, note that one can allow an energy dependence of the form

$$E_n(p_n) = \frac{\hbar^2 p_n^2}{2m_t} + \langle V \rangle + b_0 p_n^2 \tag{29a}$$

$$= \frac{\hbar^2 p_n^2}{2(m_t^*)} + \langle V \rangle, \tag{29b}$$

where

$$\frac{1}{2m_t^*} = \frac{1}{2m_t} + \frac{2b_0}{\hbar^2} \tag{30}$$

thus introducing an effective mass in the problem. The computational difficulty that such a term introduces appears as an angle dependence on $\vec{t}'$ in the denominator of Eq. (26).

In the following discussion we relate the result given in Eq. (27) to that obtainable from the bound collision matrix defined by KMT.[5] KMT[5] define a bound collision matrix $\tau(E)$ by the expression

$$\tau_{\pi n}(E = E_k + E_0) = v_{\pi n} - v_{\pi n} \frac{a}{E(t) - E(k) + H_A - E_0 - i\epsilon} \tau_{\pi n}(E), \tag{31}$$

where a projects onto antisymmetrized intermediate states of the target nucleus and H_A is the full many-body nuclear Hamiltonian. In the formalism adopted by KMT one takes matrix elements of the operator $(A-1)\tau(E)$ between initial and final free projectile states and ground state initial and final nuclear wave functions to obtain the optical potential. Our result, Eq. (27), can be obtained in the KMT formalism by (a) representing $v_{\pi n}$ by a Galilean invariant separable potential, (b) neglecting the projection operator a, (c) using plane wave single particle intermediate states for the nucleon interacting with the pion, and (d) assume $H_A - E_0$ can be written as a difference of single particle energies expressible as a difference of single nucleon kinetic energies plus a constant binding term. After these approximations, the complicated integrals over the various momenta are evaluated using correct kinematics assuming an infinitely heavy nucleus.

B. Optical potential

Now we consider the problem of a projectile scattering from a nucleus composed of a system of finite mass constituents. For simplicity we shall treat the *nucleus* as infinitely heavy so that the many-body center-of-mass and laboratory systems coincide. The projectile-constituent nucleon interaction is represented by a nonlocal separable Galilean invariant potential and the following many-body Lippmann-Schwinger equation is assumed appropriate for describing the scattering in the laboratory system

$$\psi_{\vec{k}}^{(+)}(\vec{x}_1^0,\ldots,\vec{x}_A^0,\vec{x}_p^0)$$
$$=\Phi_0(\vec{x}_1^0\cdots\vec{x}_A^0)e^{i\vec{k}\cdot\vec{x}_p^0}$$
$$-\sum_n\Phi_n(\vec{x}_1^0,\ldots,\vec{x}_A^0)\int\cdots\int d\vec{x}_1^1\cdots d\vec{x}_A^1\int\frac{d\vec{t}}{(2\pi)^3}\frac{e^{i\vec{t}\cdot(\vec{x}_p^0-\vec{x}_p^1)}}{E(t)-E(k)+E_n-E_0-i\epsilon}\Phi_n^*(\vec{x}_1^1,\ldots,\vec{x}_A^1)$$
$$\times\sum_{i=1}^{A}\sum_{lm}4\pi\lambda_l v_l(|\vec{x}_p^1-\vec{x}_i^1|)v_l(|\vec{x}_p^2-\vec{x}_i^2|)Y_{lm}(\Omega_{(\vec{x}_p^1-\vec{x}_i^1)})Y_{lm}^*(\Omega_{(\vec{x}_p^2-\vec{x}_i^2)})$$
$$\times\delta\left[\frac{1}{m_p+m_i}(m_p\vec{x}_p^1+m_i\vec{x}_i^1)-\frac{1}{m_p+m_i}(m_p\vec{x}_p^2+m_i\vec{x}_i^2)\right]$$
$$\times\psi_{\vec{k}}^{(+)}(\vec{x}_1^1,\ldots,\vec{x}_i^2,\ldots,\vec{x}_A^1,\vec{x}_p^2)d\vec{x}_i^2d\vec{x}_p^1d\vec{x}_p^2\,. \tag{32}$$

The states Φ_n represent a set of energy eigenstates of the A particle target satisfying

$$H_n|\Phi_n\rangle=E_n|\Phi_n\rangle\;, \tag{33}$$

where E_0 is the energy of the initial many-body nuclear target state. In Eq. (32) all variables referring to the projectile have a subscript p (i.e., x_p), while variables which refer to the ith target particle have a subscript i (i.e., x_i). Superscripts are used to distinguish between different variables referring to the same particle. The elastic scattering amplitude $f(\vec{k}',\vec{k})$ may be written

$$f(\vec{k}',\vec{k})=-\frac{1}{4\pi}\frac{2m_p}{\hbar^2}\int\cdots\int d\vec{x}_1^0\cdots d\vec{x}_A^0d\vec{x}_p^0\Phi_0^*(\vec{x}_1^0,\ldots,\vec{x}_A^0)e^{-i\vec{k}'\cdot\vec{x}_p^0}$$
$$\times\sum_{i=1}^{A}V(\vec{x}_p^0,\vec{x}_i^0;\vec{x}_p^1,\vec{x}_i^1)\psi_{\vec{k}}^{(+)}(\vec{x}_1^0,\ldots,\vec{x}_i^1,\ldots,\vec{x}_A^0,\vec{x}_p^1)d\vec{x}_p^1d\vec{x}_i^1\,, \tag{34a}$$

where V is given by Eq. (1) and appears explicitly in Eq. (32). Defining

$$T=\frac{\hbar^2}{2m_p}f \tag{34b}$$

we observe

$$\frac{d\sigma}{d\Omega}=\frac{\hbar^2}{4m_p^2}|T|^2\,. \tag{34c}$$

One procedure for obtaining the optical potential is to first iterate Eq. (32) for $\psi^{(+)}$ and insert the resulting series expression into Eq. (34a). This yields a complicated series expansion for the elastic scattering amplitude. In order to obtain from this an expression for the optical potential one must compare the many-body scattering amplitude series expansion [Eq. 34(a)] with iteration of Eq. (32) inserted for $\psi^{(+)}$ with the result obtained from considering an equivalent "one-body problem," suppressing the degrees of freedom of the many-body nuclear target. If we consider nonlocal optical potentials, what one wishes is an optical potential $U(\vec{r},\vec{r}')$ which when inserted into the "one-body" (projectile) Lippmann-Schwinger equation

$$\psi_{\vec{k}}^{(+)}(\vec{r})=e^{i\vec{k}\cdot\vec{r}}-\int\iint\frac{d\vec{t}}{(2\pi)^3}\frac{e^{i\vec{t}\cdot(\vec{r}-\vec{r}')}}{E(t)-E(k)-i\epsilon}\times U(\vec{r}',\vec{r}'')\psi_{\vec{k}}^{(+)}(\vec{r}'')d\vec{r}'d\vec{r}'' \tag{35}$$

will yield a scattering amplitude

$$-\frac{4\pi\hbar^2}{2m_p}f(\vec{k}',\vec{k})=\iint e^{-i\vec{k}'\cdot\vec{r}}U(\vec{r},\vec{r}')\psi_{\vec{k}}^{(+)}(\vec{r}')d\vec{r}d\vec{r}' \tag{36}$$

$$=\iint e^{-i\vec{k}'\cdot\vec{r}}U(\vec{r},\vec{r}')e^{i\vec{k}\cdot\vec{r}'}d\vec{r}d\vec{r}'$$
$$+\int\cdots\int e^{-i\vec{k}'\cdot\vec{r}}U(\vec{r},\vec{r}')\frac{e^{i\vec{t}\cdot(\vec{r}'-\vec{r}'')}}{E(k)-E(t)+i\epsilon}U(\vec{r}'',\vec{r}''')e^{i\vec{k}\cdot\vec{r}'''}\frac{d\vec{t}}{(2\pi)^3}d\vec{r}d\vec{r}'d\vec{r}''d\vec{r}'''+\cdots \tag{37}$$

which is identical with the elastic scattering amplitude calculated from the series expansion in the full-many-body problem. What is often done in practice (see Refs. 1 and 2) is to make approximations or assumptions that simplify the series expansion for Eq. (34a), so that a term by term comparison with Eq. (37) is possible and one can thus identify the expression for the optical potential. We shall adopt some of these approximations below.

