Solutions of the radial equation for scattering by a nonlocal potential

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Nonlocality is characteristic of potentials describing processes in which degrees of freedom are eliminated, and provides for a description of a much wider variety of phenomena than that encountered with short range local potentials. In this paper, properties of the radial equation for symmetric nonlocal potentials are investigated, using a configuration space approach and restricting the analysis to real k. Emphasis is placed on identifying those constraints associated with a local potential which are relaxed in going from a local to a nonlocal potential. The Fredholm determinants associated with the integral equations for the physical, regular, and Jost solutions are central to the development. Unlike the case for a short range local potential, for a nonlocal potential these Fredholm determinants can vanish for real $k \neq 0$. It is shown that when $D^{\pm}(k) = 0$ the other Fredholm determinants are zero as well. The properties of the solutions at the zeroes of the Fredholm determinants are discussed in this context, and the concepts "spurious state" and "continuum bound state" are clarified. The behavior of solutions is illustrated with examples.

[NUCLEAR REACTIONS Scattering by a nonlocal potential, Fredholm determinants and their zeroes, continuum bound states, spurious states.]

I. INTRODUCTION

Formalisms whereby an equation of motion for a many-particle system is reduced to an effective equation for the relative motion of two particles invariably result in an effective interaction that is nonlocal. Thus, the effective nucleon-nucleus interaction is nonlocal by virtue of its taking into account excitations of the target nucleus. It is nonlocal even when target excitations are neglected if the Pauli principle is imposed. The effective interaction between composite nuclear systems is likewise nonlocal for the same reasons. In these cases, or in any case of an effective interaction, the interaction is nonlocal independent of the basic interaction between the constituents of the many-particle system; rather, nonlocality is characteristic of processes in which degrees of freedom are eliminated. In this context, the nonlocality of an effective interaction provides the means of taking into account phenomena characteristic of a many-particle system within the framework of a one-particle description.

For a short range local potential analytic constraints associated with the radial equation provide an important albeit indirect link between the potential and solution of the equation for bound state and scattering wave functions. One consequence of the nonlocality of an effective interaction is that the radial equation obtained from a partial wave decomposition of the effective equation is not governed by certain of these constraints. With this in mind, the purpose of the present work is to initiate the development of a description of phenomena characteristic of nonlocal potentials in terms of the analytic properties of the radial equation for such potentials. Properties of the wave functions associated with nonlocal potentials are developed and the extent to which the behavior of these wave functions is expanded relative to that of wave functions for short range local potentials is discussed. In particular, emphasis is placed on identifying those constraints associated with a local potential which are relaxed in going from a local to a nonlocal potential.

Phenomena characteristic of short range local potentials have been studied extensively in the past, and at the present time the connection between these phenomena and the analytic properties of the radial equation is well understood.[1,2] The approach followed here for nonlocal potentials is similar to the approach[1] used for local potentials in that Fredholm determinants of certain integral equations are central to the development. In contrast to the class of local potentials, where only one Fredholm determinant is sufficient, several Fredholm determinants are important to the description of the analytic properties of the radial equation for a nonlocal potential. The standard results for a local potential[1] are recovered, however, since all but one of the Fredholm determinants are identically equal to unity in the limit as the potential becomes local. Thus, the analytic constraints specific to the class of local potentials follow in the usual way from the behavior of the Fredholm determinants in this limit.

The analytic constraints associated with the radial equation for a short range local potential are automatically incorporated in local potential models of an effective interaction, and such models have been successful in describing a wide variety of experimental results. Nevertheless, it is possible to isolate situations where relaxation of the constraints is necessary in order to accommodate experimental results. The unified theory of nuclear reactions[3] is an example of a general situation where the nonlocality of the interaction provided by the theory is necessary. The nonlocal effective nucleon-nucleus interaction obtained from the unified theory is able to describe all types of resonant behavior observed in nucleon-nucleus collisions. This would not be possible if the effective interaction obtained from the theory were a short range local potential. For a local potential, the reduced width of a resonance, roughly $3\hbar^2/2mR^2$, depends on the range of the potential. Once the range has been fixed, the radial equation with a local potential cannot describe an isolated resonance which has a reduced width very much smaller than $3\hbar^2/2mR^2$. The nonlocal effective interaction of the unified theory encompasses all types of resonant behavior, from the broad potential scattering resonances to the very narrow compound resonances. In the vicinity of an isolated compound resonance, the phase shift for the resonant partial wave undergoes a rapid, almost discontinuous,[4] π change of phase. The change of phase is discontinuous in the limit as the width of the resonance approaches zero. This type of behavior is inconsistent with scattering by a short range local potential, where the continuity of the phase shift is a consequence of the analytic properties of the radial equation.

The resonating group method[5,6] of deriving effective interactions between composite nuclear systems is another example of a situation where the nonlocality of an effective interaction is important. The effective interaction between two α particles is a good illustration since the ^{8}Be nucleus is unstable and the low-lying states show up as resonances in α-α scattering. Resonating group calculations[7,8] of the relative motion of two α particles show that the Pauli principle for nucleons requires the wave functions in the S and D partial waves to have additional nodes at small distances while the wave functions in the G and higher partial waves are not required to have additional nodes. The ground state of ^{8}Be is a $3S$ radial configuration[9] and the first excited state is a $2D$ radial configuration. The $1S$, $2S$, and $1D$ radial configurations for the relative motion of two α particles are excluded by the Pauli principle.

The connection between the Pauli principle and the nonlocality of the effective interaction stems from the fact that excluded radial configurations are not compatible with local potentials. A general requirement on the solutions of the radial equation for a local potential is that the solution corresponding to the lowest energy eigenstate in each partial wave does not have nodes within the range of the potential except for the node at the origin.[10] Thus, a local potential model wave function for the ground state of ^{8}Be is required by an analytic constraint to be a $1S$ radial configuration. Otherwise it would not be a ground state wave function. Similarly, a local potential model wave function for the first excited state of ^{8}Be is required to be a $1D$ radial configuration. It is the nonlocality of the effective interaction between two α particles which provides the means by which this constraint associated with the radial equation for a local potential is relaxed and the Pauli principle for nucleons properly incorporated.

The examples just considered show that the nonlocality of an effective interaction does provide a description of a much wider variety of phenomena than that encountered with short range local potentials, and that some of the expanded capabilities of nonlocal potentials are related to a relaxation of analytic constraints associated with the radial equation for a local potential. Work is presently in progress on understanding the specific mechanisms by which nonlocal potentials provide the flexibility for describing these phenomena. However, the state of development of the connection between phenomena associated with nonlocal potentials and the analytic properties of the radial equation for this class of potentials is not at present satisfactory. One aspect of this problem has been discussed by Bolsterli,[11] who noted the use of different definitions of the phase shift for partial wave scattering by a nonlocal potential. These definitions are equivalent for a local potential; for a nonlocal potential, the phase shifts obtained from the different definitions lead to quite different interpretations. In addition, it is not clear which of the definitions of the phase shift is related to the number of excess nodes of the wave function within the range of the interaction. The present paper is a report on an investigation of the effects on the wave function of the relaxation of constraints associated with a local potential. The consequences of this relaxation for the phase shifts will be discussed in a subsequent paper.

Preliminary work has indicated that understanding the effects of antisymmetrization may require analysis of nonlocal potentials which are not symmetric.[12] However, in this paper discussions of the behavior of scattering wave functions are limited to nonlocal potentials which are symmetric and real. None of the aspects to be considered here requires the extension of k to the complex plane; thus, the analysis is limited for convenience to real values of the wave number. Furthermore, discussions are limited to the $l=0$ partial wave.

The general results presented in this paper are illustrated by specific examples of nonlocal potentials which have wave functions that exhibit distinctive features associated with nonlocality.

II. BASIC EQUATIONS AND NOMENCLATURE

The radial equation for s-wave scattering of a particle with wave number k by a nonlocal potential $V(r,s)$ is

$$u(k,r)''+k^2u(k,r)=\int_0^\infty V(r,s)u(k,s)ds\,. \tag{1}$$

The condition that there exists a $\beta>0$ such that

$$e^{\beta r}\int_0^\infty V(r,s)ds<\infty \tag{2}$$

is imposed on the potential; this condition is sufficient to insure the convergence of all integrals given in this paper.

Through the use of an appropriate Green's function the radial equation can be converted to an integral equation for a solution which, if it exists, satisfies a given set of boundary conditions. Several integral equations, their solutions, and related quantities are defined in the following paragraphs. These definitions are a straightforward extrapolation to the class of nonlocal potentials of the standard integral equations and solutions for the class of short range local potentials as given by Newton.[1] The limitations of this extrapolation procedure have been noted previously[13]; they are essential to the aim of this work and are discussed more fully in subsequent sections.

The physical solution $\psi^+(k,r)$ is defined by the mixed boundary conditions that $\psi^+(k,r)$ have the asymptotic form

$$\psi^+(k,r)\rightarrow\tfrac{1}{2}i[e^{-ikr}-S^+(k)e^{ikr}] \tag{3}$$

as $r\rightarrow\infty$, and that $\psi^+(k,r)$ be regular at $r=0$. The function $S^+(k)$ in Eq. (3) is the s-wave scattering matrix element. The physical solution $\psi^+(k,r)$ and its conjugate $\psi^-(k,r)$ satisfy the integral equations

$$\psi^\pm(k,r)=\sin kr+\int_0^\infty\int_0^\infty G^\pm(k,r,r')V(r',s)\psi^\pm(k,s)ds\,dr'\,, \tag{4}$$

where

$$G^\pm(k,r,r')=-k^{-1}e^{\pm ikr_>}\sin kr_<\,. \tag{5}$$

The asymptotic form of $\psi^-(k,r)$ for large r is given by Eq. (3) with $S^+(k)$ replaced by $S^-(k)\equiv S^+(k)^*$. The Fredholm determinants associated with the kernels of Eqs. (4) are denoted by $D^+(k)$ for the physical solution and $D^-(k)$ for its conjugate.

