Nuclear level densities

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An improved expression for the nuclear level density is obtained by introducing a higher term in the expansion of the excitation energy in terms of nuclear temperature. The new term leads to better fitting with the experimental results especially at the high excitation energy part of the spectrum. The level density parameters were calculated for the nuclei ^{48}Ti, ^{53}Mn, ^{56}Fe, ^{58}Co, ^{65}Zn, ^{113}Sn, and ^{116}Sb. For the two last nuclei the spin dependent level density was considered.

[NUCLEAR REACTIONS New expressions calculated for level density parameters derived for ^{48}Ti, ^{53}Mn, ^{56}Fe, ^{58}Co, ^{65}Zn, ^{113}Sn, ^{116}Sb.]

I. INTRODUCTION

According to the statistical model[1-7] the differential cross section for a given nuclear reaction is given by

$$\frac{d^2\sigma}{d\Omega\, d\epsilon} = c\epsilon\sigma_{\text{inv}}(\epsilon)w(E) \equiv \text{constant}Mw(E)\,, \qquad (1.1)$$

where ϵ is the kinetic energy of the outgoing particle, $\sigma_{\text{inv}}(\epsilon)$ is the inverse cross section for the formation of the compound nucleus from the outgoing particle and the residual nucleus, and $w(E)$ is the level density of the residual nucleus as a function of its excitation energy E. c is a constant and $M = \epsilon\sigma_{\text{inv}}(\epsilon)$.

The concepts of the nuclear temperature and entropy were introduced in the derivation of the expressions for level density.[1,8,9] Following Weisskopf,[1] one may expand the excitation energy, E, in powers of the nuclear temperature Θ, around $\Theta = 0$. Since the specific heat is zero for $\Theta = 0$, the power series expansion must start at least with the term containing Θ^2. Neglecting terms higher than Θ^2, one gets

$$E = a_0\Theta^2\,, \qquad (1.2)$$

where a_0 is the level density parameter. One defines the entropy S as follows:

$$S = \ln w(E) = \int \frac{dE}{\Theta}\,. \qquad (1.3)$$

Substituting for Θ from Eq. (1.2) into Eq. (1.3) one gets[1,7]

$$w(E) = R\exp[2(a_0E)^{1/2}]\,, \qquad (1.4)$$

where R and a_0 are parameters depending on the nucleus whose level density is w and excitation energy is E. It was found that the parameter R depends on the excitation energy E,[3,10] and hence the level density $w(E)$ may be written as

$$w(E) = \frac{R}{E^2}\exp[2(a_0 E)^{1/2}]\,, \qquad (1.5)$$

where R in Eq. (1.5) is another parameter depending on the nucleus.

Refinements were introduced to Eq. (1.5) by taking into consideration shell and pairing effects.[3,5,11-17] Due to residual interactions, pairing energy ΔE is subtracted from the excitation energy of the nucleus. Pairing energy ΔE was taken to be equal to zero, 1.4 and 2.8 MeV for odd-odd, even-odd or odd-even, and even-even nuclei, respectively.[18] Introducing pairing energy correction, Eq. (1.5) becomes

$$w(E) = \frac{R}{(E-\Delta E)^2}\exp\{2[a_0(E-\Delta E)]^{1/2}\}\,. \qquad (1.6)$$

In Eqs. (1.1) and (1.6) there is no dependence on angular momentum and so the spectrum of the emitted particles from the compound nucleus is generally isotropic. The situation is not so simple in case of heavy-ion induced reactions[3] where angular distribution of the emitted particle was found to be anisotropic.[19-21] The angular distributions depend on the angular momentum in the inlet and outlet channels, hence the spin dependent level density should be considered.[7,19,22,23] The spin dependent level density for a nucleus at excitation energy E and spin J, denoted by $w(E,J)$, was found to be

$$w(E,J) = w(E)\frac{2J+1}{[\pi(2\sigma^2)^3]^{1/2}}\exp[-(J+\tfrac{1}{2})^2/2\sigma^2]\,, \qquad (1.7)$$

where $w(E)$ is given by Eq. (1.6) and it actually represents the total state density summed over all spins of the nucleus. σ^2 is the spin cutoff parameter, given by[24,25]

$$\sigma^2 = \mathscr{I}\Theta/\hbar^2 . \tag{1.7a}$$

Θ is the nuclear temperature of the nucleus and $\mathscr{I}$ is its moment of inertia which is taken usually as equal to that of a rigid body, i.e.,[26-28]