First we make the never-come-back approximation in the series expansion of the elastic scattering amplitude [Eq. 34(a)]. This means that although a given target nucleon may be multiply struck (in the sense that we replace a series of v_i interactions by the ith nucleon-projectile t matrix) that once the projectile leaves the ith particle in the nucleus and scatters from the jth particle ($j \neq i$), the projectile does not return to interact again with the ith particle. This allows the multiple scattering series to be written symbolically as

$$-4\pi T = \left\langle \Phi_0 \left| e^{-i\vec{k}'\cdot\vec{r}_p} \left(\sum_{i=1}^{A} v_i + \sum_{i\neq j=1}^{A} v_i G v_j + \sum_{i=1}^{A} v_i G v_i + \sum_{i\neq j\neq k=1}^{A} v_i G v_j G v_k + \sum_{i\neq j=1}^{A} v_i G v_i G v_k + \sum_{i\neq j=1}^{A} v_i G v_j G v_j + \sum_{i=1}^{A} v_i G v_i G v_i + \cdots \right) e^{+i\vec{k}\cdot\vec{r}'_p} \right| \Phi_0 \right\rangle, \tag{38}$$

where G is a shorthand notation for the sum (or integral) over a set of intermediate states divided by appropriate energy denominators. Earlier in this section we obtained an expression for the t-matrix for scattering from a particle bound in a fixed potential. In terms of a series of v interactions we may write the t matrix for scattering from the ith particle as

$$t_i \equiv v_i + v_i G v_i + v_i G v_i G v_i + \cdots \, . \tag{39}$$

Using this definition of t_i we may rewrite the multiple scattering series Eq. (38) as

$$\left\langle \Phi_0 \left| e^{-i\vec{k}'\cdot\vec{r}_p} \left(\sum_{i=1}^{A} t_i + \sum_{i\neq j=1}^{A} t_i G t_j + \sum_{i\neq j\neq k=1}^{A} t_i G t_j G t_k + \cdots \right) e^{+i\vec{k}\cdot\vec{r}'_p} \right| \Phi_0 \right\rangle. \tag{40}$$

Now we wish to make a connection between the projectile-bound nucleon t matrix introduced above and the t matrices obtained for a projectile scattering from a nucleon bound in a fixed potential in Sec. II A. To do this we must adopt a similar model for the intermediate states in both cases. Also we must largely ignore the effects of correlations induced, for example, by the fact that in Eq. (38), (39), and (40) the initial, intermediate, and final states of the nucleus must be properly antisymmetrized many-particle states. We assume the intermediate states appearing in Eq. (40) are simple products of single particle orbitals.

There are two kinds of terms we need to consider. First, a term involving repeated v_i interactions from the same (ith) particle. We will, in general, have an expression of the form

$$\langle \Phi_n(\vec{x}_1 \cdots \vec{x}_A) | e^{-i\vec{t}'_p\cdot\vec{r}_p} (v_i + v_i G v_i + v_i G v_i G v_i + \cdots) \times e^{+i\vec{t}_p\cdot\vec{r}'_p} | \Phi_{n'}(\vec{x}_1 \cdots \vec{x}_A) \rangle. \tag{41}$$

Assuming simple product many-body states we see that all particles except the ith one must be in the same state in Φ_n, $\Phi_{n'}$ and all intermediate states or else the expression above vanishes. If we write the energy denominators appearing in G in terms of sums of single particle energies this allows the nuclear difference $E_n - E_0$ to be written as E_{t_n}(ith nucleon) $- E_0$ (ith nucleon). Here t_n is the momentum associated with the assumed intermediate plane wave state appearing in G for the ith nucleon. If we actually substitute in the separable interaction, then Eq. (41) becomes identical with the scattering amplitude calculated for a projectile scattering from a nucleon bound in a fixed potential, i.e., Eq. (41) may be rewritten as

$$-4\pi t_k^i(\vec{t}'_p, \vec{t}_p) = \frac{\hbar^2}{2\mu} \sum_{lm} \int d\vec{p}'_i \lambda_l v_{lm}^*(\beta\vec{t}'_p - \alpha\vec{p}'_i) \frac{\varphi_0^*(\vec{p}'_i)}{(2\pi)^{3/2}} \times \left[(2\pi)^{3/2} v_{lm}(\vec{t}_p - \alpha(\vec{t}'_p + \vec{p}'_i)) \varphi_0(\vec{p}'_i + \vec{t}'_p - \vec{t}_p) - \frac{\hbar^2}{2\mu} \sum_{l'm'} \int \frac{d\vec{t}}{(2\pi)^3} \frac{v_{lm}(\vec{t} - \alpha(\vec{p}'_i + \vec{t}'_p)) \lambda_{l'} v_{l'm'}(\vec{t} - \alpha(\vec{p}'_i + \vec{t}'_p))}{E(t) - E(k) + E_n(\vec{t}'_p + \vec{p}'_i - \vec{t}) - E_0 - i\epsilon} \psi_{l'm'}^{(+)}(\vec{t}_p, \vec{t}'_p + \vec{p}'_i) \right] \tag{42}$$

[compare Eqs. (13a)–(17)].

The other type of term we need to consider arises in Eq. (40) when one considers the intermediate states and the associated energies contained in a G sandwiched between t matrix interactions involving two different particles. A sufficient example for our purposes is given by

$$\left\langle \Phi_0 \middle| e^{-i\vec{k}'\cdot\vec{r}_p} t_i \int \frac{d\vec{t}_p}{(2\pi)^3} e^{+i\vec{t}_p\cdot\vec{r}_p} \sum_\alpha \frac{|\Phi_\alpha(\vec{x}_1\cdots\vec{x}_A)\rangle\langle\Phi_\alpha(\vec{x}_1'\cdots\vec{x}_A')|}{E(t_p)-E(k)+E_\alpha-E_0-i\epsilon} e^{-i\vec{t}_p\cdot\vec{r}_p'} t_j e^{+i\vec{k}\cdot\vec{r}_p'} \middle| \Phi_0 \right\rangle. \tag{43}$$

Because of the never-come-back approximation the intermediate ket wave function $|\Phi_\alpha\rangle$ must have the jth target nucleon in the same state as it is in the ground state (the final state $\langle\Phi_0|$) of the nucleus. Similarly, the intermediate bra wave function $\langle\Phi_\alpha^*|$ must have particle i in the same state as it is in the ground state (the initial state $|\Phi_0\rangle$). This result coupled with the fact that all the other particles ($k \neq i$ or j) must be in the same state as in the ground state means, Φ_α must be identical to Φ_0. Thus $E_\alpha - E_0$ is equal to zero. Because of the assumption of single particle product wave functions and the never-come-back approximation, this argument generalizes immediately to any term $t_iGt_jGt_k$ etc. The conclusion is that the only intermediate nuclear many-body state contribution to such terms is the ground state. It is interesting that in the simple model the intermediate states contribution to v_iGv_i can be single particle type excitations, while the ground state only contributes to t_iGt_j type terms. We now continue in the spirit of the FW formalism. Treating the particles equivalently in the single nucleon densities, assuming the number of particles $A \gg n$ where n is the number of iterations of the t_i needed to yield the cross section to sufficient accuracy, and using the considerations above allows the multiple scattering series Eq. (40) to be written

$$-4\pi A t_k(\vec{k}',\vec{k}) + A^2 \int \frac{d\vec{t}}{(2\pi)^3} \frac{t_k(\vec{k}',\vec{t})t_k(\vec{t},\vec{k})}{E(k)-E(t)+i\epsilon}(4\pi)^2 + A^3 \int \frac{d\vec{t}}{(2\pi)^3} \frac{t_k(\vec{k}',\vec{t}_1)t_k(\vec{t}_1,\vec{t}_2)t_k(\vec{t}_2,\vec{k})(-4\pi)^3}{[E(k)-E(t)+i\epsilon][E(k)-E(t)+i\epsilon]} + \cdots . \tag{44}$$

If Eq. (44) is compared with the series expansion for the optical potential given by Eq. (37), then the optical potential may be identified as

$$U_k(\vec{t}_1,\vec{t}_2) = -4\pi A t_k(\vec{t}_1,\vec{t}_2) \tag{45}$$

with t_k given by Eq. (42) [the index i appearing in Eq. (42) is no longer needed]. Equation (45) gives the optical potential which is the generalization of the FW optical potential including the effects of nucleon binding, recoil, and intermediate nuclear excitation in the bound nucleon t matrix.

One can eliminate the never-come-back approximation and the $A \gg n$ condition by simply *requiring* that the nucleus is in its ground state between *t-matrix* interactions. One simply states that part of the *definition* of the first order optical potential includes this requirement. Relaxing the never-come-back approximation and using the ground state requirement for intermediate states between t matrices (but not for terms like $v_iGv_jGv_i$) allows the operator appearing in Eq. (40) to be written

$$\sum_{i=1}^{A} t_i + \sum_{i\neq j=1}^{A} t_i \,|\,\text{nuclear g.s.}\rangle g \langle \text{nuclear g.s.}|\, t_j + \sum_{i=1}^{A}\sum_{i\neq j=1}^{A}\sum_{j\neq k=1}^{A} t_i \,|\text{nuclear g.s.}\rangle g \langle\text{nuclear g.s.}\; |t_j|\; \text{nuclear g.s.}\rangle g\langle \text{nuclear g.s.}\; |t_k + \cdots , \tag{46}$$

where g denotes an integral over intermediate *projectile* states divided by the appropriate *projectile* energy difference. Now treating the particles equivalently allows Eq. (44) to be rewritten

$$(-4\pi)A t_k(\vec{k}',\vec{k}) + A(A-1)\int \frac{d\vec{t}}{(2\pi)^3}\frac{t_k(\vec{k}',\vec{t})t_k(\vec{t},\vec{k})}{E(k)-E(t)+i\epsilon}(4\pi)^2 + A(A-1)^2 \int \frac{d\vec{t}_1}{(2\pi)^3}\frac{d\vec{t}_2}{(2\pi)^3} \frac{t_k(\vec{k}',\vec{t}_1)t_k(\vec{t}_1,\vec{t}_2)t_k(\vec{t}_2,\vec{k})(-4\pi)^3}{[E(k)-E(t_1)+i\epsilon][E(k)-E(t_2)+i\epsilon]} + \cdots + A(A-1)^n[\quad](-4\pi)^{n+1} + \cdots . \tag{47}$$

Now by defining a many-body scattering amplitude T' by (as in Ref. 5)

$$T' \equiv \frac{A-1}{A} T , \tag{48}$$

where T is defined in Eq. (34b), we obtain a series expansion for T' which may be written in the form

$$T' = U_k'(\vec{k}', \vec{k}) + \int \frac{d\vec{t}}{(2\pi)^3} \frac{U_k'(\vec{k}', \vec{t}) U_k'(\vec{t}, \vec{k})}{E(k) - E(t) + i\epsilon} + \int \frac{d\vec{t}_1}{(2\pi)^3} \frac{d\vec{t}_2}{(2\pi)^3} \frac{U_k'(\vec{k}', \vec{t}_1) U_k'(\vec{t}_1, \vec{t}_2) U_k'(\vec{t}_2, \vec{k})}{[E(k) - E(t_1) + i\epsilon][E(k) - E(t_2) + i\epsilon]} , \tag{49}$$

where

$$U_k'(\vec{t}_1, \vec{t}_2) = -4\pi(A-1) t_k(\vec{t}_1, \vec{t}_2) . \tag{50}$$

This form of the first order optical potential is that obtained in Ref. 5 and with further approximations has been used for pion-nucleus scattering in Refs. 11 and 12. Apparently it explicitly assumes the intermediate ground state requirement but has the advantage of allowing one to drop the $n \ll A$ requirement. In pion-nucleus scattering near the (3, 3) resonance our experience has been that the number of iterations required of the optical potential is *certainly not small* compared to A (the series frequently does not converge). Of course, in practical calculations one does not use the series expansion but adopts a matrix inversion technique.