The regular solution $\varphi(k,r)$ is defined by the boundary conditions

$$\varphi(k,0)=0\,, \tag{6a}$$

$$\varphi(k,0)'=1\,. \tag{6b}$$

The integral equation for the regular solution is

$$\varphi(k,r)=k^{-1}\sin kr+\int_0^r\int_0^\infty G(k,r,r')V(r',s)\varphi(k,s)ds\,dr'\,, \tag{7}$$

where

$$G(k,r,r')=k^{-1}\sin k(r-r')\,. \tag{8}$$

The Fredholm determinant associated with the kernel of Eq. (7) is denoted by $D(k)$.

The conjugate irregular or Jost solutions $f^\pm(k,r)$ are defined[1] by the boundary conditions

$$\lim_{r\rightarrow\infty}e^{\mp ikr}f^\pm(k,r)=1\,. \tag{9}$$

The integral equations for the Jost solutions are

$$f^\pm(k,r)=e^{\pm ikr}-\int_r^\infty\int_0^\infty G(k,r,r')V(r',s)f^\pm(k,s)ds\,dr'\,. \tag{10}$$

The Fredholm determinant associated with the kernel of Eqs. (10) is denoted by $\Delta(k)$.

Questions about the linear independence of the solutions defined above for a nonlocal potential require somewhat more attention than for a local potential. This is due to the fact that the Wronskian of two solutions to Eq. (1) is, in general, r dependent, whereas the Wronskian of two solutions for a local potential is independent of r. The Wronskian of the Jost solutions is

$$W(k,r)=f^-(k,r)f^+(k,r)'-f^+(k,r)f^-(k,r)'\,, \tag{11}$$

which can be written[13,14]

$$W(k,r)=2ik\left[1-\int_r^\infty\int_0^\infty V(r',s)Q(k,r',s)ds\,dr'\right], \tag{12}$$

where

$$Q(k,r,s)=\frac{f^-(k,r)f^+(k,s)-f^+(k,r)f^-(k,s)}{2ik}\,. \tag{13}$$

The value of this Wronskian at $r=\infty$ is $2ik$; since only symmetric potentials are considered here, it has the same value at $r=0$.[13,14] While the Wronskian of the Jost solutions for a nonlocal potential can be zero for some values of r, $r=0$ and ∞ excepted, it cannot be identically zero for all r. Thus, the Jost solutions for a nonlocal potential are linearly independent when $k\neq 0$.

The Jost function $\mathcal{L}^+(k)$ and its conjugate $\mathcal{L}^-(k) \equiv \mathcal{L}^+(k)^*$ are defined according to

$$\mathcal{L}^{\pm}(k)=f^{\pm}(k,0)\,. \tag{14}$$

Since $W(k,0)=2ik$, it follows that the Jost function has the property

$$\mathcal{L}^{\pm}(k)\neq 0 \quad (k\neq 0)\,. \tag{15}$$

This restriction on the behavior of the Jost function holds for both local and symmetric nonlocal potentials. The behavior of the Jost function at $k=0$ when $\mathcal{L}^{\pm}(0)=0$ requires special attention[15] for both types of potentials and is not considered in this paper.

When they exist, the physical, regular, and Jost solutions are related through the Jost function; i.e.,

$$\psi^{\pm}(k,r)=k\varphi(k,r)/\mathcal{L}^{\pm}(k) \tag{16}$$

and

$$\varphi(k,r)=\frac{\mathcal{L}^-(k)f^+(k,r)-\mathcal{L}^+(k)f^-(k,r)}{2ik}\,. \tag{17}$$

These relations follow the linear independence of the Jost solutions and demonstrate the linear dependence of the physical and regular solutions.

The Jost function has the following integral representations:

$$\mathcal{L}^{\pm}(k)=1+\int_0^\infty\int_0^\infty k^{-1}\sin kr\,V(r,s)f^{\pm}(k,s)ds\,dr \tag{18}$$

and

$$\mathcal{L}^{\pm}(k)=1+\int_0^\infty\int_0^\infty e^{\pm ikr}V(r,s)\varphi(k,s)ds\,dr\,. \tag{19}$$

The first representation given follows from the definitions (10) and (14), while the second is obtained from Eq. (17) evaluated in the limit as $r\to\infty$. Both representations bear a close resemblance to the integral representations of the Jost function for a local potential.[1]

In addition to the conjugate irregular solutions $f^{\pm}(k,r)$ it is convenient to introduce an irregular solution $\theta(k,r)$, defined by

$$\theta(k,r)=\tfrac{1}{2}[\mathcal{L}^+(k)^{-1}f^+(k,r)+\mathcal{L}^-(k)^{-1}f^-(k,r)]\,. \tag{20}$$

By definition, $\theta(k,r)$ obeys the mixed boundary conditions

$$\theta(k,0)=1 \tag{21}$$

and, as $r\to\infty$,

$$\theta(k,r)\to|\mathcal{L}^+(k)|^{-1}\cos(kr+\delta)\,, \tag{22}$$

where

$$\delta(k)=-\operatorname{phase}[\mathcal{L}^+(k)]\,. \tag{23}$$

It follows that $\theta(k,r)$ is real. The integral equation for the irregular solution $\theta(k,r)$ is

$$\theta(k,r)=\tfrac{1}{2}[\mathcal{L}^+(k)^{-1}e^{ikr}+\mathcal{L}^-(k)^{-1}e^{-ikr}] - \int_r^\infty\int_0^\infty G(k,r,r')V(r',s)\theta(k,s)ds\,dr'\,. \tag{24}$$

The Fredholm determinant associated with the kernel of Eq. (24) is $\Delta(k)$.

The solutions $\theta(k,r)$ and $\varphi(k,r)$ are linearly independent. Their Wronskian is

$$W_{\theta\varphi}(k,r)=\theta(k,r)\varphi(k,r)'-\varphi(k,r)\theta(k,r)'\,, \tag{25}$$

which can be written

$$W_{\theta\varphi}(k,r) =1+\int_r^\infty\int_0^\infty V(r',s)[\theta(k,r')\varphi(k,s) -\varphi(k,r')\theta(k,s)]ds\,dr'\,. \tag{26}$$

The integral in this equation vanishes for a symmetric potential in the limit as $r\to 0$. Thus

$$W_{\theta\varphi}(k,0)=W_{\theta\varphi}(k,\infty)=1\,. \tag{27}$$

The integral equations for the solutions defined in the preceding paragraphs are inhomogeneous Fredholm equations of the second kind. The conditions for which a unique solution to such an equation exists are well known. In simplest form, a solution exists and is unique if the Fredholm determinant associated with the kernel of the integral equation is not zero. The fact that the Fredholm determinants $D(k)$, $\Delta(k)$, and $D^{\pm}(k)$ may have zeroes for real $k\neq 0$ when the potential is nonlocal represents a basic difference between local and nonlocal potentials. It is a difference which affects even the modest aim of defining a solution to Eq. (1) by prescribing boundary conditions. The properties of the Fredholm determinants and their zeroes are discussed in the next section, and in the following sections the solutions to Eq. (1) are considered when some of the Fredholm determinants are zero.

III. FREDHOLM DETERMINANTS AND THEIR ZEROES

The Fredholm determinants have the following properties: $\operatorname{Re}D^{\pm}(k)$, $D(k)$, and $\Delta(k)$ are even functions of k, while $\operatorname{Im}D^{\pm}(k)$ are odd functions of

k. Both $D(k)$ and $\Delta(k)$ are real for real k, and $D(k)=\Delta(k)$ for local and symmetric nonlocal potentials.[16] These properties of the Fredholm determinants reflect corresponding properties of the Green's functions in Eqs. (5) and (8).

For a *local* potential, the kernels of each of the integral equations defined previously, with the exception of those for $\psi^{\pm}(k,r)$, become Volterra kernels, with Fredholm determinants identically equal to unity.[1] Thus $D(k)$ and $\Delta(k)$ have no explicit role in the description of the scattering process. For a local potential, all phenomena associated with scattering are determined by the Fredholm determinants $D^{\pm}(k)$. It is well known that $D^{\pm}(k)$ and the Jost functions $\mathfrak{L}^{\pm}(k)$ are identically equal[1]:

$$\mathfrak{L}^{\pm}(k)=D^{\pm}(k) \qquad \text{[local potential]}. \tag{28}$$

Since $\mathfrak{L}^{\pm}(k)\neq 0$ for real $k\neq 0$ [Eq. (15)], it follows that $D^{\pm}(k)$ has this same property for a local potential.

For a *nonlocal* potential, it has been shown[17-20] that $D^{\pm}(k)$ and $\mathfrak{L}^{\pm}(k)$ are not identically equal; rather, they are related by

$$\mathfrak{L}^{\pm}(k)=D^{\pm}(k)/D(k)\,. \tag{29}$$

The local potential result, Eq. (28), is a special case of Eq. (29) since $D(k)=1$ for a local potential. The integral equations with which $D(k)$ and $\Delta(k)$ are associated have Fredholm rather than Volterra kernels when the potential is nonlocal. In general, the Fredholm determinants of integral equations which have Fredholm kernels may have zeroes. Thus the Fredholm determinants $D(k)$ and $\Delta(k)$ for a nonlocal potential *may* have zeroes for *any* real value of k. Furthermore, in contrast to the results stated above for a local potential, the Fredholm determinants $D^{\pm}(k)$ may also have zeroes for any real $k\neq 0$ when the potential is nonlocal.