$$\mathscr{I} = \tfrac{2}{5} mAR^2 , \tag{1.7b}$$

where R is the nuclear radius given by $R = r_0 A^{1/3}$, $r_0 = 1.5$ fm, m the mass of a nucleon, and A is the mass number of the nucleus. The differential cross section, which depends now on the angle of emission θ, was calculated[19,22,27-31] and found to run as

$$\frac{d^2\sigma}{d\Omega\, d\epsilon} = \text{constant} \sum_{J=0}^{\infty} (2J+1)(T_J/\Gamma_J)(2S_\gamma+1)\frac{w(E)}{[\pi(2\sigma^2)^3]^{1/2}} \times \sum_{l=0}^{\infty}(2l+1)(T_l^\gamma)\exp\{-[(J+\tfrac{1}{2})^2+(l+\tfrac{1}{2})^2]/2\sigma^2\} J_0 \frac{i(J+\frac{1}{2})(l+\frac{1}{2})}{\sigma^2} W_{Jl}(\theta,E)$$
$$\equiv \text{constant } Mw(E) , \tag{1.8}$$

where

$$W_{Jl}(\theta,E) = \frac{1}{4\pi}\left(J_0 \frac{i(J+\frac{1}{2})(l+\frac{1}{2})}{\sigma^2}\right)^{-1} \sum_k (-)^k (4k+1)\left(\frac{(2k!)}{(2^k k!)^2}\right)^2 J_{2k}\frac{i(J+\frac{1}{2})(l+\frac{1}{2})}{\sigma^2} P_{2k}(\cos\theta) \tag{1.8a}$$

and

$$\Gamma_J \equiv \text{total width} = \sum_\gamma (2S_\gamma+1)\int_0^\infty \frac{d\epsilon\, w(E)}{[\pi(2\sigma^2)^3]^{1/2}} \sum_{l=0}^{\infty}(2l+1)T_l^\gamma J_0\frac{i(J+\frac{1}{2})(l+\frac{1}{2})}{\sigma^2}\exp\left(-\frac{(J+\frac{1}{2})^2+(l+\frac{1}{2})^2}{2\sigma^2}\right),$$

where γ refers to the outgoing particle, s to its spin, and ϵ_{max} is the maximum possible kinetic energy for the outgoing particle. T_J is the transmission coefficient for the Jth partial wave in the inlet channel. T_l^γ is the transmission coefficient for the lth partial wave in specified outlet channel, J_{2k} denote the spherical Bessel functions, P_{2k} are Legendre polynomials, and θ is the angle of emission.

The infinity in the upper limit of the integral over the energy is replaced by the maximum kinetic energy available, ϵ_{max}, which is given by $(M_c - M_R - M_b)C^2 + E_c$ where Mc^2 is the energy equivalent to the mass of the nucleus M. The infinity in the upper limit of the sum over J is replaced by J_{max} determined by the equation[24]

$$E = J_{max}{}^2\hbar/2\mathscr{I} : \tag{1.9}$$

Also for each value of J the maximum value for the orbital angular momentum is $J + j_{max}$ where j_{max} is given by Eq. (1.9) for the residual nucleus with excitation energy E, i.e., $l_{max} = J + j_{max}$ replaces the infinity in the upper limit of summation over l. In fact the replacement of the infinite integrals by finite sums does not change the results since it was found by calculations that terms which are significant in the integrals or the sums are smaller than the proposed upper limits.

The level density parameter a_0 was found to be proportional to the mass number of the nucleus A.[32,33] It was found that $a_0 \simeq .05A$,[32] $a_0 \simeq A/13$ (Ref. 34) and $a_0 \simeq A/8$ (Ref. 3).

By plotting the experimental values of $\ln w(E) \times (E - \Delta E)^2$ versus $[(E-\Delta E)]^{1/2}$, according to Eq. (1.6)—the relation must be a straight line whose slope is $2\sqrt{a_0}$.

It was found that the relation deviates from the straight line at high and low excitation energies.[18] These were attributed possibly to the cascade emission and direct reaction, respectively.[18,35]

II. IMPROVED EXPRESSION FOR LEVEL DENSITY

While the expression (1.2) may be valid for reactions initiated by light projectiles, one might expect the second term in the expansion, containing Θ^3, to have appreciable values for heavy-ion reactions.