In the next section we shall present results of various calculations using the optical potential given in Eq. (50) and adopting different forms for $t_k(\vec{t}_1, \vec{t}_2)$. The particular analytic form assumed for the microscopic Galilean invariant projectile-nucleon separable potential is given by

$$V(\vec{r}, \vec{r}') = \frac{\hbar^2}{2\mu} \sum_m 4\pi\lambda v_1(r) v_1(r') Y_{1m}(\Omega_{\vec{r}}) Y^*_{1m}(\Omega_{\vec{r}'}), \tag{51a}$$

where

$$v_1(r) = e^{-\gamma r} \tag{51b}$$

and μ is the projectile-nucleon reduced mass. This separable p wave potential has the Fourier transform form factor

$$v_1(k) \equiv 4\pi \int_0^\infty j_1(kr) v_1(r) r^2 dr = \frac{8\pi k}{(k^2+\gamma^2)^2} . \tag{52}$$

The particular values chosen for the constants will be discussed in the next section. A p wave interaction was adopted in order to allow a connection with pion-nucleus scattering in our model problems.

Actually, the potential given in Eqs. (45) or (50) and (42) has not been previously used in calculations. Instead, authors[11-13] who have carried out practical calculations of pion-nucleus elastic scattering have adopted further approximations. For example, Piepho and Walker (PW)[13] have used a semirelativistic generalization of the Foldy-Walecka formalism. In order to obtain a fixed-scatterer pion-nucleon scattering amplitude, (PW) transform the pion-nucleon data to the laboratory system, then assume the laboratory data resulted from the scattering of a pion from an infinitely heavy nucleon (which allows one to obtain fixed scatterer phase shifts) and proceeded to solve the inverse scattering problem using Eq. (3). An optical potential is then obtained via the FW formalism. The analogous form of the optical potential used by PW (in our discussion here which uses nonrelativistic kinematics and ignores spin and isospin degrees of freedom) is given by [see Eqs. (45), (42), and (13a)–(17) with $\alpha = 0$, $\beta = 1$, and $E_n - E_0 = 0$]

$$U_k(\vec{t}_1, \vec{t}_2) = -4\pi A \sum_l \frac{\hbar^2}{2m_\pi} \frac{\lambda_l \rho(\vec{t}_1 - \vec{t}_2) v_l'(t_1) v_l'(t_2)(2l+1) P_l(\cos\theta_{t_1, t_2})}{1 + \frac{\lambda_l}{(2\pi)^3} \int \frac{d\vec{t}\, [v_l'(t)]^2}{t^2 - k^2 - i\epsilon}} , \tag{53}$$

where here ρ is the single nucleon density defined by

$$\rho(\vec{t}_1 - \vec{t}_2) \equiv \int \varphi_0^*(\vec{p}') \varphi_0(\vec{p}' + \vec{t}_1 - \vec{t}_2)\, d\vec{p}' . \tag{54}$$

The separable potential form factors $v'(t)$ have primes on them to remind us that they resulted from fitting the fixed scatterer phase shifts and so may be somewhat different than the $v(t)$ which fit the center-of-mass phase shifts. Equation (53) will be adopted in the next section to obtain the fixed scatterer PW results to be compared with the results obtained adopting other forms of the optical potential.

If we adopt the approximations leading to Eq. (50) and assume that the form of the energy intermediate nuclear state dependence used allows $l = l'$ and $m = m'$ in Eqs. (42) or (13), then the optical potential U_k' may be written in the form [see Eq. (27)]

$$U_k'(\vec{t}_1,\vec{t}_2) = -4\pi(A-1)\frac{\hbar^2}{2\mu}\sum_{l,m}\left[-\frac{1}{4\pi}\int\frac{d\vec{p}'\varphi_0^*(\vec{p}')\varphi_0(\vec{p}'+\vec{t}_1-\vec{t}_2)v_{lm}^*(\beta\vec{t}_1-\alpha\vec{p}')v_{lm}(\vec{t}_2-\alpha(\vec{t}_1+\vec{p}'))}{1+\frac{\lambda_l}{(2\pi)^3}\frac{\hbar^2}{2\mu}\int\frac{d\vec{t}'|v_l(t')|^2}{\hbar^2t'^2/2\mu-(\beta\vec{t}_1-\alpha\vec{p}')^2/2\mu-B-i\epsilon}}\right], \quad (55)$$

where B is a term that appears due to binding related effects [see Eq. (28)]. Note that $\vec{p}'$, the struck nucleon final momentum, and $\vec{t}_2$ appear in the denominator of Eq. (55). This makes the optical potential given by Eq. (55) considerably more difficult to use in practical calculations. However, by using the p wave analytic form for the projectile-nucleon separable potential (and by using *considerable* computer time) we have used Eq. (55) to obtain results (which we call the model correct results) that are compared to other predictions in the next section.

Now we wish to consider approximations to the optical potential, Eq. (55), which lead to simplified forms that are analogous to ones used by authors in practical studies of pion-nucleus scattering. Although linked by integration variables, Eq. (55) has the form of a single particle density multiplied by an "off-shell" form of the projectile-nucleon t matrix. We therefore write Eq. (55) in the more suggestive form

$$U_k'(\vec{t}_1,\vec{t}_2) = -4\pi(A-1)\int\langle\vec{t}_1,\vec{p}'|T(\omega_0)|\vec{t}_2,\vec{p}\rangle F(\vec{p}',\vec{p})d\vec{p}d\vec{p}' \times\delta(\vec{p}'+\vec{t}_1-\vec{p}-\vec{t}_2)\ , \quad (56)$$

where

$$F(\vec{p}',\vec{p})=\varphi_0^*(\vec{p}')\varphi_0(\vec{p})\ . \quad (57)$$

Now, if one *assumes* that the dependence on the nucleon's momentum $\vec{p}'$ in the two-body t matrix (after using the δ function to eliminate the $\vec{p}$ integration) can be replaced by some *average value* $\vec{p}_0$, then one obtains

$$U_k'(\vec{t}_1,\vec{t}_2) = -4\pi(A-1)\langle\vec{t}_1,\vec{p}_0|T(\omega_0)|\vec{t}_2,\vec{p}_0+\vec{q}\rangle\rho(q)\ , \quad (58)$$

where

$$\vec{q}=\vec{t}_1-\vec{t}_2 \quad (59a)$$

and

$$\rho(q)=\int F(\vec{p}',\vec{p}'+\vec{q})\,d\vec{p}'\ . \quad (59b)$$

Using our *model* for the t matrix, one can use this optical potential without further approximation. Of course, one would like to directly relate the two-body t matrix $\langle\,|T(\omega_0)|\,\rangle$, appearing in Eq. (58) in the projectile-nucleus c.m. frame (which coincides with the lab frame here since we assume an infinitely heavy nucleus) to the projectile-nucleon t matrix in the projectile-nucleon c.m. system (thus eliminating model ambiguities). If we denote the projectile-nucleon t matrix and the appropriate kinematic variables by $\langle\vec{K}'|T(\tilde{\omega}_0)|\vec{K}\rangle$, then we wish to obtain a relation between $\vec{t}_1,\vec{t}_2$, $\vec{p}_0$, and ω_0 and $\vec{K}'$, $\vec{K}$, and $\tilde{\omega}_0$ and we wish to find a γ such that

$$\langle\vec{t}_1,\vec{p}_0|T(\omega_0)|\vec{t}_2,\vec{p}_0+\vec{q}\rangle=\gamma\langle\vec{K}'|T(\tilde{\omega}_0)|\vec{K}\rangle\ . \quad (60)$$

In order to find such a relationship we now make further approximations, reduced to their nonrelativistic limits, adopted in previous practical calculations[11] (for comparative purposes). [See Ref. 11 for a more detailed discussion.] For nonrelativistic kinematics $\gamma=1$. If we use the frozen nucleon approximation of Ref. 11 and the Galilean invariant relative velocity between the pion projectile (p) and nucleon (n) we obtain

$$\vec{p}_0=\frac{-\vec{t}_i}{A}\quad\text{(frozen nucleon approximation)}, \quad (61a)$$

$$\vec{v}_p-\vec{v}_n=\vec{v}_p'-\vec{v}_n'\ ,$$

so

$$\frac{\vec{t}_i}{m_p}-\frac{\vec{p}_0}{m_n}=\frac{\vec{K}}{m_p}+\frac{\vec{K}}{m_n}=\vec{t}_i\left(\frac{1}{m_p}+\frac{1}{Am_n}\right) \quad (61b)$$

and therefore

$$\vec{K}=\vec{t}_i(\beta+\alpha/A)\xrightarrow[A\to\infty]{}\beta\vec{t}_i\ . \quad (61c)$$