Connection between zeroes of $D^{\pm}(k)$ and $D(k)$

The occurrence of zeroes of $D^{\pm}(k)$ and $D(k)$ is not independent since, through Eq. (29), the Fredholm determinants are subject to the condition (15) that the Jost function have no zeroes for $k\neq 0$. When taken together, Eqs. (15) and (29) suggest that a zero of $D^{\pm}(k)$ is always accompanied by a zero of $D(k)$. A proof of the suggestion follows from the assertion that at wave number $k_0\neq 0$, $D(k_0)\neq 0$. This assertion implies that the regular and Jost solutions are definable at k_0 without complications which may be associated with a zero of $D(k_0)$; therefore, the Jost solutions are linearly independent at k_0 and the Jost function satisfies $\mathfrak{L}^{\pm}(k_0)\neq 0$. It then follows from Eq. (29) that $D^{\pm}(k_0)\neq 0$ since both $\mathfrak{L}^{\pm}(k_0)\neq 0$ and $D(k_0)\neq 0$. Thus the condition $D(k_0)=0$ is necessary in order to have $D^{\pm}(k_0)=0$.

Continuum bound states

A zero of $D^{\pm}(k)$ for real $k\neq 0$ has been called a continuum bound state.[21,22] As just demonstrated, $D(k)$ must also be zero when $D^{\pm}(k)=0$; thus a continuum bound state is characterized by simultaneous zeroes of both $D^{\pm}(k)$ and $D(k)$. The nomenclature continuum bound state derives from the fact that the homogeneous integral equations associated with Eqs. (4), namely

$$\psi_h^{\pm}(k,r)=\int_0^{\infty}\int_0^{\infty} G^{\pm}(k,r,r')V(r',s)\psi_h^{\pm}(k,s)ds\,dr'\,, \tag{30}$$

admit discrete normalizable solutions for real $k\neq 0$ when $D^{\pm}(k)=0$. Trivial solutions $\psi_h^{\pm}(k,r)=0$ are the only solutions allowed when $D^{\pm}(k)\neq 0$. The properties of continuum bound state wave functions are discussed in Sec. IV.

Spurious states

A zero of $D(k)$ for real $k\neq 0$ has been called a spurious state.[13,14] There is no local potential analogy for a spurious state since $D(k)=1$ for a short range local potential. Since $D(k)$ must also be zero at a continuum bound state, it follows that a continuum bound state is a special case of a spurious state. Nevertheless, in order to maintain existing nomenclature,[11,21,22] the term continuum bound state is used in the following discussion to denote simultaneous zeroes of $D^{\pm}(k)$ and $D(k)$, while the term spurious state is reserved for a zero of $D(k)$ with $D^{\pm}(k)\neq 0$.

IV. SOLUTIONS AT ZEROES OF FREDHOLM DETERMINANTS

In this section the behavior of the wave functions defined previously is discussed at spurious state and continuum bound state wave numbers. It is demonstrated that at a spurious state the regular solution $\varphi(k,r)$ does not exist, although it is possible to obtain a solution to Eq. (1) regular at the origin. In addition, the Jost solutions $f^{\pm}(k,r)$ do not exist at a spurious state. On the other hand, the physical solution $\psi^{+}(k,r)$, its conjugate $\psi^{-}(k,r)$, and the irregular solution $\theta(k,r)$ continue to have meaning at a spurious state. At a continuum bound state the solutions $\psi^{\pm}(k,r)$ and $\varphi(k,r)$ always exist, while the existence of $\theta(k,r)$ and $f^{\pm}(k,r)$ depends upon the circumstances. The homogeneous solutions associated with the integral equations for $\psi^{\pm}(k,r)$, $\varphi(k,r)$, $\theta(k,r)$, and $f^{\pm}(k,r)$ are also discussed.

The question of whether or not an inhomogeneous integral equation has a solution when the Fredholm determinant associated with its kernel is zero requires closer inspection of the existence conditions mentioned at the end of Sec. II. For this purpose, these conditions are now stated more precisely.[23] The integral equations under discussion are of the form

$$\chi(r)=F(r)+\int_0^\infty K(r\,|\,s)\chi(s)ds\,. \tag{31}$$

The homogeneous equation associated with Eq. (31),

$$\chi_h(r)=\int_0^\infty K(r\,|\,s)\chi_h(s)ds\,, \tag{32}$$

has a nontrivial solution if and only if the Fredholm determinant associated with the kernel $K(r\,|\,s)$ is zero. The inhomogeneous equation (31) then has a solution if and only if the inhomogeneous term $F(r)$ is orthogonal to the solution $\overline{\chi}_h(r)$ of the transposed homogeneous equation

$$\overline{\chi}_h(r)=\int_0^\infty K(s\,|\,r)\overline{\chi}_h(s)ds\,. \tag{33}$$

Thus the existence of $\chi(r)$ when the Fredholm determinant is zero reduces to an orthogonality condition involving the inhomogeneous term $F(r)$ and the solution $\overline{\chi}_h(r)$ of the transposed homogeneous equation associated with Eq. (31). This result is known as Fredholm's third theorem.

Solutions when $D(k)=0$, $D^\pm(k)\neq 0$

The definitions of the physical solution $\psi^+(k,r)$ and its conjugate $\psi^-(k,r)$ given in Sec. II are unaltered at a spurious state wave number since $D^\pm(k)\neq 0$. That is, if at wave number k, $D(k)=0$ and $D^\pm(k)\neq 0$, there exists a unique solution to Eq. (1) which is regular at the origin and has asymptotic form in the limit as $r\to\infty$ given by Eq. (3).

The question of whether or not the inhomogeneous equation for the regular solution $\varphi(k,r)$, Eq. (7), has a solution at a spurious state is now discussed in terms of Fredholm's third theorem. The homogeneous equation associated with Eq. (7) is

$$\varphi_h(k,r)=\int_0^r\int_0^\infty G(k,r,r')V(r',s)\varphi_h(k,s)ds\,dr'\,. \tag{34}$$

Before taking the transpose of the kernel of Eq. (34), it is convenient to use the Green's function identity

$$\int_0^r G(k,r,r')h(r')dr'=\int_0^\infty G^\pm(k,r,r')h(r')dr'+k^{-1}\sin kr\int_0^\infty e^{\pm ikr'}h(r')dr'\,, \tag{35}$$

valid for arbitrary $h(r)$, to rewrite Eq. (34) as

$$\varphi_h(k,r)=\int_0^\infty\left[\int_0^\infty G^\pm(k,r,r')V(r',s)dr'+\frac{\sin kr}{k}\int_0^\infty e^{\pm ikr'}V(r',s)dr'\right]\varphi_h(k,s)ds\,. \tag{36}$$

Thus the transposed homogeneous equation with solution $\overline{\varphi}_h(k,r)$ is

$$\overline{\varphi}_h(k,r)=\int_0^\infty\left[\int_0^\infty G^\pm(k,s,r')V(r',r)dr'+\int_0^\infty e^{\pm ikr'}V(r',r)dr'\,\frac{\sin ks}{k}\right]\overline{\varphi}_h(k,s)ds\,. \tag{37}$$

That the inhomogeneous term of Eq. (7), namely $\sin kr$, is not orthogonal to $\overline{\varphi}_h(k,r)$ follows immediately from Eq. (37). If $\overline{\varphi}_h(k,r)$ and $\sin kr$ were orthogonal, then Eq. (37) would reduce to

$$\overline{\varphi}_h(k,r)=\int_0^\infty\int_0^\infty G^\pm(k,s,r')V(r',r)\overline{\varphi}_h(k,s)ds\,dr'\,. \tag{38}$$

But Eq. (38) is identical to the transposed homogeneous equation associated with Eq. (4) for the solutions $\psi^\pm(k,r)$, and will have a solution if and only if the Fredholm determinant $D^\pm(k)$ is zero. Since at a spurious state $D^\pm(k)\neq 0$, it follows that $\sin kr$ is not orthogonal to $\overline{\varphi}_h(k,r)$ and, therefore, that Eq. (7) has no solution at a spurious state.