This may be due to the large amount of kinetic energy carried out by the projectile which will produce excessive heating on collision. The value of the nuclear temperature Θ may thus be expected to be larger than the corresponding value for lighter projectiles. One may thus be not completely justified in neglecting higher terms than Θ^2 in the expansion of the excitation energy E in powers of the nuclear temperature as was assumed in Eq. (1.2) above. It is the purpose of the present work to study, generally, the effect of higher terms in the expansion of the excitation energy E as functions of Θ, on the level density. Thus one may write

$$E = a\Theta^2 + b\Theta^3, \tag{2.1a}$$

where a and b are parameters to be determined. Dividing Eq. (2.1a) by $\Theta^3 E$ one gets

$$\left(\frac{1}{\Theta}\right)^3 - \frac{a}{E}\left(\frac{1}{\Theta}\right) - \frac{b}{E} = 0. \tag{2.1b}$$

Equation (2.1b) is a cubic equation in I/Θ, and its solution will depend on the value of the excitation energy E (see Appendix). The solution for the whole energy range is as follows:

(a) For values of E less than $4a^3/(27b^2)$,

$$\frac{1}{\Theta} = 2\left(\frac{a}{3E}\right)^{1/2} \cos\left(\tfrac{1}{3}\cos^{-1}\frac{b\sqrt{27E}}{2a^{3/2}}\right); \tag{2.2}$$

(b) for $E = 4a^3/(27b^2)$,

$$\frac{1}{\Theta} = 2\left(\frac{a}{3E}\right)^{1/2}; \tag{2.3}$$

(c) and for E larger than $4a^3/(27b^2)$,

$$\frac{1}{\Theta} = 2\left(\frac{a}{3E}\right)^{1/2} \cosh\left(\tfrac{1}{3}\cosh^{-1}\frac{b(27E)^{1/2}}{2a^{3/2}}\right). \tag{2.4}$$

It is clear that when the energy inequalities in Eqs. (2.2) and (2.4) become equalities both of them will reduce to Eq. (2.3).

The parameters a and b are expected to depend on the mass number of the nucleus. Thus for a given nucleus the factor $4a^3/27b^2$ defines the range of excitation energies for which Eq. (2.2), (2.3), or (2.4) is applied. The excitation energy E which equals $4a^3/27b^2$ will be called the critical energy, E_c, as it corresponds to the point at which expression (2.2) is replace by expression (2.4) for the nuclear temperature. If $b = 0$, all the energy range satisfies Eq. (2.2) and will be given by

$$\frac{1}{\Theta} = \lim_{b\to 0} 2\left(\frac{a}{3E}\right)^{1/2} \cos\left(\tfrac{1}{3}\cos^{-1}\frac{b(27E)^{1/2}}{2a^{3/2}}\right),$$

which is again Eq. (1.2), proposed by Weisskopf[1] for light nuclei and may be considered as the zero approximation for Eqs. (2.1a) and (2.2). As b/a decreases, i.e., as $4a^3/27b^2$ increases, the range of energies for which Eq. (2.2) is valid increases until b/a goes to zero (i.e., the Weisskopf zero approximation holds). In this case the critical value $E_c = 4a^3/(27b^2)$ goes to infinity and Eq. (2.2) with $b \to 0$ holds for the whole energy range. Hence again, one gets the Weisskopf formula for nuclear temperature and consequently the level density, for all the energy range.

To obtain level density, consider again Eq. (1.3)

$$\ln w(E) = \int \frac{dE}{\Theta}.$$

From Eq. (2.1) one gets

$$\frac{dE}{\Theta} = (2a + 3b\Theta)d\Theta,$$

$$\ln w = 2a\Theta + \tfrac{3}{2}b\Theta^2 + \ln R. \tag{2.5}$$

where R is a constant. Substituting for Θ in Eq. (2.5) by Eqs. (2.2), (2.3), or (2.4) according to the values of E, introducing the pairing energy correction ΔE, and replacing the constant R by $R/(E-\Delta E)^2$ as discussed above one gets,

$$w(E) = \frac{R}{(E-\Delta E)^2} e^N, \tag{2.6}$$

where

$$N = (3a(E-\Delta E)^{1/2} \sec\left(\tfrac{1}{3}\cos^{-1}\frac{b[27(E-\Delta E)]^{1/2}}{2a^{3/2}}\right) + \frac{gb}{8a}(E-\Delta E)\sec^2\left(\tfrac{1}{3}\cos^{-1}\frac{b[27(E-\Delta E)]^{1/2}}{2a^{3/2}}\right)$$