The variable ω_0 is the parametric energy appropriate for the projectile-nucleon t matrix in the many-body environment, while $\tilde{\omega}_0$ is the total energy available in the projectile-nucleon c.m. when free, two-body kinematics are appropriate. As the binding term $B\to0$ in our model interaction, Eq. (55), the parametric energy becomes the same as the energy one would have identified in the projectile-nucleon c.m. and is related to the momentum by

$$\tilde{\omega}_0=\frac{\hbar^2(K')^2}{2\mu}\Big|_{p-\text{nucleon}}\quad\text{(free, two body)}. \quad (62)$$

In order to express the t matrix in the appropriate variables for the many-body problem, one must make an "angle-transformation" from the projectile-nucleon c.m. to the projectile-nucleus c.m. The actual procedure is to make the following partial wave decompositions (P_l is the usual Legendre polynomial)

$$\langle \vec{K}'|T(\tilde{\omega}_0)|\vec{K}\rangle = \sum_l (2l+1)(K'|T_l|K)P_l(\hat{K}'\cdot\hat{K}), \tag{63a}$$

$$\rho(\vec{t}_1 - \vec{t}_2) = \sum_{l'} \rho_{l'}(t_1,t_2)P_{l'}(\hat{t}_1\cdot\hat{t}_2) \tag{63b}$$

and then using Eqs. (60), (63a), and (63b) in Eq. (58) we obtain

$$U_k'(\vec{t}_1,\vec{t}_2) = -4\pi(A-1)\gamma \times\sum_{l,l'}(2l+1)(K'|T_l|K)\,\rho_{l'}(t_1,t_2) \times P_l(\hat{K}'\cdot\hat{K})P_{l'}(\hat{t}_1\cdot\hat{t}_2). \tag{64}$$

In the Legendre polynomials appearing in Eq. (64) we have both the angles in the projectile-nucleon c.m. system ($\hat{K}'\cdot\hat{K}$) and the scattering angles in the projectile-nucleus system ($\hat{t}_1\cdot\hat{t}_2$). We can eliminate the projectile-nucleon angles by using the mathematical *identity*

$$P_l(\hat{K}'\cdot\hat{K}) = \sum_{L=0}^{\infty} d_{lL}P_L(\hat{t}_1\cdot\hat{t}_2), \tag{65}$$

where, in general, the d_{lL} would *be a function of the scattering angle*. Using Eq. (65) and the relation

$$P_L(\hat{t}_1\cdot\hat{t}_2)P_{l'}(\hat{t}_1\cdot\hat{t}_2) = \sum_{l''}(2l''+1)\begin{pmatrix} l' & l'' & L\\ 0 & 0 & 0\end{pmatrix}^2 \times P_{l''}(\hat{t}_1\cdot\hat{t}_2), \tag{66}$$

we may rewrite Eq. (64) as

$$U_k'(\vec{t}_1,\vec{t}_2) = -4\pi(A-1)\left[\sum_{l''}(2l''+1)\sum_{l,l',L}\begin{pmatrix} l' & l'' & L\\ 0 & 0 & 0\end{pmatrix}^2(2l+1)d_{lL}\,\rho_{l'}(t_1,t_2)(K'|t_l|K)\right]P_{l''}(\hat{t}_1\cdot\hat{t}_2). \tag{67}$$

Now, using the definitions of $\vec{K}'$ and $\vec{K}$ given just above Eq. (60), we may write

$$\vec{K}' - \vec{K} = \beta(\vec{t}_1 - \vec{t}_2) - \alpha(\vec{p}\,' - \vec{p}) \tag{68a}$$

or

$$\vec{K}' - \vec{K} = \vec{t}_1 - \vec{t}_2 - \alpha(\vec{t}_1 - \vec{t}_2 + \vec{p}\,' - \vec{p}), \tag{68b}$$

since $\alpha+\beta=1$.
If we obey the momentum conserving δ function that appears in Eq. (56), Eq. (68b) becomes

$$\vec{K}' - \vec{K} = \vec{t}_1 - \vec{t}_2 \tag{68c}$$

and we can relate the angles in the projectile-nucleon c.m. to the angles in the projectile-nucleus c.m. by squaring Eq. (68c) to obtain

$$\cos\theta_{p\text{-nucleon}} = \frac{(K')^2 + (K)^2 - (t_1)^2 - (t_2)^2}{2K'K} + \frac{t_1t_2}{K'K}\cos\theta_{p\text{-nucleus}}. \tag{68d}$$

[To parallel the discussion in Ref. 11 more closely, one may note that Eq. (68c) is a consequence of the *on-shell* invariance of relative velocities which is the nonrelativistic analog of the four-vector invariance used in Ref. 11 to relate the angles in the two different c.m. systems.]

Making use of the frozen nucleus approximation Eq. (61), we obtain

$$\cos\theta_{p\text{-nucleon}} = \left[1 - \frac{A^2}{(\beta A+\alpha)^2}\right]\frac{(t_2{}^2 + t_1{}^2)}{2t_2t_1} + \frac{A^2}{(\beta A+\alpha)^2}\cos\theta_{p\text{-nucleus}}. \tag{69}$$

Since we shall adopt a p wave ($l=1$) separable potential this is sufficient to determine the d_{lL} appearing in Eq. (65). They are angle independent using the frozen nucleon approximation and are given by

$$d_{10} = \left[1 - \frac{A^2}{(\beta A+\alpha)^2}\right]\frac{(t_2{}^2 + t_1{}^2)}{2t_2t_1} \tag{70a}$$

and

$$d_{11} = \frac{A^2}{(\beta A+\alpha)^2}. \tag{70b}$$

Given $\vec{t}_1$ and $\vec{t}_2$ we can determine the $\vec{K}'$ and $\vec{K}$ to be used in the t matrix appearing in Eq. (67) by using Eq. (61). Thus all the terms appearing in the square brackets in Eq. (67) are independent of the angles of $\vec{t}_1$ and $\vec{t}_2$. We may thus make a partial wave decomposition of $U_k(\vec{t}_1,\vec{t}_2)$ such as

$$U_k'(\vec{t}_1,\vec{t}_2) = \sum_{l''}(2l''+1)(t_1|U_{l''}'|t_2)P_{l''}(\hat{t}_1\cdot\hat{t}_2), \tag{71}$$

where from Eq. (67) we can identify

$$(t_1|U_{l''}'|t_2) = -4\pi(A-1)\sum_{l,l',L}\begin{pmatrix} l' & l'' & L\\ 0 & 0 & 0\end{pmatrix}^2(2l+1)d_{lL}\,\rho_{l'}(t_1,t_2)\,(t_1(\beta+\alpha/A)|\,T_l\,|\,t_2(\beta+\alpha/A)). \tag{72}$$

A method for "Fermi-folding" is obtained in Ref. 11 by noting that the off-shell t matrix appearing above in Eq. (72), $(K'|t_l|K)$, can actually be written for separable models as an on-shell t-matrix, $(K_0|t_l|K_0)$, multiplied by the ratio $v_l(K')v_l(K)/[v_l(K_0)]^2$. The nucleon's momentum in the nucleus is then approximately taken into account by doing a two-dimensional integral over the on-shell t matrix, $(K_0|t_l|K_0)$, times a nucleon momentum weighting function. The variable K_0, appearing in the on-shell t matrix, is related to the nucleon's momentum $(\vec{p})$, in the projectile-nucleus c.m. by $\vec{K}_0=\beta\vec{t}_2-\alpha\vec{p}$ in our nonrelativistic analogue. The weighting function used in Ref. 11 is the Fourier transform of the radial part of the single particle density. We use this approximate Fermi-folding technique in some of our calculations and the results will be discussed in the next section.

Using the decomposition of $U_k(\vec{t}_1,\vec{t}_2)$ given by Eq. (71) results in considerable simplification compared to performing the multidimensional integrations involved in obtaining $t_k(\vec{t}_1,\vec{t}_2)$ from Eq. (42) and then using the optical potential given by Eq. (55).

One has required two particularly important approximations to obtain Eqs. (71) and (72). The first is the factorization leading to Eq. (58), while the second is the frozen nucleon approximation. In the next section we shall compare the results obtained using the optical potentials given by Eqs. (71) and (72) and Eq. (58) with the results obtained from other optical potentials. It is important to keep in mind that the frozen nucleon approximation is actually, in general, inconsistent with the relation $\vec{K}'=\beta\vec{t}_1-\alpha\vec{p}'$ along with the δ function $\delta(\vec{t}_1+\vec{p}'-\vec{t}_2-\vec{p})$ using in reducing equation (56).

More recently several authors have suggested alternatives to the procedure suggested in earlier work for treating the kinematics approximately. Miller[16] notes that his suggested angle transformation is currently being adopted in some momentum space optical potential calculations using separable pion-nucleon interactions. Recently Landau[17] has reported calculations that do not assume the frozen nucleon approximation, Eq. (61a), but uses the relation

$$p_0=\frac{-t_i}{A}+q\,\frac{(A-1)}{A}\,. \tag{73}$$

Applied consistently, the more recent prescriptions make the "constants" d_{10} and d_{11}, Eq. (70), functions of angle. One advantage of using Eq. (73), is that it is consistent with the relation $k'=\beta t_1-\alpha p'$ and the momentum conserving δ function $\delta(t_1+p'-t_2-p)$. We note that our results indicate that such calculations, based on Eq. (73), which do an approximate Fermi-folding, are only negligibly different from our Eq. (55) *without* including intermediate state excitation.