Since the regular solution $\varphi(k,r)$ does not exist when $D(k)=0$, $D^\pm(k)\neq 0$, it is convenient to introduce a modified regular solution $\Phi(k,r)$, related to $\psi^\pm(k,r)$ by

$$\psi^\pm(k,r)=k\Phi(k,r)/D^\pm(k)\,; \tag{39}$$

$\Phi(k,r)$ is well defined independent of the occurrence of zeroes of $D(k)$. Equations (16) and (29) may be used to show that when $\varphi(k,r)$ is well defined it is related to $\Phi(k,r)$ by $\Phi(k,r)=D(k)\varphi(k,r)$. The modified regular solution $\Phi(k,r)$ satisfies the integral equation

$$\Phi(k,r)=D(k)k^{-1}\sin kr+\int_0^r\int_0^\infty G(k,r,r')V(r',s)\Phi(k,s)ds\,dr'\,. \tag{40}$$

The boundary conditions associated with $\Phi(k,r)$ follow from Eq. (40), and are

$$\Phi(k,0)=0\,, \tag{41a}$$

$$\Phi(k,0)'=D(k)\,. \tag{41b}$$

It also follows from Eqs. (3) and (39) that, in the limit as $r\to\infty$,

$$\Phi(k,r)\to[D^-(k)e^{ikr}-D^+(k)e^{-ikr}]/2ik\,. \tag{42}$$

At a wave number such that $D(k)=0$, Eq. (40) becomes a homogeneous equation for $\Phi(k,r)$, namely

$$\Phi(k,r)=\int_0^r\int_0^\infty G(k,r,r')V(r's)\Phi(k,s)ds\,dr' \qquad [D(k)=0]. \tag{43}$$

Thus when $D(k)=0$ the integral equation for $\Phi(k,r)$ is identical to Eq. (34), the homogeneous equation associated with Eq. (7) for $\varphi(k,r)$. The solution $\varphi_h(k,r)$ of Eq. (34) satisfies the boundary conditions

$$\varphi_h(k,0)=0\,, \tag{44a}$$

$$\varphi_h(k,0)'=0\,; \tag{44b}$$

it should be noted from Eq. (41) that $\Phi(k,r)$ satisfies these same boundary conditions when $D(k)=0$. Further examination of Eqs. (34) and (43) reveals that, in fact, the modified regular solution $\Phi(k,r)$ is, apart from normalization, identical to $\varphi_h(k,r)$ when $D(k)=0$. Thus, it follows from these remarks and from Eq. (39) that when $D(k)=0$ the physical solution $\psi^+(k,r)$ and its conjugate $\psi^-(k,r)$ are related not to the solution of the inhomogeneous equation for $\varphi(k,r)$, Eq. (7), but to the solution $\varphi_h(k,r)$ of the associated homogeneous equation, Eq. (34). In particular, the homogeneous solution $\varphi_h(k,r)$ is that solution which contributes to the phase shift at a spurious state.

Since the regular solution $\varphi(k,r)$ of the inhomogeneous equation does not exist at a spurious state, it seems reasonable to question also the existence of the irregular solution $\theta(k,r)$. Because at a spurious state $D(k)=0$ and $D^\pm(k)\neq 0$, it follows from Eq. (29) that the inhomogeneous term in Eq. (24) for $\theta(k,r)$ is identically zero, and the equation becomes

$$\theta(k,r)=-\int_r^\infty\int_0^\infty G(k,r,r')V(r',s)\theta(k,s)ds\,dr' \qquad [D(k)=0,\quad D^\pm(k)\neq 0]. \tag{45}$$

As pointed out earlier, the Fredholm determinant $\Delta(k)$ associated with the kernel of Eq. (45) is equal to $D(k)$ for a symmetric nonlocal potential. Thus $D(k)=0$ is the necessary and sufficient condition which insures the existence of a solution to Eq. (45), and $\theta(k,r)$ is therefore established as well defined at a spurious state by Eqs. (21) and (45).

It follows from the asymptotic behavior given in Eq. (22) or from Eq. (45) that, as $r\to\infty$,

$$\theta(k,r)\to 0 \quad [D(k)=0,\quad D^\pm(k)\neq 0]. \tag{46}$$

The behavior of $\Phi(k,r)$ as $r\to\infty$ is established in Eq. (42). Since as $r\to\infty$ $\theta(k,r)$ goes to zero and $\Phi(k,r)$ does not, it is clear that they are linearly independent. The Wronskian of $\theta(k,r)$ and the modified regular solution $\Phi(k,r)$, namely

$$W_{\theta\Phi}(k,r)=\theta(k,r)\Phi(k,r)'-\Phi(k,r)\theta(k,r)'\,, \tag{47}$$

is

$$W_{\theta\Phi}(k,r)=D(k)+\int_r^\infty\int_0^\infty V(r',s)[\theta(k,r')\Phi(k,s)-\Phi(k,r')\theta(k,s)]ds\,dr'\,. \tag{48}$$

As in Eq. (26) for $W_{\theta\varphi}(k,r)$, the integral in this equation vanishes for a symmetric potential in the limit as $r\to 0$. Thus,

$$W_{\theta\Phi}(k,0)=W_{\theta\Phi}(k,\infty)=D(k)\,. \tag{49}$$

When $D(k)=0$, $W_{\theta\Phi}(k,r)$ vanishes at $r=0$ and ∞; however, $W_{\theta\Phi}(k,r)$ cannot vanish for all r.

Thus, $\Phi(k,r)$ and $\theta(k,r)$ form a pair of linearly independent solutions of Eq. (1) valid even at spurious states. A general solution to Eq. (1) can be constructed according to

$$u(k,r)=\alpha\Phi(k,r)+\beta\theta(k,r)\,, \tag{50}$$

where α and β are constants to be determined by the boundary conditions. However, it is clear that at a spurious state one cannot impose arbitrary boundary conditions on $u(k,r)$. For example, it has already been demonstrated that no solution $u(k,r)$ exists with boundary conditions given by Eq. (6).

It is also not possible to construct at a spurious state the Jost solutions $f^\pm(k,r)$. That this is the case follows from the behavior of $\Phi(k,r)$ and $\theta(k,r)$ in the limit as $r\to\infty$. Equations (42) and (46) clearly show that it is not possible to choose α and β such that Eq. (50) satisfies the Jost boundary conditions of Eq. (9) at a spurious state.[24] On the other hand, the fact that $D(k)\equiv\Delta(k)$ is zero at a spurious state implies the existence of a solution $f_h^\pm(k,r)$ of the homogeneous equation associated with Eq. (10), namely,

$$f_h^\pm(k,r)=-\int_r^\infty\int_0^\infty G(k,r,r')V(r',s)f_h^\pm(k,s)ds\,dr'\,. \tag{51}$$

Comparison of Eqs. (51) and (45), however, shows that apart from normalization $f_h^\pm(k,r)$ is identical with $\theta(k,r)$.

In the discussion of $D(k)$ the possibility of zeroes of higher order was not mentioned. If Eq. (32) admits m solutions, then it follows from Fredholm theory that Eq. (33) admits exactly m solutions also. In this case, existence of solutions of inhomogeneous equations requires that orthogonality conditions be satisfied with respect to each of the m solutions of the associated transposed homogeneous equation.

Solutions when $D(k)=0$, $D^{\pm}(k)=0$

As demonstrated in Sec. III, both $D^{\pm}(k)$ and $D(k)$ are zero for a continuum bound state. The fact that $D^{\pm}(k)$ is zero immediately implies the existence of the solution $\psi_h^{\pm}(k,r)$ of Eq. (30). The fact that $D(k)\equiv\Delta(k)$ is zero immediately implies the existence of a solution $f_h^{\pm}(k,r)$ of Eq. (51). In order to compare Eq. (51) with Eq. (30), it is convenient to use the Green's function identity

$$-\int_r^\infty G(k,r,r')h(r')dr' = \int_0^\infty G^{\pm}(k,r,r')h(r')dr' + k^{-1}e^{\pm ikr}\int_0^\infty \sin kr'h(r')dr', \quad (52)$$

valid for arbitrary $h(r)$, to rewrite Eq. (51) in the form

$$f_h^{\pm}(k,r)=\int_0^\infty\int_0^\infty G^{\pm}(k,r,r')V(r',s)f_h^{\pm}(k,s)ds\,dr' + k^{-1}e^{\pm ikr}\int_0^\infty\int_0^\infty \sin kr'V(r',s) \times f_h^{\pm}(k,s)ds\,dr'. \quad (53)$$

Important properties of the solutions $\psi_h^{\pm}(k,r)$ and $f_h^{\pm}(k,r)$ can be obtained by multiplying Eq. (30) by $\int_0^\infty f_h^{\pm}(k,s)V(s,r)ds$ and integrating over r from zero to infinity, multiplying Eq. (53) by $\int_0^\infty \psi_h^{\pm}(k,s)V(s,r)ds$ and integrating over r, and subtracting the two resulting equations. This gives, after making use of the symmetry of $V(r,r')$ and $G^{\pm}(k,r,r')$,

$$\int_0^\infty\int_0^\infty e^{\pm ikr'}V(r',s')\psi_h^{\pm}(k,s')ds'\,dr' \times\int_0^\infty\int_0^\infty \sin krV(r,s)f_h^{\pm}(k,s)ds\,dr=0. \quad (54)$$

Equation (54) implies that either

$$\int_0^\infty\int_0^\infty e^{\pm ikr'}V(r',s')\psi_h^{\pm}(k,s')ds'\,dr'=0 \quad (55)$$

or

$$\int_0^\infty\int_0^\infty \sin krV(r,s)f_h^{\pm}(k,s)ds\,dr=0, \quad (56)$$

or both. If Eq. (56) is true, then it follows that the second term on the right-hand side of Eq. (53) is zero, in which case the resulting equation for $f_h^{\pm}(k,r)$ becomes identical with Eq. (30) for $\psi_h^{\pm}(k,r)$. Thus the validity of Eq. (56) implies that $f_h^{\pm}(k,r)$ is identical with $\psi_h^{\pm}(k,r)$, apart from normalization. On the other hand, Eq. (55) would also be sufficient to satisfy condition (54). But if Eq. (55) is true, then substituting $\int_0^\infty V(r',s)\psi_h^{\pm}(k,s)ds$ for $h(r')$ in the Green's function identity of Eq. (35) leads to the condition

$$\int_0^\infty\int_0^\infty G^{\pm}(k,r,r')V(r',s)\psi_h^{\pm}(k,s)ds\,dr' = \int_0^r\int_0^\infty G(k,r,r')V(r',s)\psi_h^{\pm}(k,s)ds\,dr', \quad (57)$$

from which it follows that $\psi_h^{\pm}(k,r)$ would have to satisfy the integral equation

$$\psi_h^{\pm}(k,r)=\int_0^r\int_0^\infty G(k,r,r')V(r',s)\psi_h^{\pm}(k,s)ds\,dr'. \quad (58)$$