$$\text{for } E - \Delta E \leq \frac{4a^3}{27b^2} \tag{2.6a}$$

and

$$N = [3a(E-\Delta E)]^{1/2} \operatorname{sech}\left(\tfrac{1}{3}\cosh^{-1}\frac{b[27(E-\Delta E)^{1/2}}{2a^{3/2}}\right) + \frac{gb}{8a}(E-\Delta E)\operatorname{sech}^2\left(\tfrac{1}{3}\cosh^{-1}\frac{b[27(E-\Delta E)]^{1/2}}{2a^{3/2}}\right)$$

$$\text{for } E - \Delta E \geq \frac{4a^3}{27b^2}. \tag{2.6b}$$

The energy equality signs used in Eqs. (2.6a) and (2.6b) will replace the substitution for Θ from Eq. (2.3) as mentioned above. To get the complete spin dependent level density, $w(E)$ in Eq. (1.7) is now replaced by the expressions (2.6), (2.6a), and (2.6b). Moreover, the nuclear temperature defined in Eq. (1.7a) is now given by Eqs. (2.2), (2.3), and (2.4), i.e.; one finally gets

$$w(E,J) = \frac{R}{(E-\Delta E)^2}\frac{2J+1}{[\pi(2\sigma^2)^3]^{1/2}}\exp\left(-\frac{(J+\frac{1}{2})^2}{2\sigma^2}\right)e^N, \tag{2.7}$$

where N is given by Eqs. (2.6a) and (2.6b).

TABLE I. Values of parameters.

Parameter values	^{48}Ti	^{53}Mn	^{56}Fe	^{58}Co	^{65}Zn	^{113}Sn	^{116}Sb
a_0 MeV^{-1} ($b=0$)	5	6.5	6.7	8.7	7	22.95	23.91
a MeV^{-1}	4.25	5.65	6.5	7.3	6.25	20.1	20.25
b MeV^{-2}	1.2	2.15	2.075	3.04	2.602	4.7	4.73
E_c MeV	7.6	5.7	9.45	6.25	5.304	54.45	54.98

III. CALCULATIONS

To obtain the parameters a and b of the improved new formula of level density, $w(E)$, the equations (2.6), (2.6a), and (2.6b) are used to fit the experimental values for level density. Experimental values of level density are obtained by considering the experimental values for the differential cross section $d^2\sigma/d\Omega d\epsilon$ which is related to $w(E)$ by Eqs. (1.1) or (1.8). The second equation is used when the angular momentum effect is taken into consideration; it is clear that the nucleus for which the level density is obtained is the residual nucleus in the nuclear reaction under consideration.

The experimental values for the level density as a function of excitation energy which are chosen for testing the new formulas, are considered for the nuclei ^{58}Co, ^{65}Zn,[18] ^{48}Ti, ^{56}Fe, ^{53}Mn,[6] ^{116}Sb, and ^{113}Sn.[19] For the first five nuclei Eq. (1.1) was used in the calculations, while for the two last nuclei Eq. (1.8) was used. The representation of $(d^2\sigma/d\Omega d\epsilon)[(E-\Delta E)^2/M]$ which is equal to a constant k multiplied by the exponential term in the level density expressions, versus $(E-\Delta E)^{1/2}$ on a semilogarithmic scale are considered for these nuclei. Since Weisskopf's formula (i.e., $b=0$) is assumed to be a good zero approximation, one may start the fitting procedure by assuming first $b=0$, i.e., we use Eq. (1.6) for $w(E)$. Fitting the experimental points by a straight line, the constants k and a_0 are determined. One then takes these values of a and b as starting points, changing their values gradually and using Eqs. (2.6a) and (2.6b) until a good fit to the experimental results is obtained. The pairing effect is considered for the nuclei ^{65}Zn, ^{58}Co, ^{116}Sb, and ^{113}Sn where ΔE is taken as 1.4, 0, 0, and 1.4 MeV, respectively. The experimental data from Ref. 6 were multiplied by the square of the corresponding excitation energy so that experimental values of the level density could be fitted by Eq. (1.6).

For calculating M the spin cutoff parameters σ^2 given by Eq. (1.7a) were calculated for residual nuclei in all exit channels for the reaction given in Ref. 19 where the excitation energy of the compound nucleus ^{117}Te is 71 MeV. The outgoing particles in the exit channels were protons, α particles, deutrons, tritons, ^{6}Li, and ^{7}Li. The widths for last two channels were calculated and found negligible. The nuclear temperature and the spin cut off parameters, used in calculating Γ_J and $W_{Jl}(\theta, E)$ were calculated using initial values for the parameter a_0, i.e.,

$$\Theta = (E/a_0)^{1/2}, \quad \text{where } a_0 \simeq \tfrac{1}{6}A.$$

Transmission coefficients for the reaction given in Ref. 19 were calculated using Eqs. (5.5), (5.9), and (5.9) of Ref. 1 which are based on the WKB approximation. The error due to the discontinuity at the barrier was treated by interpolation of the results after and before the barrier. From all these data the values of M were calculated by Eq.