III. RESULTS, COMPARISONS, AND DISCUSSION

A. Results

In the previous section, using various approximations, several forms were obtained for the optical potential. In this section, we compare the results of calculations of elastic differential cross sections and total cross sections using these potentials. All the optical potentials were obtained assuming a separable microscopic projectile-nucleon interaction. As noted earlier, nonrelativistic kinematics and the Lippmann-Schwinger equation have been used. In all our calculations we have assumed the nucleus to be composed of 16 nucleons (^{16}O) and have adopted a single particle harmonic oscillator model to obtain the relevant nuclear single particle density. We have assumed the target nucleons fill the $1s$ and $1p$ oscillator shells and have used the value $b=1.77$ fm for the oscillator parameter. Thus the actual form taken for

$$\varphi_0^*(\vec{p}')\varphi_0(\vec{p}'+\vec{t}_1-\vec{t}_2) \tag{74a}$$

was

$$\tfrac{1}{4}\phi_{1s}^*(\vec{p}')\phi_{1s}(\vec{p}'+\vec{t}_1-\vec{t}_2)+\tfrac{3}{4}\phi_{1p}^*(\vec{p}')\phi_{1p}(\vec{p}'+\vec{t}_1-\vec{t}_2) \tag{74b}$$

$$=(b^3/4\pi)^{3/2}\exp\{-(b^2/2)[(\vec{p}')^2+(\vec{p}'+\vec{q})^2]\}[1+2b^2(\vec{p}'+\vec{q})\cdot\vec{p}'], \tag{74c}$$

where $\vec{q}=\vec{t}_1-\vec{t}_2$.

The basic projectile-nucleon interaction is assumed to be a separable p wave interaction as presented in Eqs. (51) and (52). The particular parameters chosen for the potential [see Eqs. (51a) and (51b)] are $\lambda=-6112$ fm^{-5} and $\gamma=3.5$ fm^{-1}. These parameters have been used previously in Ref. 19 in a fit to the pion-nucleon $P33$ data *below* the (3, 3) resonance. The total cross section obtained from the on-shell pion-nucleon t matrix generated by this separable potential is shown as a solid line in Fig. 1. For comparative purposes the true pion-nucleon total cross section is shown as a dotted line in the same figure. The model t matrix yields a total cross section energy dependence that is similar to that observed for the pion-nucleon system. The model interaction total cross section peaks at about 180 MeV [at about the (3, 3) resonance energy] and is broader and slightly larger than the observed total cross section. The model interaction provides a good fit to the observed pion-nucleon $P33$ phase shifts up to 150 MeV. Beyond this point the observed phase shift continues to rise while the model phase shift levels off at $\sim 80°$ and eventually begins to decrease above 400 MeV. Our model p wave interaction t matrix is only meant to be suggestive of some of the features that would be present below the (3, 3) resonance in more detailed fits to the pion-nucleon data, as in Refs. 11 and 13.

We obtained a form for the optical potential Eq. (55) [using single nucleon plane wave intermediate states] that incorporates effects due to nucleon recoil, the target nucleon momentum distribution, and, in our model, intermediate nuclear state excitation (ISE). We refer to the results using Eq. (55) with B [given by Eq. (28)] set equal to zero as the "model correct" results. This optical potential (with $B=0$) is the one that one might select to include rigorously nucleon recoil and effects of the struck nucleon momentum distribu-

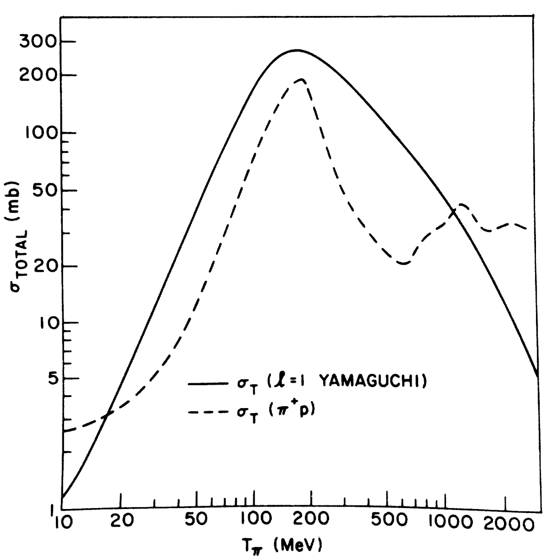

FIG. 1. A comparison of the energy dependence of the total pion-nucleon cross section obtained using the p wave separable potential introduced in the text and the observed (π^+p) total cross section. The separable potential is used as microscopic input in the different optical potentials discussed in the text.

tion in a model incorporating the two-body off-shell t matrix derivable from a separable potential. Our model of the intermediate states [see Eqs. (24a) and (28)] allows B to be written as

$$-(p')^2/2m_t+A' , \qquad (75a)$$

where

$$A'=\langle T\rangle_0+\langle V\rangle_0-\langle V\rangle \qquad (75b)$$

and where $\langle T\rangle_0+\langle V\rangle_0$ is the energy of the initial bound nucleon. Relative to a free nucleon of zero momentum, if we take $\langle T\rangle_0+\langle V\rangle_0 \simeq -15$ to -20 MeV, then $A' \simeq -20$ MeV $-\langle V\rangle$. The symbol $\langle V\rangle$ is the average potential energy felt by the intermediate excited plane wave states. It is consistent in our model to assume $\langle V\rangle \simeq 0$. In order to demonstrate the effects of a $B\neq 0$ term, we have calculated optical potentials, elastic differential cross sections, and total cross sections using $A'=-20$ and -40 MeV. These are now compared with the "model correct" result which neglects the ISE terms. The results for the real and imaginary parts of the optical potentials for T_π (lab energy) = 150 MeV ($|\vec{k}|=|\vec{k}'|$) are shown in Fig. 2. The results obtained for the energies (T_π) in the range 100–180 MeV are similar to those shown. One effect of ISE on the momentum space optical potential is to increase the magnitude of the real part and decrease the imaginary part at forward angles. The angular distributions obtained when these optical potentials are used in a Lippmann-Schwinger equation to describe the scattering are shown in Fig. 3. It is seen that the greater the ISE term, A', the more the differential cross section is decreased at forward angles and enhanced at back angles. We note that, compared to the model correct result, the differential cross sections including the A' term are significantly larger at back angles and that the effect increases with increasing energy in the energy range shown. A calculation of the absolute square of the first order optical potential reveals the same enhancement feature at back angles for the potentials containing the ISE term (compared to the model correct result). In the energy region 100–180 MeV, the total cross sections obtained (using the optical theorem), employing t matrices generated by the optical potentials shown in Fig. 2, become smaller as the ISE term A' becomes more negative. In fact, σ_T obtained from the model correct result is $\sim 10\%$ greater than σ_T obtained when $A'=-20$ MeV. Of course, there may be little connection between U and T in this energy range, but, as shown in Fig. 2, $\mathrm{Im}U(\vec{k}',\vec{k})$ at $\theta=0$ decreases as A' increases magnitude. Of course, this is the same feature present in $\mathrm{Im}T(\vec{k}',\vec{k})$ at $\theta=0$ yielding the general

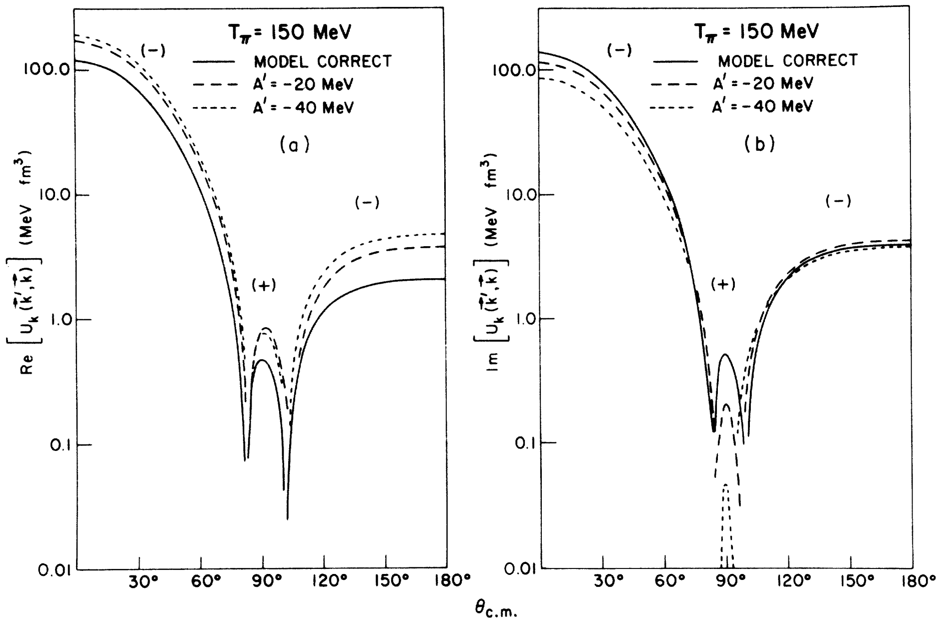

FIG. 2. (a) Comparison of the real parts of momentum space optical potentials obtained using the assumptions leading to (1) the model correct optical potential and (2) optical potentials incorporating the effect of intermediate nuclear state excitation ($A' = -20$ and -40 MeV). (b) Same as Fig. 2(a) except the imaginary parts of the optical potentials are compared.