Since there exists a real solution $\psi_h^{\pm}(k,r)$ to Eq. (58), and since if Eq. (55) holds it must hold for all solutions $\psi_h^{\pm}(k,r)$, it thus follows that condition (55) implies that

$$\int_0^\infty\int_0^\infty \sin krV(r,s)\psi_h^{\pm}(k,s)ds\,dr=0. \quad (59)$$

But if Eq. (59) is true, then this allows $k^{-1}\exp(\pm ikr)$ times the integral in Eq. (59) to be added to the right-hand side of Eq. (30), in which case the resulting equation for $\psi_h^{\pm}(k,r)$ is identical with Eq. (53) for $f_h^{\pm}(k,r)$. Thus the validity of Eq. (55) also implies that, apart from normalization, $\psi_h^{\pm}(k,r)$ is identical with $f_h^{\pm}(k,r)$. From Eq. (54) it therefore follows that

$$\psi_h^{\pm}(k,r)\propto f_h^{\pm}(k,r) \quad [D^{\pm}(k)=0,\ D(k)=0], \quad (60)$$

and that

$$\int_0^\infty\int_0^\infty \sin krV(r,s)\psi_h^{\pm}(k,s)ds\,dr=0 \quad [D^{\pm}(k)=0,\ D(k)=0]. \quad (61)$$

The conditions under which Eq. (55) holds can be obtained by differentiating Eq. (30) and setting $r=0$. The result is

$$\psi_h^{\pm}(k,0)'=-\int_0^\infty\int_0^\infty e^{\pm ikr}V(r,s)\psi_h^{\pm}(k,s)ds\,dr \quad (62)$$

or, using Eq. (61),

$$\int_0^\infty\int_0^\infty \cos krV(r,s)\psi_h^{\pm}(k,s)ds\,dr=-\psi_h^{\pm}(k,0)' \quad [D^{\pm}(k)=0,\ D(k)=0]. \quad (63)$$

Thus Eq. (55) can be expected to hold only for a continuum bound state wave function $\psi_h^\pm(k,r)$ which has zero slope at the origin. This requirement also follows from Eq. (58). Equation (58) is a conditional equation, true if Eq. (55) is true. Direct differentiation of Eq. (58) gives $\psi_h^\pm(k,0)'=0$.

The inhomogeneous integral equations associated with the physical solution $\psi^+(k,r)$ and its conjugate $\psi^-(k,r)$ are given by Eq. (4). Since the Fredholm determinant $D^\pm(k)=0$ at a continuum bound state, it is necessary to examine the existence of solutions of Eq. (4) under this condition. In investigating the properties of solutions of inhomogeneous equations at a continuum bound state it is necessary, as in the case of spurious states, to consider the solution of the transposed homogeneous equation. The equation for $\bar\psi_h^\pm(k,r)$ is

$$\bar\psi_h^\pm(k,r)=\int_0^\infty\int_0^\infty G^\pm(k,s,r')V(r',r)\bar\psi_h^\pm(k,s)ds\,dr\,. \tag{64}$$

Following Bertero, Talenti, and Viano,[25] the existence of the solutions $\psi^\pm(k,r)$ can be established by multiplying Eq. (30) for $\psi_h^\pm(k,r)$ by V and integrating over r from zero to infinity. The resulting equation is

$$\int_0^\infty V(r,s')\psi_h^\pm(k,s')ds' =\int_0^\infty\int_0^\infty G^\pm(k,s,r')V(r',r) \times\int_0^\infty V(s,s')\psi_h^\pm(k,s')ds'ds\,dr\,. \tag{65}$$

Comparison of Eqs. (64) and (65) shows that

$$\bar\psi_h^\pm(k,r)\propto\int_0^\infty V(r,s)\psi_h^\pm(k,s)ds \qquad [D^\pm(k)=0,\quad D(k)=0]. \tag{66}$$

The necessary and sufficient condition that the physical solution $\psi^+(k,r)$ and its conjugate $\psi^-(k,r)$ defined by Eqs. (4) exist is that the solution $\bar\psi_h^\pm(k,r)$ of the transposed homogeneous equation associated with Eq. (4) be orthogonal to the inhomogeneous term of Eq. (4), namely $\sin kr$. But Eq. (66) demonstrates that this is just the condition established in Eq. (61). Thus at a continuum bound state the solutions $\psi^\pm(k,r)$ of the inhomogeneous equations (4) exist in addition to the solution $\psi_h^\pm(k,r)$.

However, the solutions $\psi^\pm(k,r)$ are not unique. From Fredholm theory it is known that when the solution of the inhomogeneous integral equation exists it is in general arbitrary with respect to the addition of any amount of the solutions of the associated homogeneous equation, consistent with the boundary conditions. That the solution $\psi_h^\pm(k,r)$ goes to zero for large r is well known, a fact that can be seen, for example, immediately from Eq. (51). This is the origin of the terminology continuum bound state.

It has been pointed out[26] that the existence of a continuum bound state reflects a cancellation between the Green's function and the potential. That this cancellation take place at a given wave number demands a particular relationship between the potential and Green's function at that wave number. The boundary conditions on the continuum bound state wave function $\psi_h^\pm(k,r)$ will not be governed by the properties of the Green's function $G^\pm$. Rather, the asymptotic behavior of $\psi_h^\pm(k,r)$ will depend upon the properties of the residual kernel after the cancellation between $G^\pm$ and V, with the boundary conditions at the origin depending upon the choice of the potential V.[27]

Having established the existence and nonuniqueness of the physical solution of Eq. (1) at a continuum bound state, it is convenient to consider now the existence of the regular and Jost solutions under these conditions. Equations satisfied by $\varphi_h(k,r)$ and by $\bar\varphi_h(k,r)$ have been given as Eqs. (36) and (37), respectively. Equation (53) is satisfied by the homogeneous Jost solution $f_h^\pm(k,r)$, from which follows for $\bar f_h^\pm(k,r)$ the equation

$$\bar f_h^\pm(k,r)=\int_0^\infty\int_0^\infty G^\pm(k,s,r')V(r',r)\bar f_h^\pm(k,s)ds\,dr'+k^{-1}\int_0^\infty \sin kr'V(r',r)dr'\int_0^\infty e^{\pm iks}\bar f_h^\pm(k,s)ds\,. \tag{67}$$

Multiplying Eqs. (36) and (53) by V and integrating on r from zero to infinity results in equations which can be written in the respective forms

$$\int_0^\infty V(r,s')\varphi_h(k,s')ds'=\int_0^\infty\int_0^\infty G^\pm(k,s,r')V(r',r)\int_0^\infty V(s,s')\varphi_h(k,s')ds'ds\,dr +k^{-1}\int_0^\infty \sin kr'V(r',r)dr'\int_0^\infty e^{\pm iks}\int_0^\infty V(s,s')\varphi_h(k,s')ds'ds \tag{68}$$

and

$$\int_0^\infty V(r,s')f_h^\pm(k,s')ds' = \int_0^\infty\int_0^\infty G^\pm(k,s,r')V(r',r)\int_0^\infty V(s,s')f_h^\pm(k,s')ds'ds\,dr$$
$$+k^{-1}\int_0^\infty e^{\pm ikr'}V(r',r)dr'\int_0^\infty \sin ks\int_0^\infty V(s,s')f_h^\pm(k,s')ds'ds\,. \qquad (69)$$

Comparing Eqs. (37) and (69) leads to the conclusion

$$\overline{\varphi}_h(k,r)\propto\int_0^\infty V(r,s)f_h^\pm(k,s)ds\,, \qquad (70)$$

while comparing Eqs. (67) and (68) leads to the conclusion

$$\overline{f}_h^\pm(k,r)\propto\int_0^\infty V(r,s)\varphi_h(k,s)ds\,. \qquad (71)$$

The regular solution $\varphi(k,r)$ will exist if and only if $\overline{\varphi}_h(k,r)$ is orthogonal to the inhomogeneous term $k^{-1}\sin kr$ in Eq. (7). But Eqs. (60), (61), and (70) assure this orthogonality, and thus establish the existence of $\varphi(k,r)$.

In discussing solutions of Eq. (1) at a spurious state, it was found that $\varphi(k,r)$ did not exist. Thus, a modified regular solution $\Phi(k,r)$ was introduced. At a continuum bound state, however, $\varphi(k,r)$ exists. The relationship $\Phi(k,r)=D(k)\varphi(k,r)$ shows that $\Phi(k,r)$, which would still be expected to satisfy Eq. (43), becomes the trivial solution $\Phi(k,r)\equiv 0$.

It has already been demonstrated that $\psi_h^\pm(k,r)$ and $\varphi_h(k,r)$ satisfy Eq. (1) at a continuum bound state. Thus the existence of the solution $\varphi(k,r)$ brings into question the relationships among these solutions. The discussion of this question depends upon the behavior of $\psi_h^\pm(k,r)$ at the origin. It follows from Eqs. (51) and (60) that $\psi_h^\pm(k,0)=0$ in all cases. The investigation of the solutions can then be separated according to the conditions $\psi_h^\pm(k,0)'\neq 0$ and $\psi_h^\pm(k,0)'=0$.