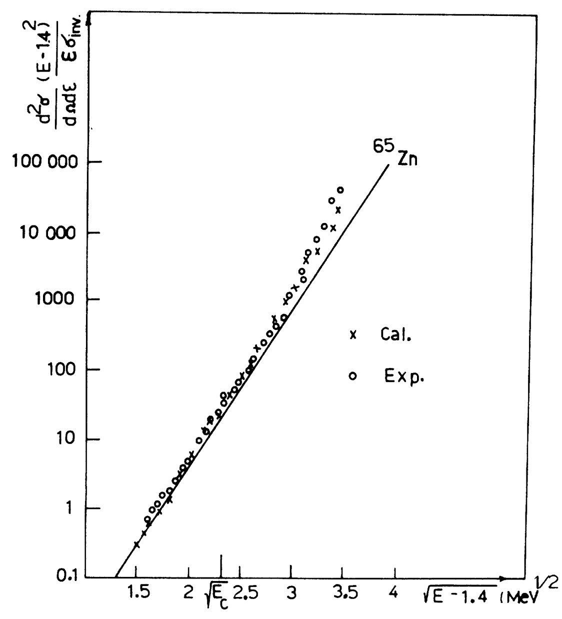

FIG. 1. The relation between $(d^2\sigma/d\Omega d\epsilon)[(E-\Delta E)^2/\epsilon\sigma_{inv}]$ and $(E-\Delta E)^{1/2}$ for ^{65}Zn: ΔE is the pairing energy taken as 1.4 MeV (Ref. 18). The circles represent experimental data obtained from Ref. 18. The ×'s represent calculated values using Eqs. (2.6a) and (2.6b) for N where $a=6.25$ MeV^{-1} and $b=2.602$ MeV^{-2} and $\sqrt{E_c}=2.3$ MeV$^{1/2}$. The straight line represents the results given by zero order formula (1.6) using $a_0=7$ MeV^{-1} ($b=0$).

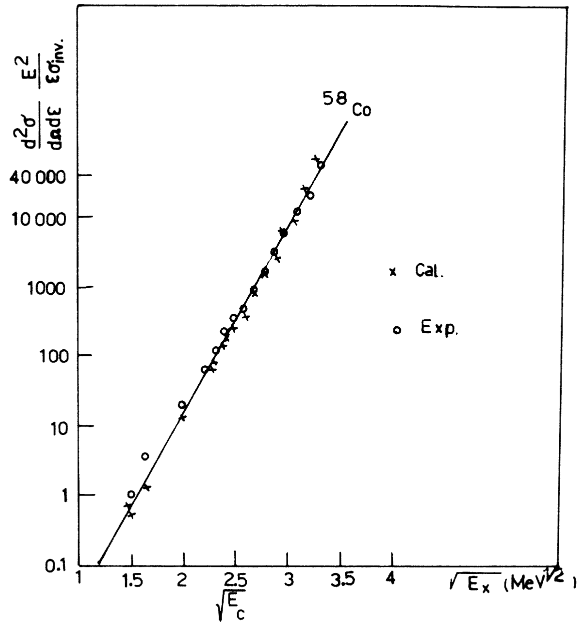

FIG. 2. The relation between $(d^2\sigma/d\Omega d\epsilon)[(E-\Delta E)^2/\epsilon\sigma_{inv}]$ and $(E-\Delta E)^{1/2}$ for ^{58}Co, ΔE is taken to be zero (Ref. 18). The circles represent experimental data obtained from Ref. 18. The ×'s represent calculated values using Eqs. (2.6a) and (2.6b) for N, where $a=7.3$ MeV^{-1} and $b=3.04$ MeV^{-2}. $\sqrt{E_C}=2.5$ MeV$^{1/2}$. The straight line represents the results given by zero order formula (1.6) using $A_0=8.7$ MeV^{-1} $(b=0)$.

(1.8b) for ^{113}Sn and ^{116}Sb.

The values of a, b, a_0 and the critical energy $(E_c=4a^3/27b^2)$ are given for the chosen nuclei in Table I.