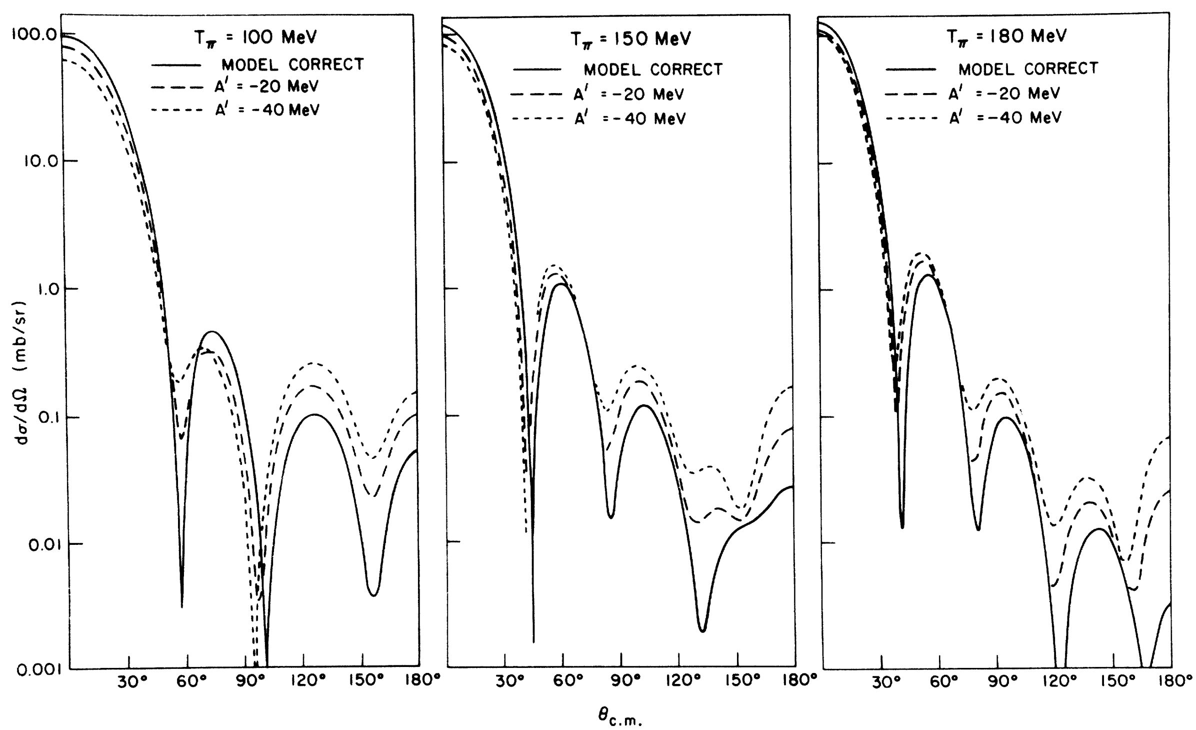

FIG. 3. Comparisons of the angular distributions obtained using (1) the model correct optical potential and (2) the optical potentials incorporating the effect of intermediate nuclear state excitation ($A' = -20$ and -40 MeV). Results are shown for the pion laboratory kinetic energies of 100, 150, and 180 MeV.

behavior of σ_T as a function of A'.

Next we consider the effects of a "$t\rho$" factorization of the optical potential. B is set equal to zero below. Now, instead of using Eq. (55) for the optical potential we use expression (58) with $\vec{p}_0$, the average target nucleon momentum, set equal to zero. This allows a factorization of the optical potential into a product of two terms, $\rho(q)$ the Fourier transform of the struck nucleon momentum distribution and $\langle \vec{t}_1, \vec{p}_0 = 0 \,|\, T(\omega_0) \,|\, \vec{t}_2, \vec{p}_0 + \vec{q} = \vec{q} \rangle$ the pion-nucleon t matrix. Using our analytic expression for the t matrix we can evaluate this optical potential quite easily without further approximation. All results obtained using the optical potential given by Eq. (58) are designated by the title, $t\rho$ factorization. We also consider an approximation where one removes t from the $\vec{p}'$ integral and carries out the angular integrations over $\vec{p}'$ in $\varphi^*(\vec{p}')\varphi(\vec{p}' + \vec{t}_1 - \vec{t}_2)$ and then subsequently brings t back into the integral and integrates over $\vec{p}'$ (the angles of $\vec{p}'$ now only appear in t). This approximation is equivalent to not setting $\vec{p}_0' = 0$ but using a nucleon momentum distribution function (form factor) having only a radial p' dependence. We call this the "radial form factor" approximation. Comparisons of the model correct, $t\rho$ factorization, and radial form factor approximation results are shown in Figs. 4 and 5. Clearly the model correct and radial form factor results are quite similar. Unfortunately, the radial form factor approximation does not cause appreciable simplification (approximately equal computer time) in actual calculations. What the results indicate is that the angle dependence of the single nuclear form factor does not play an appreciable role when one is considering closed shell nuclei, as would be expected. [Of course, the effects do not exactly vanish.] The situation for the $t\rho$ factorization is quite different. This approximation results in considerable simplification and has been adopted previously. Unfortunately, as shown in Fig. 5, there is considerable difference between the model correct and $t\rho$ factorization results at back angles. The $t\rho$ factorization causes the predicted back angle cross section to be enhanced, sometimes by an order of magnitude. Nonnegligible deviations from the model correct result also occur in the first order optical potential. A general conclusion from our studies (representative results are shown in Fig. 5) is that a *complete* neglect of Fermi-folding or the nucleon momentum distribution results in inaccuracies at back angles that are as important as those deviations resulting from the neglect of intermediate nuclear state excitation or, as we shall discuss below, approximate angle transformations. On the positive side, we note that total cross sections calculated with the $t\rho$ factorization potential do not deviate from those obtained using the model correct method by more than 10%

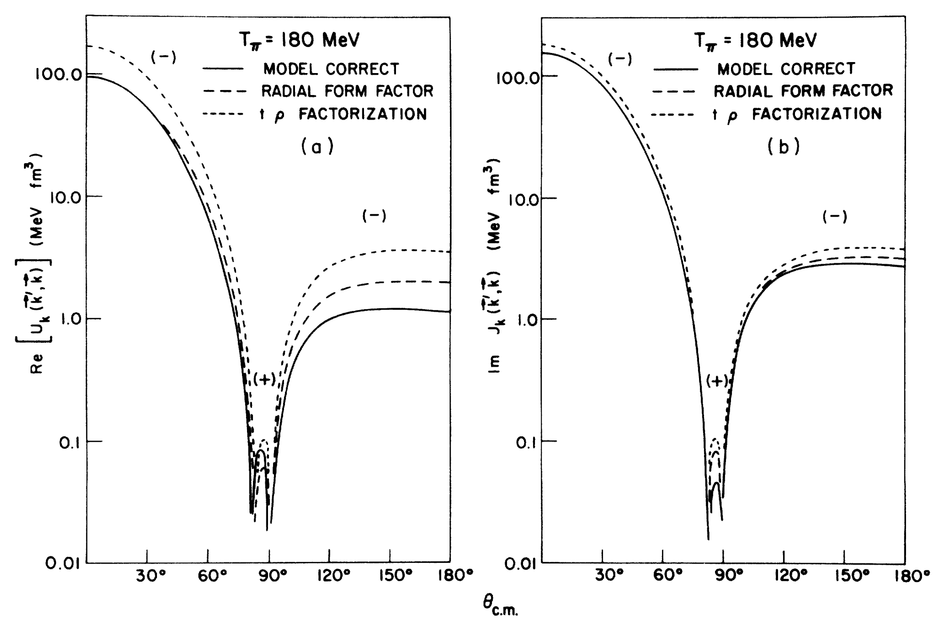

FIG. 4. (a) Comparison of the real parts of momentum space optical potentials obtained using two approximate forms of Fermi-folding (radial form factor and $t\rho$ factorization) with an exact treatment (model correct). (b) Same as Fig. 4(a) except the imaginary parts of the optical potentials are compared.