Case (a), $\psi_h^\pm(k,0)'\neq 0$

Using the Green's function identity of Eq. (35), Eq. (7) for $\varphi(k,r)$ can be rewritten in the form

$$\varphi(k,r)=k^{-1}\sin kr + \int_0^\infty\int_0^\infty G^\pm(k,r,r')V(r',s)\varphi(k,s)ds\,dr' + k^{-1}\sin kr\int_0^\infty\int_0^\infty e^{\pm ikr'}V(r',s)\varphi(k,s)ds\,dr'\,. \qquad (72)$$

When $\psi_h^\pm(k,0)'\neq 0$, direct substitution of $\psi_h^\pm(k,r)[\psi_h^\pm(k,0)']^{-1}$ for $\varphi(k,r)$ in Eq. (72) and the use of Eqs. (62) and (30) shows it to be a solution. To this solution can, of course, be added any amount of the homogeneous solution $\varphi_h(k,r)$. Thus the most general solution $\varphi(k,r)$ is

$$\varphi(k,r)=A\varphi_h(k,r)+\psi_h^\pm(k,r)[\psi_h^\pm(k,0)']^{-1}\,, \qquad (73)$$

where A is an arbitrary constant. It follows from Eq. (34) for $\varphi_h(k,r)$ that

$$\varphi_h(k,0)=0 \qquad (74a)$$

and

$$\varphi_h(k,0)'=0\,. \qquad (74b)$$

Since the boundary conditions on $\psi_h^\pm(k,r)$ at $r=0$ have been explicitly chosen such that $\psi_h^\pm(k,0)'\neq 0$, the solutions $\varphi_h(k,r)$ and $\psi_h^\pm(k,r)$ of Eq. (1) must be linearly independent. Equation (73) shows the most general solution $\varphi(k,r)$ to be a linear combination of these two solutions.

The nonuniqueness of the solution $\varphi(k,r)$ at a continuum bound state brings into serious question the validity of Eqs. (19) for $\mathfrak{L}^\pm(k)$ under these conditions. After substitution for $\varphi(k,r)$ from Eq. (73), Eqs. (19) reduce to

$$\mathfrak{L}^\pm(k)=A\int_0^\infty\int_0^\infty e^{\pm ikr}V(r,s)\varphi_h(k,s)ds\,dr\,. \qquad (75)$$

Thus it follows that if Eqs. (19) for the Jost functions $\mathfrak{L}^\pm(k)$ are valid at a continuum bound state, $\mathfrak{L}^\pm(k)$ are not unique. The resolution of this difficulty rests in the realization that the Jost solutions $f^\pm(k,r)$ do not exist at a continuum bound state of this type. The solutions $f^\pm(k,r)$ exist if and only if $\overline{f}_h^\pm(k,r)$ is orthogonal to $e^{\pm ikr}$, the inhomogeneous term in Eq. (10). If this is to be the case, then if follows from Eq. (71) that

$$\int_0^\infty\int_0^\infty e^{\pm ikr}V(r,s)\varphi_h(k,s)ds\,dr=0\,. \qquad (76)$$

But comparison of Eqs. (75) and (76) shows that this would demand that $\mathfrak{L}^\pm(k)$ be zero. However, this is in contradiction to Eq. (15), which states that if the Jost solutions $f^\pm(k,r)$ exist, $\mathfrak{L}^\pm(k)$ must not be zero. Thus at a continuum bound state of this type Jost boundary conditions cannot be imposed upon the general solution of Eq. (1). Since the derivation of Eqs. (19) depended upon expressing $\varphi(k,r)$ in terms of the Jost solutions through Eqs. (17), they are not valid when $\psi_h^\pm(k,0)'\neq 0$.

Discussion of the solution $\theta(k,r)$ at a continuum bound state of this type presents special difficulties in that the defining equation for $\theta(k,r)$, Eq. (20), makes explicit use of the Jost functions

$\mathcal{L}^{\pm}(k)$. Since the Jost solutions $f^{\pm}(k,r)$ do not exist, $\mathcal{L}^{\pm}(k)$ cannot be obtained from them in the usual manner. For this reason, it is useful to recast the integral equation for $\theta(k,r)$, Eq. (24), as an equation in which the Jost functions do not appear. This can be accomplished by evaluating $\theta(k,0)$ and $\theta(k,0)'$ using Eqs. (20) and (24), and solving for $[\mathcal{L}^{\pm}(k)]^{-1}$. The result is

$$[\mathcal{L}^{\pm}(k)]^{-1}=1\pm(ik)^{-1}\theta(k,0)' \pm(ik)^{-1}\int_0^\infty\int_0^\infty e^{\mp ikr}V(r,s)\theta(k,s)ds\,dr\,. \tag{77}$$

If this expression is now substituted for $[\mathcal{L}^{\pm}(k)]^{-1}$ in Eq. (24), the equation for $\theta(k,r)$ becomes

$$\theta(k,r)=\cos kr+k^{-1}\theta(k,0)'\sin kr +\int_0^r\int_0^\infty G(k,r,r')V(r',s)\theta(k,s)ds\,dr'\,. \tag{78}$$

This result must be valid away from spurious and continuum bound states; it is more difficult to use than Eq. (24), however, since the value of $\theta(k,0)'$ must be supplied from other considerations. It should be noted that $\theta(k,0)'$ cannot be arbitrarily chosen, but is fixed by the definition of $\theta(k,r)$, Eq. (20). Equation (78) yields no information about $\theta(k,0)'$, in that direct differentiation leads only to the identity $\theta(k,0)'=\theta(k,0)'$.

However, if $\theta(k,r)$ exists at a continuum bound state, it must have a well-defined slope $\theta(k,0)'$, and must satisfy Eq. (78). But from Eq. (78) it follows that $\theta(k,r)$ can be written as

$$\theta(k,r)=\Theta(k,r)+\theta(k,0)'\varphi(k,r)\,, \tag{79}$$

where $\Theta(k,r)$ is defined by the integral equation

$$\Theta(k,r) =\cos kr+\int_0^r\int_0^\infty G(k,r,r')V(r',s)\Theta(k,s)ds\,dr'\,. \tag{80}$$

The existence of $\varphi(k,r)$ at a continuum bound state has already been established, and the existence of $\theta(k,r)$ at a continuum bound state of this type thus rests on that of $\Theta(k,r)$. The transposed homogeneous equation for $\overline{\Theta}_h(k,r)$ associated with Eq. (80) is identical with that for $\overline{\varphi}_h(k,r)$. From Eqs. (60) and (70), then, it follows that the condition for the existence of $\Theta(k,r)$ is

$$\int_0^\infty\int_0^\infty \cos kr\,V(r,s)\psi_h^{\pm}(k,s)ds\,dr=0\,. \tag{81}$$

Since by virtue of Eq. (63) this would contradict the assumption $\psi_h^{\pm}(k,0)'\neq 0$, $\Theta(k,r)$ and thus $\theta(k,r)$ cannot exist.

Thus at a continuum bound state, when $\psi_h^{\pm}(k,0)'\neq 0$ the most general solution $u(k,r)$ of Eq. (1) can be written in terms of the linearly independent solutions $\varphi_h(k,r)$ and $\psi_h^{\pm}(k,r)$ as

$$u(k,r)=\alpha\varphi_h(k,r)+\beta\psi_h^{\pm}(k,r)\,. \tag{82}$$

The linear independence of $\varphi_h(k,r)$ and $\psi_h^{\pm}(k,r)$ has already been established in terms of their boundary conditions at the origin. This can also be demonstrated in terms of the Wronskian

$$\mathcal{W}(k,r)=\varphi_h(k,r)\psi_h^{\pm}(k,r)'-\psi_h^{\pm}(k,r)\varphi_h(k,r)'\,, \tag{83}$$

which can be put in the form

$$\mathcal{W}(k,r)=k^{-1}\int_0^r\int_0^\infty \sin kr'V(r',s')\varphi_h(s')ds'dr \times\int_r^\infty\int_0^\infty \cos kr\,V(r,s) \times\psi_h(k,s)ds\,dr\,. \tag{84}$$

Inspection shows $\mathcal{W}(k,r)$ to be zero at $r=0$ and $r=\infty$, but it cannot be zero for all r.

It remains to discuss the behavior of $\varphi_h(k,r)$ for large r. From Eq. (34) it follows that, as $r\to\infty$,

$$\varphi_h(k,r)\to k^{-1}\sin kr\int_0^\infty\int_0^\infty \cos kr'V(r',s) \times\varphi_h(k,s)ds\,dr' -k^{-1}\cos kr\int_0^\infty\int_0^\infty \sin kr'V(r',s) \times\varphi_h(k,s)ds\,dr'\,. \tag{85}$$

But it was pointed out that Eq. (76) held if and only if the Jost solutions $f^{\pm}(k,r)$ existed. Since it has been demonstrated that they do not exist under the conditions imposed, it is clear that the coefficients of $\sin kr$ and $\cos kr$ in Eq. (85) cannot simultaneously be zero. Hence it must follow that $\varphi_h(k,r)$ oscillates at infinity.

Case (b), $\psi_h^{\pm}(k,0)'=0$

The condition $\psi_h^{\pm}(k,0)'=0$ results in a considerable simplification of the relationships among solutions, since then it follows from Eq. (63) that

$$\int_0^\infty\int_0^\infty \cos kr\,V(r,s)\psi_h^{\pm}(k,s)ds\,dr=0\,. \tag{86}$$

As a result, Eqs. (61) and (86) imply that Eq. (36) for $\varphi_h(k,r)$, Eq. (53) for $f_h^{\pm}(k,r)$, and the homogeneous equation for $\theta_h(k,r)$ associated with Eq. (78) are each satisfied by the continuum bound state wave function $\psi_h^{\pm}(k,r)$. Furthermore, Eqs. (66), (70), and (71) demonstrate that the solutions $\overline{\psi}_h^{\pm}(k,r)$, $\overline{\varphi}_h(k,r)$, $\overline{f}_h^{\pm}(k,r)$, and $\overline{\theta}_h(k,r)$ are also proportional to one another.