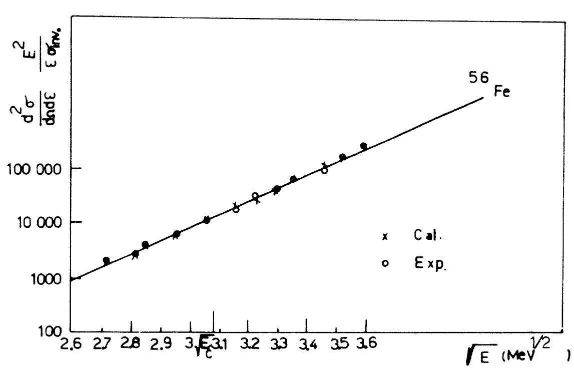

FIG. 3. The relation between $(d^2\sigma/d\Omega d\epsilon)(E^2/\epsilon\sigma_{inv})$ and $\sqrt{E}$ for ^{56}Fe. The experimental data (denoted by circles) are obtained from Ref. 6. The calculated values according to Eqs. (2.6a) and (2.6b) for N are denoted by ×'s, using $a=6.5$ MeV^{-1} and $b=2.075$ MeV^{-2}, $\sqrt{E_c}=3.07$ MeV$^{1/2}$. The straight line zero approximation gives $a_0=6.7$ MeV^{-1} $(b=0)$.

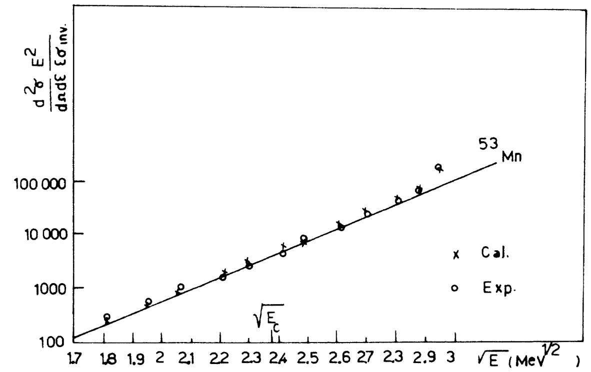

FIG. 4. Same relation as in Figs. 1–3 for ^{53}Mn, using same notation for the experimental data, Ref. 6 and the calculated values using $a=5.65$ MeV^{-1} and $b=2.15$ MeV^{-2}, $\sqrt{E_c}=2.38$ MeV$^{1/2}$. The straight line relation gives $a_0=6.5$ MeV^{-1} $(b=0)$.

IV. DISCUSSION

From Figs. 1–5 it is clear that if the expression (1.6) for the level density is used the experimental data can be fitted by a straight line whose slope is $2\sqrt{a_0}$ multiplied by $\log_{10}e$. But the experimental points deviate from the straight line relation at high (e.g., Figs. 1 and 4) and low (Fig. 2) excitation energies. Using Eqs. (2.6), (2.6a), and (2.6b) for level densities, developed in the present work, we notice that in Figs. 1 and 4 the calculated points agree very well with the experimental points in the regions where the straight line, representing the case with $b=0$, deviates from the experimental data. This shows clearly that the above derived formulas are especially valid for regions of high excitation energy. This is expected as high excitation energy means high nuclear temperature so the neglect of the term $b\Theta^3$ in the expansion (2.1)

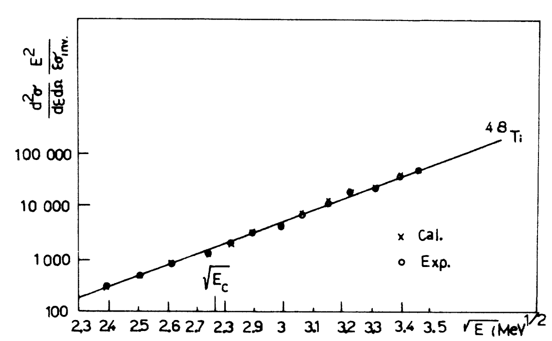

FIG. 5. Same relation as in the previous figures, for ^{48}Ti using same notation of the experimental data, Ref. 6, and the calculated values using $a=4.25$ MeV^{-1}, $b=1.2$ MeV^{-2}, and $\sqrt{E_c}=2.75$ MeV$^{1/2}$. The straight line relation gives $a_0=5$ MeV^{-1} $(b=0)$.

may not be completely justified. In the other graphs (2, 3, and 5) the zero approximation [i.e., Eq. (1.6)] and the present expressions for the level density [i.e., Eqs. (2.6), (2.6a), and (2.6b)] give nearly the same results. We notice that in Fig. 3 at low excitation the experimental points do not agree completely with the calculated values which may suggest that direct reactions might contribute to this range.