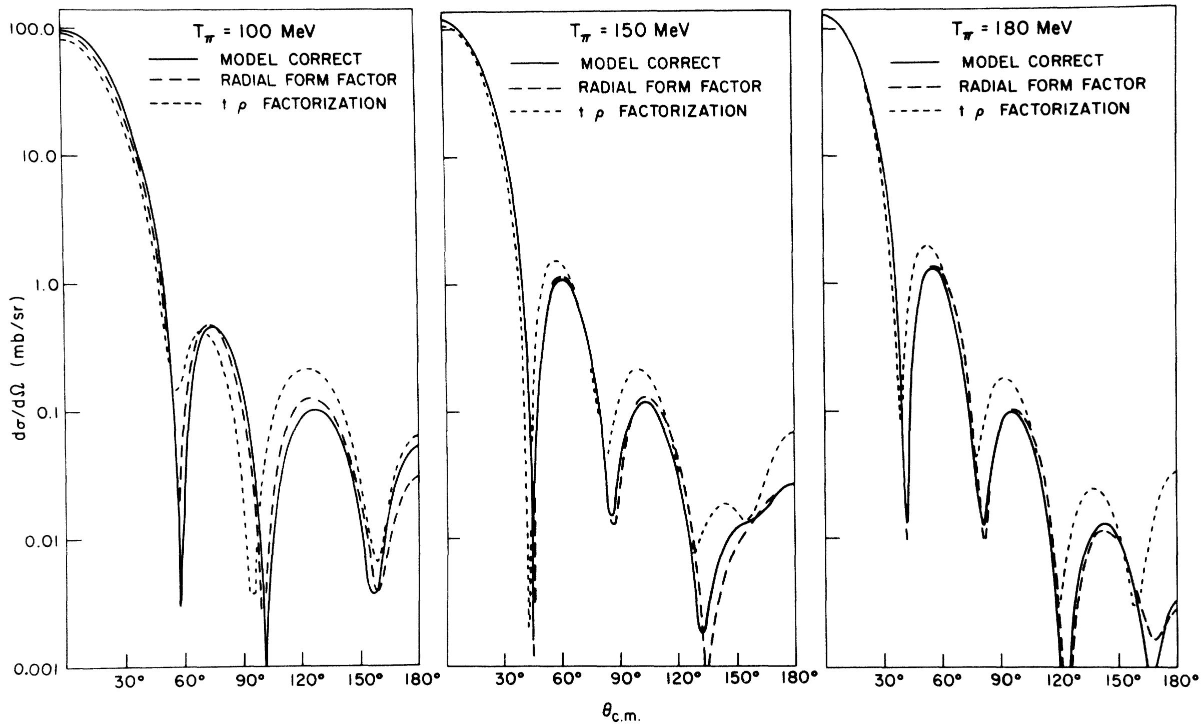

FIG. 5. Comparison of the angular distributions obtained using two approximate forms of Fermi-folding and the result obtained in an exact treatment (designated as model correct). Results are shown for pion laboratory kinetic energies of 100, 150, and 180 MeV.

in the energy range 50–400 MeV.

It is of interest to compare the total and elastic differential cross sections obtained when standard approximations are used to simplify [Eq. (55)] for the optical potential. The results of a study using the approximations introduced in Refs. 11 and 13 will now be summarized. As discussed following Eq. (52) Piepho and Walker[13] have used the fixed scatterer formalism elaborated in Ref. 1 to make detailed predictions for intermediate energy pion-nucleus elastic scattering. PW employ an optical potential of the form given by Eq. (53). In Figs. 6 and 7 the results obtained using the optical potential given by Eq. (53), are referred to as the "PW" approximation. Landau, Phatak, and Tabakin (LPT) have used a procedure discussed between Eqs. (56) and (72) which includes an approximate angle transformation relying on the "frozen nucleon approximation" [see Eq. (61a)]. The optical potential given by Eq. (72) with the approximate Fermi-folding and angle transformation adopted by LPT is referred to as the "LPT" approximation. We compare in Figs. 6 and 7 the model correct, LPT, and PW approximations. The real and imaginary parts of the optical potential obtained in the PW approximation deviate from both model correct and LPT at forward angles. Model correct and LPT can be shown to be *mathematically identical* for the forward scattering $|\vec{k}| = |\vec{k}'|$ first order optical potential. At back angles, both the LPT and FW optical potentials deviate by roughly the same amount from the model correct result. Examination of the angular distributions shown in Fig. 7 demonstrates that both PW and LPT deviate from the model correct results. The magnitude of the deviation depends on the energy, but, in general, PW tends to be smaller than model correct at backward angles.

The LPT approximation yields a good fit at forward angles but in the range 90° to ~150° generally overestimates the differential cross section. The total cross sections obtained with LPT and FW do not deviate from model correct by more than 5% in the range 100 to 400 MeV. The total cross sections calculated from all of the optical potentials discussed in this paper are quite broad and change by only ~50% in the range 100 to 400 MeV.

In order to demonstrate how the errors due to the $t\rho$ factorization are magnified when a relatively heavy particle scatters from a nucleus we have

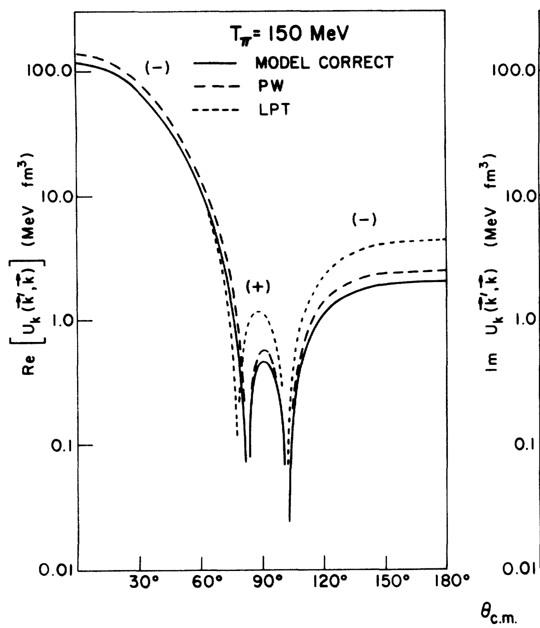

FIG. 6. (a) Comparison of the real parts of momentum space optical potentials obtained using approximations adopted in previous detailed calculations of pion-nucleus elastic scattering (by Peipho-Walker PW see Ref. 6 and Landau, Phatak, and Tabakin LPT see Ref. 4) and the model correct optical potential discussed in the text. (b) Same as 6(a) except the imaginary parts of the optical potentials are compared.

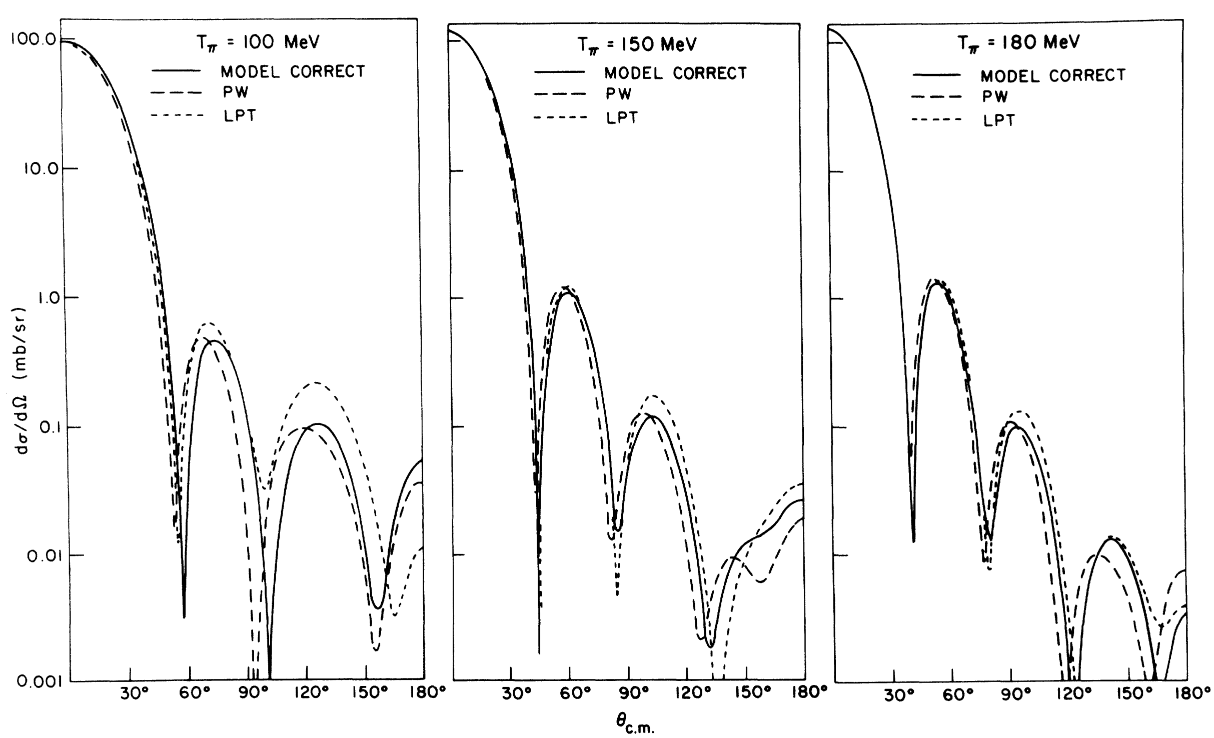

FIG. 7. Comparison of the angular distributions obtained using the model correct optical potential and optical potentials obtained using approximations previously adopted in detailed studies of pion-nucleus elastic scattering (see Fig. 6).

considered a model problem where a 100 MeV particle having the α particle mass scatters from ^{16}O. We have used parameters $\lambda = 100$ fm^{-5}, $\gamma = 1.5$ fm^{-1} in our p wave separable potential. The behavior of the resulting completely fictitious "model" α-nucleon total cross section is shown in Fig. 8. In Fig. 9 we compare the results of the new model correct, $t\rho$ factorization, and radial form factor approximations. Certainly neither of the approximate optical potential's differential cross sections is satisfactorily close to the model exact results. The critical point is that for fixed incident energy in the projectile-nucleus system, the range over which the two-body t matrix (inside the integral) is evaluated becomes *larger* as the projectile becomes *heavier*. From the relation $\vec{K} = \beta \vec{t}_2 - \alpha \vec{p}$ we can evaluate the range over which the momentum $|\vec{K}|$ in the projectile-nucleon c.m. varies. Taking the nucleon's momentum to be the Fermi momentum $|\vec{p}| \approx 1$ fm^{-1}, we find for the pion case (100 MeV pion in the pion-nucleus c.m.) 0.6 fm$^{-1} < |\vec{K}| < 0.86$ fm^{-1}, while for the α particle $0 < |\vec{K}| < 1.5$ fm^{-1}. When one makes the $t\rho$ factorization, one chooses *a* value of $|\vec{K}|$ to evaluate the two-body t matrix. Obviously, the larger the range of $|\vec{K}|$ to choose from, the less likely $t\rho$ will approximate model correct. For our α particle case, the $t\rho$ factorization gives *a* value of $|\vec{K}| = 0.88$ fm^{-1} or in terms of energy in the two-body c.m. $T_{4N} = 20$ MeV, while the range is $0 < T_{4N} < 58$ MeV. By looking at the total cross section in the two-body system (see Fig. 8) one might argue that the 20 MeV value is representative, but the information from σ_T is not sufficient and, for example, the behavior of the phase shift must also be considered. A larger range of $\vec{K}$ and $\vec{K}'$ values means a wider band of two-body phase shifts are important and also that the angles of $\vec{K}$ and $\vec{K}'$ over which the two-body t matrix is evaluated for fixed $\vec{k}$ and $\vec{k}'$ in the many-body environment is significantly broadened.