The proofs that the physical solution $\psi^{+}(k,r)$, its conjugate $\psi^{-}(k,r)$, and the regular solution

$\varphi(k,r)$ exist are unaltered by the condition $\psi_h^\pm(k,0)'=0$. However, the additional orthogonality given in Eq. (86) is sufficient to guarantee the existence of the Jost solutions as well. That is, conditional Eq. (76), which would insure the existence of the Jost solutions and which was not true for $\psi_h^\pm(k,0)'\neq 0$, is in fact satisfied for $\psi_h^\pm(k,0)'=0$. However, none of the solutions $\psi^\pm(k,r)$, $\varphi(k,r)$, or $f^\pm(k,r)$ is uniquely the solution of its appropriate integral equation. In each case, the solution is arbitrary with respect to the addition of any amount of the continuum bound state solution $\psi_h^\pm(k,r)$.

Since the Jost and regular solutions exist under the condition $\psi_h^\pm(k,0)'=0$, it would be expected that Eqs. (18) and (19) would be valid for $\mathcal{L}^\pm(k)$. But as $f^\pm(k,r)$ and $\varphi(k,r)$ are not unique, investigation is required to determine whether Eqs. (18) and (19) produce meaningful results. That indeed they do follows from the fact that the term $\psi_h^\pm(k,r)$ giving rise to the nonuniqueness of $f^\pm(k,r)$ and $\varphi(k,r)$ is, by virtue of the conditions imposed, orthogonal to both $\int_0^\infty \sin kr' V(r',r)dr'$ and $\int_0^\infty e^{\pm ikr'}V(r',r)dr'$, and thus cannot contribute to $\mathcal{L}^\pm(k)$.

The fact that the expressions for $\mathcal{L}^\pm(k)$ are well defined implies that the equation for $\theta(k,r)$ is well defined, and will be of the form given in either Eq. (24) or Eq. (78). The existence of the solution $\theta(k,r)$ then follows from the assertion that $\int_0^\infty V(r,s)\psi_h^\pm(k,s)ds$ is orthogonal both to $\sin kr$ and $\cos kr$. As is the case with other solutions at a continuum bound state of this type, $\theta(k,r)$ is arbitrary with respect to the addition of any amount of the continuum bound state solution $\psi_h^\pm(k,r)$.

This concludes the discussion of cases (a) and (b) for simple zeroes of $D^\pm(k)$. However, as mentioned in connection with spurious states, the possibility of zeroes of higher order of the Fredholm determinant must be considered. Such consideration would, of course, affect both case (a) and case (b). If there exists a zero of $D^\pm(k)$ of multiplicity m, then there are m solutions of the associated homogeneous equation at the continuum bound state energy. Such a situation occurs when considering the scattering of a nucleon antisymmetrized with respect to single-particle states of the target nucleus. In this case the n occupied single-particle states with respect to which the wave function of the incident nucleon is antisymmetrized are said to be "redundant" in that arbitrary amounts of these states can appear in the single-particle scattering wave function. It has been demonstrated[28] that these redundant states are continuum bound state solutions of multiplicity n which appear for every energy of the incident particle.

V. EXAMPLES

In order to illustrate the behavior of solutions of Eq. (1) at spurious and continuum bound states, solutions defined and discussed in the previous sections have been obtained for cases of one-term and two-term separable nonlocal potentials. First, parameters for a one-term separable potential with a Yamaguchi form factor are chosen such that it has a spurious state. Next, the two-term separable potential of the Mongan case IV form is considered for two choices of the parameters, one leading to a spurious state, the other to a continuum bound state.

One-term separable potential with Yamaguchi form factor

Yamaguchi[29] has introduced a one-term separable nonlocal potential to describe nucleon-nucleon scattering. In configuration space his potential is of the form

$$V(r,r')=\lambda g(r)g(r'), \tag{87}$$

where

$$g(r)=e^{-\alpha r}. \tag{88}$$

For this potential

$$D(k)\equiv\Delta(k)=1-\frac{\lambda}{2\alpha(\alpha^2+k^2)} \tag{89}$$

and

$$D^\pm(k)=D(k)+\frac{\lambda\alpha}{(\alpha^2+k^2)^2}\pm\frac{i\lambda k}{(\alpha^2+k^2)^2}. \tag{90}$$

From these expressions it is clear that no values of λ and α will make $D^\pm(k)$ zero, and therefore that a continuum bound state cannot be associated with the Yamaguchi form factor. On the other hand, $D(k)$ can be zero for a wide range of values of λ and α. Thus, although the values of λ and α used by Yamaguchi do not generate a spurious state at any energy, if $\lambda>2\alpha^3$ a spurious state will occur.

The solutions for the potential defined in Eqs. (87) and (88) are

$$\psi^\pm(k,r)=\sin kr+\frac{\lambda k}{D^\pm(k)(\alpha^2+k^2)^2}[e^{-\alpha r}-e^{\pm ikr}], \tag{91}$$

$$\varphi(k,r)=k^{-1}\sin kr+\frac{\lambda}{D(k)(\alpha^2+k^2)^2}\left[\frac{\alpha}{k}\sin kr-\cos kr+e^{-\alpha r}\right], \tag{92}$$

$$\Phi(k,r)=k^{-1}D(k)\sin kr+\frac{\lambda}{(\alpha^2+k^2)^2}\left[\frac{\alpha}{k}\sin kr-\cos kr+e^{-\alpha r}\right], \tag{93}$$

$$\theta(k,r)=\frac{D(k)}{D^+(k)D^-(k)}\left\{\frac{\lambda k}{(\alpha^2+k^2)^2}\sin kr+\left[D(k)+\frac{\lambda\alpha}{(\alpha^2+k^2)^2}\right](\cos kr-e^{-\alpha r})\right\}+e^{-\alpha r}, \tag{94}$$

and

$$f^\pm(k,r)=e^{\pm ikr}+\frac{\lambda}{D(k)}\left[\frac{\alpha}{(\alpha^2+k^2)^2}\pm\frac{ik}{(\alpha^2+k^2)^2}\right]e^{-\alpha r}. \tag{95}$$

At a spurious state, since $D(k)=0$ and $D^\pm(k)\neq 0$, it is clear that $\psi^\pm(k,r)$, $\Phi(k,r)$, and $\theta(k,r)$ remain well defined, whereas $\varphi(k,r)$ and $f^\pm(k,r)$ do not. The expression for $\Phi(k,r)$ becomes

$$\Phi(k,r)=\frac{\lambda}{(\alpha^2+k^2)^2}\left[\frac{\alpha}{k}\sin kr-\cos kr+e^{-\alpha r}\right]\quad [D(k)=0,\ \ D^\pm(k)\neq 0], \tag{96}$$

and as expected this function satisfies the homogeneous integral equation for $\varphi_h(k,r)$ and exhibits the boundary conditions $\Phi(k,0)=0$ and $\Phi(k,0)'=0$. Also when $D(k)=0$ the solution $\theta(k,r)$ is such that $\theta(k,0)=1$ and that $\theta(k,\infty)=0$.

The solutions $\Phi(k,r)$ and $\theta(k,r)$ are linearly independent, as can be seen by inspecting their Wronskian, which is

$$W_{\theta\Phi}(k,r)=D(k)+\frac{\lambda}{k(\alpha^2+k^2)}e^{-\alpha r}\sin kr\,. \tag{97}$$

When $D(k)=0$ the Wronskian in Eq. (97) vanishes at $r=0$ and $r=\infty$, but clearly does not vanish for all r.

The behavior of the Fredholm determinants $D(k)$ and $D^\pm(k)$ for this potential is illustrated in Fig. 1, for which the values

$\lambda=21.219\ \text{fm}^{-3}$

and

$\alpha=1.5\ \text{fm}^{-1}$

have been selected, giving a spurious state at $E_{\text{lab}}=400$ MeV. In this and in the subsequent numerical calculations in this paper the energy-wave number conversion factor $1/41.47\ \text{MeV}^{-1}\ \text{fm}^{-2}$ is used. Note that neither $\text{Re}D^+(k)$ nor $\text{Im}D^+(k)$ experiences a zero at any energy, while $D(k)$ is zero at 400 MeV.

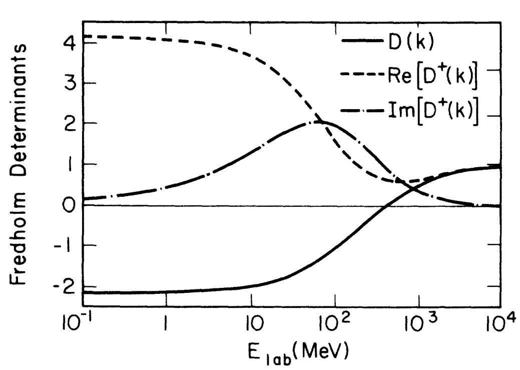

FIG. 1. Fredholm determinants for a one-term separable potential with a Yamaguchi form factor, with parameters given in the text. This potential yields a spurious state at 400 MeV.