From Figs. 6 and 7 one may observe slight differences in the values of the level density obtained from experimental values of the differential cross section at high excitation and at different angles of emission. This difference in values of $(d^2\sigma/d\Omega d\epsilon)[(E-\Delta E)^2/M]$ obtained at the different angles of emission may be attributed to the fact that using the WKB approximation in calculating transmission coefficients may not be accurate enough especially for kinetic energies far below the barrier, i.e., for high excitation energies of the residual nucleus. From Figs. 6 and 7 one may also observe, however, that using the improved expression for level density the calculated points show approximately the same behavior of the experimental points.

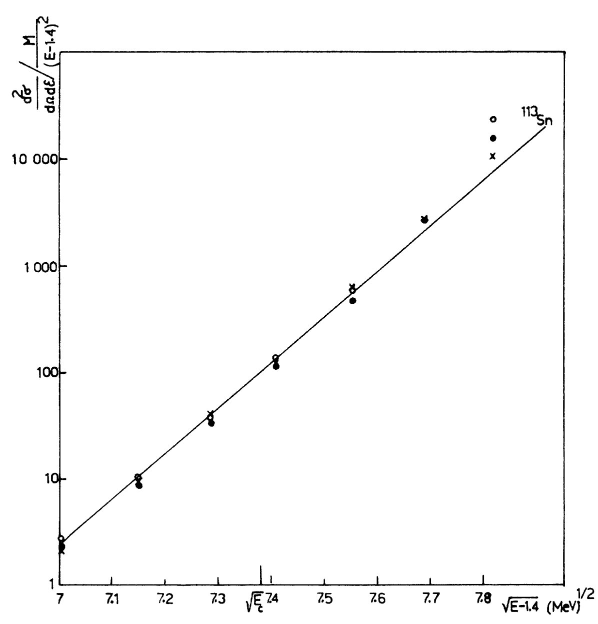

FIG. 6. The relation between $(d^2\sigma/d\Omega d\epsilon)/[M/(E-\Delta E)^2]$ and $(E-\Delta E)^{1/2}$ for ^{113}Sn. M is given by Eq. (1.8) and $\Delta E = 1.4$ MeV. The experimental values of $d^2\sigma/d\Omega d\epsilon$ are taken from Ref. 19. The open circles represent experimental values for level density $w(E)$ where $\Theta = 33°3$ and the solid circles represent same quantity but at $\Theta = 89°$. The ×'s s represent the calculated values for level density using Eqs. (2.6a) and (2.6b) for N where $a = 20.1$ MeV^{-1}, $b = 4.7$ MeV^{-2}, and $\sqrt{E_c} = 7.379$ MeV$^{1/2}$. The straight line represents the results given by zero order formula for level density, i.e., Eq. (1.6) where $a_0 = 22.954$ MeV^{-} $(b = 0)$.

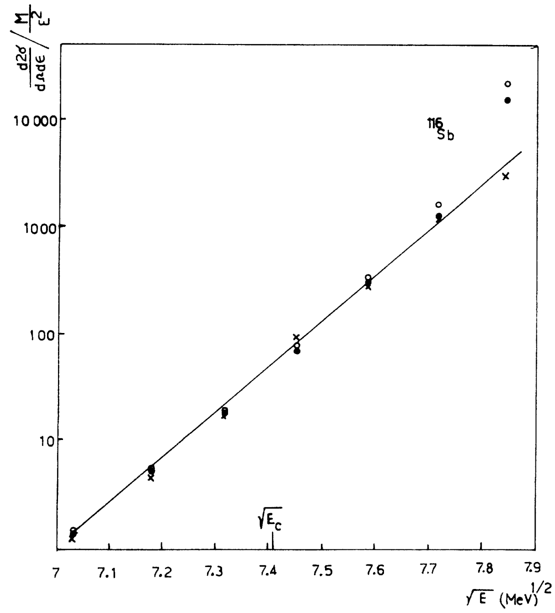

FIG. 7. Same relation as in Fig. 6 for ^{116}Sb. $\Delta E = 0$. The open circles represents experimental values, Ref. 19 of level density where $\theta = 171°$ and the solid circles represent same quantity at $\theta = 94°$ same notation for the straight line and ×'s as in Fig. 6, where $a = 20.25$ MeV^{-1}, $b = 4.73$ MeV^{-2}, $\sqrt{E_c} = 7.41$ MeV$^{1/2}$, and $a_0 = 23.91$ MeV^{-1} $(b = 0)$.