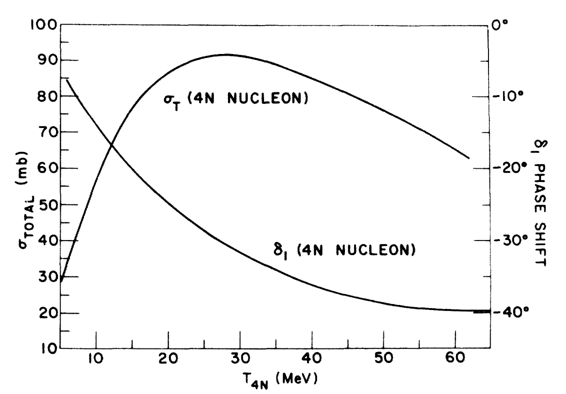

FIG. 8. The c.m. energy dependence of the "α"-nucleon total cross section and the $l = 1$ phase shift using a p wave separable potential discussed in the text. The separable potential is supposed to represent the interaction between a nucleon and some imaginary particle having the mass of four nucleons.

B. Summary

In what follows we summarize and briefly discuss our results. In Sec. II we have used a separable Galilean invariant projectile-constituent nucleon potential, multiple scattering formalisms, and various approximations to obtain several different forms for the projectile-nucleus optical potential. The use of a separable potential allowed considerable analytic progress to be made in the many-body environment which would have been otherwise impossible. Nonrelativistic kinematics and the validity of a Lippmann-Schwinger equation

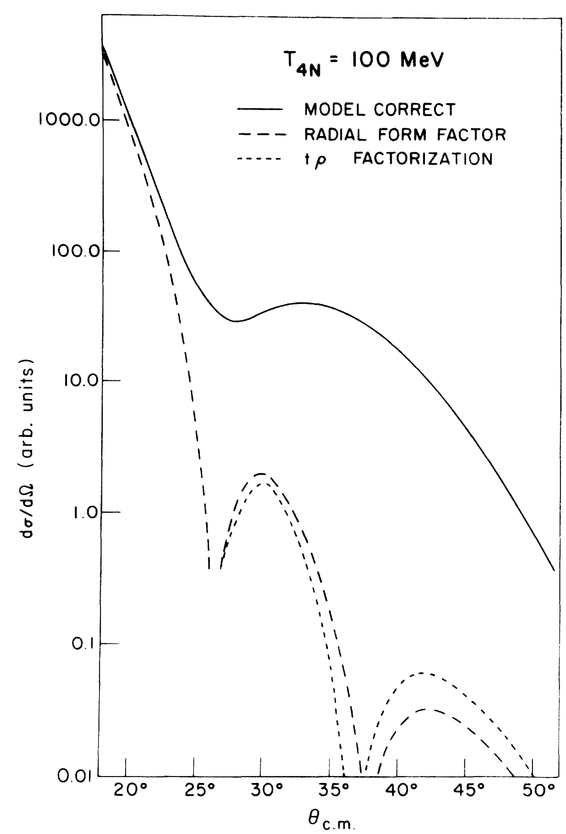

FIG. 9. A comparison of the angular distributions obtained using two approximate forms of Fermi-folding and the result obtained in the model correct treatment. All results use the microscopic separable potential discussed at the end of Sec. III. The large deviations occur because the approximate forms of Fermi-folding become inadequate as the mass of the projectile ($4M_N$) becomes larger compared to the nucleon constituent mass M_N.

were assumed throughout. We first considered a projectile scattering from single nucleon in a fixed potential. In attempting to obtain an expression for the associated t matrix it was found necessary to assume some model for the *intermediate* bound nucleon states. The resulting bound nucleon t matrix was found to enter naturally in a study of projectile-nucleus scattering and a complicated form for the optical potential was obtained. By using plane wave intermediate states with a simple energy momentum relationship (including a potential energy or binding term) it was possible to study the effect of intermediate nuclear state excitation on the projectile-bound nucleon t matrix. It was found that when simple product wave functions were used throughout along with the never-come-back approximation (or the definition that the nucleus was in its ground state between t-matrix interactions) that single particle excitations could still take place in the intermediate states contributing to the bound nucleon t matrix itself. This effect, which we denoted by ISE, to our knowledge has not received serious attention before. Such an effect *cannot* be taken into account by simply using a k_{eff} because it enters only in $v_i g v_i$ type terms and is not present in $t_i g t_j$ terms. By making further approximations adopted by authors in the past, we were able to compare and contrast the optical potentials and many-body elastic scattering predictions such approximations yield. More specifically by using the general approximations or procedures discussed above in this summary we were able to obtain the projectile-bound nucleon t matrix given by Eq. (16) supplemented by Eqs. (13)–(15).

Subsequent approximations delineated below resulted in more explicit and tractable forms for the projectile-bound nucleon t matrix:

(1) The Foldy-Walecka result[1]. The additional assumptions of $m(\text{projectile})/m(\text{nucleon}) \to 0$, neglect of the nuclear energy difference $E_n - E_0$, and closure on the nucleon states leads to Eq. (20). [See also the discussion following Eq. (20).]

(2) The result obtained in Ref. 2. The additional assumption of closure and an angle averaging of terms depending on the angles of $\vec{t}'$ in Eq. (23) is required.

(3) To obtain a new result, incorporating the effects of intermediate nuclear excitation, nucleon recoil, and the nucleon momentum distribution, it was only necessary to supplement Eqs. (13)–(16) by Eq. (24a) for the assumed intermediate nuclear state energy. This leads to Eq. (27) for the projectile-bound nucleon t matrix.

We then obtained various forms for the optical potential by making further assumptions. More specifically,

(1) To obtain the optical potential used by Piepho and Walker[13] (which is essentially the Foldy-Walecka form[1]) we assumed $\alpha = 0$, $\beta = 1$, the never-come-back approximation, $n \gg A$, a many-body $\Phi_0^* \Phi_0$ function that was a simple product of single particle densities, $E_n - E_0$ (nuclear) is ignorable, and separable potentials derived from fixed scatterer phase shifts. This optical potential is given by Eq. (53) and results obtained from it are denoted PW in the figures.

(2) To obtain the optical potential denoted by model correct we simply assumed the nucleus stays in ground state *between* projectile-bound nucleon t-matrix interactions, assumed a simple product wave function for the many-body ground state wave function and used the projectile-bound nucleon t matrix given in Eq. (27). In that equation if binding related effects are not neglected we obtain the optical potential incorporating intermediate nuclear state excitation (ISE). The model correct optical potential was also used as the starting point for further approximations to the complicated integral over the nucleon momentum distribution. Results following from the approximate treatment of the Fermi momentum are denoted $t\rho$ factorization and radial form factor in the figures.

(3) To obtain an optical potential via a procedure analogous to that followed in Ref. 11, we started with Eq. (55), then adopted Eq. (58) [in order to reduce a three dimensional momentum integral to a two dimensional one, used the frozen nucleon approximation Eq. (61a)] and the on-shell invariance of relative velocities to obtain an approximate angle transformation where the relevant coefficients are given by Eqs. (70a) and (70b). The Fermi-folding approximation used is discussed after Eq. (72). Results obtained from this optical potential are denoted LPT in the figures.

There is considerable interest in isolating effects due to higher order corrections to the optical potential. For example, it is important to identify the signature of "two nucleon correlation" effects. Also, effects due to isobar propagation are important to identify. One of our purposes in this manuscript is to demonstrate the kind of errors resulting from the usual approximations to the first order optical potential itself (even without considering higher order effects such as correlations). These first order approximations essentially result because treatments to date have not properly included one or more of the effects of ISE, Fermi-folding, and angle transformations. We hope our results will be a guide to the theoretical "error bars" to be expected when various approximations to the optical potential are adopted. Especially at medium to large angles it is important to note that LPT tended to be higher than model correct while PW tended to be low. However, *both* formalisms seem to be reasonable approximation to the model correct result. (The approximate angle transformation used by LPT is responsible for their overshoot of model correct at back angles. It is our understanding that this approximation is no longer being adopted.[20]) The difficulty is that model correct does not incorporate effects due to ISE excitation (which is certainly an important effect in raising the predicted cross section at back angles—see Fig. 3). Thus if the experimental data are higher at back angles than the predictions (as is most often the case—see Ref. 13), then in addition to looking for the resolution to the problem in higher order corrections to the optical potential, ISE effects should be more carefully studied.

Unfortunately, we have found in this study that the $t\rho$ factorization is frequently a poor approximation for $\theta > 90°$ (see Figs. 5 and 9). This implies, if one is *not* using the FW formalism, that studies of pion-nucleus elastic scattering should at least carry out the two dimensional Fermi-folding integral discussed in Sec. II and adopted by LPT.

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