Figure 8 illustrates the variation of the parameters a, b, and a_0 (i.e., when $b = 0$) with the mass number A for the first five nuclei used in the calculations.

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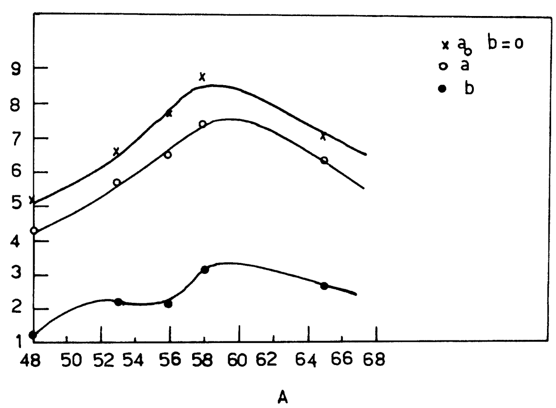

FIG. 8. Shows values of a, b, a_0 versus the mass number for the nuclei ^{65}Zn, ^{58}Co, ^{56}Fe, ^{53}Mn, and ^{48}Ti.

APPENDIX: SOLUTION OF CUBIC EQUATIONS

Consider the cubic equation in Z

$$Z^3 + 3HZ + G = 0. \tag{A1}$$

H and G are real values. The solution of Eq. (A1) has three cases[36].

(a) $G^2 + 4H^3 < 0$, in which case H is negative. The cubic equation has three distinct real roots, namely,

$$2K\cos\Theta/3, \quad 2K\cos(\Theta + 2\pi)/3, \quad 2K\cos(\Theta + 4\pi)/3,$$

where

$$K = \sqrt{-H}, \quad \Theta = \cos^{-1}(-G/2|H|^{3/2}). \tag{A2}$$

(b) $G^2 + 4H^3 = 0$. H must be negative in order to have real values for G and H. The roots are

$$2\sqrt{-H}, \quad -\sqrt{-H}, \quad -\sqrt{-H}. \tag{A3}$$

(c) $G^2 + 4H^3 > 0$. The equation has one real root, and when G and H are negative this real root is

$$2\sqrt{-H}\cosh\left(\tfrac{1}{3}\cosh^{-1}\frac{|G|}{2|H|^{3/2}}\right). \tag{A4}$$

Consider now Eq. (2.1b); it is a cubic equation in $1/\Theta$ where

$$H = -\frac{a}{3E} \quad \text{and} \quad G = -\frac{b}{E}.$$

It is clear that b must be positive, otherwise, according to Eq. (2.1a), there will be a value for Θ different from zero for which E, the excitation energy, is zero which is a contradiction to the thermodynamical analogy. So G is negative. The roots of Eq. (2.1b) will be obtained from Eqs. (2), (3), and (4) as follows:

(a) If $G^2 + 4H^3 = \frac{b^2}{E^2} - \frac{4a^3E}{27E^3} < 0$,

i.e.,

$$E < \frac{4a^3}{27b^2},$$

thus from Eq. (2)

$$\frac{1}{\Theta} = 2\left(\frac{a}{3E}\right)^{1/2}\cos\left(\tfrac{1}{3}\cos^{-1}\frac{b\sqrt{27E}}{2a^{3/2}}\right). \tag{A5}$$

The two other roots are excluded as when taking limit as b tends to zero they give negative or zero values for Θ, while for Eq. (5) when b is zero $\Theta = (E/a)^{1/2}$, i.e., Weisskopf expression (1.2):

(b) If $G^2 + 4H^3 = \frac{b^2}{E^2} - \frac{4a^3}{27E^3} = 0$,

i.e.,

$$E = \frac{4a^3}{27b^2},$$

thus from Eq. (3)

$$\frac{1}{\Theta} = 2\left(\frac{a}{3E}\right)^{1/2}. \tag{A6}$$

The two other roots are excluded as they are negative:

(c) If $G^2 + 4H^3 = \frac{b^2}{E^2} - \frac{4a^3}{27E^3} > 0$.

i.e.,

$$E > \frac{4a^3}{27b^2},$$

$$G = -\frac{b}{E}$$

is negative as stated above, so from Eq. (4)

$$\frac{1}{\Theta} = 2\left(\frac{a}{3E}\right)^{1/2}\cosh\left(\tfrac{1}{3}\cosh^{-1}\frac{b\sqrt{27E}}{2a^{3/2}}\right). \tag{A7}$$

The two other roots are excluded as they are complex.

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