Two-term separable potential of the Mongan case IV form

A two-term separable potential has been introduced by Mongan[30] in fitting the $^1S_0$ nucleon-nucleon phase shifts. The configuration space representation of the case IV Mongan potential is

$$V(r,r')=\lambda_1 e^{-\alpha_1(r+r')}+\lambda_2 e^{-\alpha_2(r+r')}. \tag{98}$$

For this potential

$$D(k)\equiv\Delta(k)=1-\frac{\lambda_1}{2\alpha_1({\alpha_1}^2+k^2)}-\frac{\lambda_2}{2\alpha_2({\alpha_2}^2+k^2)}+\frac{\lambda_1\lambda_2(\alpha_1-\alpha_2)^2}{4\alpha_1\alpha_2(\alpha_1+\alpha_2)^2({\alpha_1}^2+k^2)({\alpha_2}^2+k^2)} \tag{99}$$

and

$$D^\pm(k)=D(k)+R(k)\pm iI(k)\,, \tag{100}$$

where

$$R(k)=\frac{\lambda_1\alpha_1}{({\alpha_1}^2+k^2)^2}+\frac{\lambda_2\alpha_2}{({\alpha_2}^2+k^2)^2}-\frac{\lambda_1\lambda_2(\alpha_1-\alpha_2)^2k^2}{2\alpha_1\alpha_2({\alpha_1}^2+k^2)^2({\alpha_2}^2+k^2)^2} \tag{101}$$

and

$$I(k)=\frac{\lambda_1 k}{({\alpha_1}^2+k^2)^2}+\frac{\lambda_2 k}{({\alpha_2}^2+k^2)^2}+\frac{\lambda_1\lambda_2 k(\alpha_1-\alpha_2)^2(\alpha_1\alpha_2-k^2)}{2\alpha_1\alpha_2(\alpha_1+\alpha_2)({\alpha_1}^2+k^2)^2({\alpha_2}^2+k^2)^2}. \tag{102}$$

The solutions discussed in the previous sections can be calculated for the potential of Eq. (98), ignoring for the moment possible zeroes of the Fredholm determinants. The physical solution $\psi^+(k,r)$ and its conjugate $\psi^-(k,r)$ are given by

$$\psi^{\pm}(k,r)=\sin kr-\frac{I(k)}{D^{\pm}(k)}e^{\pm ikr}+\frac{I_1(k)}{D^{\pm}(k)}e^{-\alpha_1 r}+\frac{I_2(k)}{D^{\pm}(k)}e^{-\alpha_2 r}, \tag{103}$$

where

$$I_1(k)=\frac{\lambda_1 k}{(\alpha_1^2+k^2)^2}-\frac{\lambda_1\lambda_2(\alpha_1-\alpha_2)k}{2\alpha_2(\alpha_1+\alpha_2)(\alpha_1^2+k^2)^2(\alpha_2^2+k^2)} \tag{104}$$

and

$$I_2(k)=\frac{\lambda_2 k}{(\alpha_2^2+k^2)^2}+\frac{\lambda_1\lambda_2(\alpha_1-\alpha_2)k}{2\alpha_1(\alpha_1+\alpha_2)(\alpha_1^2+k^2)(\alpha_2^2+k^2)^2}. \tag{105}$$

It should be noted that

$$I_1(k)+I_2(k)=I(k). \tag{106}$$

The regular solution $\varphi(k,r)$ and the modified regular solution $\Phi(k,r)$ are, respectively,

$$\varphi(k,r)=k^{-1}\sin kr+\frac{1}{kD(k)}[R(k)\sin kr-I(k)\cos kr+I_1(k)e^{-\alpha_1 r}+I_2(k)e^{-\alpha_2 r}] \tag{107}$$

and

$$\Phi(k,r)=k^{-1}D(k)\sin kr+k^{-1}[R(k)\sin kr-I(k)\cos kr+I_1(k)e^{-\alpha_1 r}+I_2(k)e^{-\alpha_2 r}]. \tag{108}$$

The Jost solutions $f^{\pm}(k,r)$ are given by

$$f^{\pm}(k,r)=e^{\pm ikr}+D(k)^{-1}\{R_1(k)\pm i[I_1(k)+J(k)]\}e^{-\alpha_1 r}+D(k)^{-1}\{R_2(k)\pm i[I_2(k)-J(k)]\}e^{-\alpha_2 r}, \tag{109}$$

where

$$J(k)=\frac{\lambda_1\lambda_2 k(\alpha_1-\alpha_2)}{(\alpha_1^2+k^2)^2(\alpha_2^2+k^2)^2}, \tag{110}$$

$$R_1(k)=\frac{\lambda_1\alpha_1}{(\alpha_1^2+k^2)^2}+\frac{\lambda_1\lambda_2\alpha_2}{(\alpha_1+\alpha_2)(\alpha_1^2+k^2)(\alpha_2^2+k^2)^2}-\frac{\lambda_1\lambda_2\alpha_1}{2\alpha_2(\alpha_1^2+k^2)^2(\alpha_2^2+k^2)}, \tag{111}$$

and

$$R_2(k)=\frac{\lambda_2\alpha_2}{(\alpha_2^2+k^2)^2}+\frac{\lambda_1\lambda_2\alpha_1}{(\alpha_1+\alpha_2)(\alpha_1^2+k^2)^2(\alpha_2^2+k^2)}-\frac{\lambda_1\lambda_2\alpha_2}{2\alpha_1(\alpha_1^2+k^2)(\alpha_2^2+k^2)^2}. \tag{112}$$

The functions $R_1(k)$ and $R_2(k)$ are related by

$$R_1(k)+R_2(k)=R(k), \tag{113}$$

where $R(k)$ is defined in Eq. (101). The real irregular solution $\theta(k,r)$ follows from the definition given in Eq. (20) and Eq. (109), and is

$$\begin{aligned}\theta(k,r)=&\frac{D(k)}{D^+(k)D^-(k)}\{[D(k)+R(k)]\cos kr+I(k)\sin kr\}\\&+\frac{1}{D^+(k)D^-(k)}\{[D(k)+R(k)]R_1(k)+[I_1(k)+J(k)]I(k)\}e^{-\alpha_1 r}\\&+\frac{1}{D^+(k)D^-(k)}\{[D(k)+R(k)]R_2(k)+[I_2(k)-J(k)]I(k)\}e^{-\alpha_2 r}.\end{aligned} \tag{114}$$

Spurious state solutions

Mongan finds a good fit to the nucleon-nucleon phase shifts for the following values of the parameters:

$\lambda_1=3454.8\ \text{fm}^{-3}$,

$\lambda_2=-28.293\ \text{fm}^{-3}$,

$\alpha_1=6.157\ \text{fm}^{-1}$,

$\alpha_2=1.786\ \text{fm}^{-1}$.

Mongan's potential is known to have a zero of $D(k)$ at a laboratory energy of 19.6 BeV.[14] That $D(k)=0$ and $D^{\pm}(k)\neq 0$ at this energy, and that there are no other zeroes of $D(k)$ for these values of the parameters, can be seen by examining the expres-

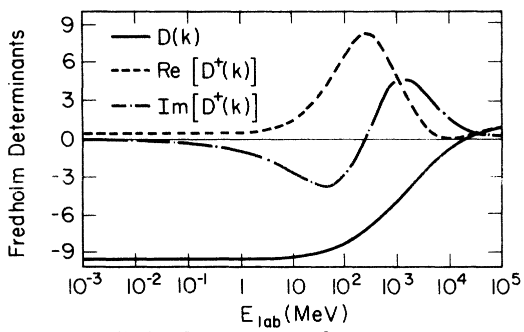

FIG. 2. Fredholm determinants for a two-term separable potential of the Mongan case IV form, with the parameters given by Mongan. This potential yields a spurious state at 19.6 BeV.

sions for the Fredholm determinants. A plot of their behavior as a function of the energy is given in Fig. 2. From the form of $D(k)$ given in Eq. (99) it is clear that this is not the only set of parameters which will lead to a spurious state; a spurious state can be expected from a wide variety of values of λ_1, λ_2, α_1, and α_2.

At a spurious state it is evident that, as in the case of the Yamaguchi potential, $\psi^{\pm}(k,r)$, $\Phi(k,r)$, and $\theta(k,r)$ remain well defined, whereas $\varphi(k,r)$ and $f^{\pm}(k,r)$ do not. Again, $\Phi(k,r)$, given for the Mongan potential at $D(k)=0$ by

$$\Phi(k,r)=k^{-1}[R(k)\sin kr - I(k)\cos kr + I_1(k)e^{-\alpha_1 r} + I_2(k)e^{-\alpha_2 r}], \qquad (115)$$

satisfies the homogeneous equation for $\varphi_h(k,r)$ and exhibits the boundary conditions $\Phi(k,0)=0$ and $\Phi(k,0)'=0$. Also, as expected and as can be seen from Eq. (114), when $D(k)=0$, $\theta(k,0)=1$ and $\theta(k,\infty)=0$.

Moreover, $\Phi(k,r)$ and $\theta(k,r)$ are linearly independent for all k. This can be seen by examining their Wronskian, which is

$$\begin{aligned} W_{\theta\Phi} = {} & D(k) + k^{-1}\{kR_1(k) - \alpha_1[I_1(k)+J(k)]\}e^{-\alpha_1 r}\cos kr \\ & + k^{-1}\{kR_2(k) - \alpha_2[I_2(k)-J(k)]\}e^{-\alpha_2 r}\cos kr \\ & + \{k^{-1}\alpha_1 R_1(k) + I_1(k) + J(k)\}e^{-\alpha_1 r}\sin kr \\ & + \{k^{-1}\alpha_2 R_2(k) + I_2(k) - J(k)\}e^{-\alpha_2 r}\sin kr \\ & + k^{-1}(\alpha_1-\alpha_2)J(k)e^{-(\alpha_1+\alpha_2)r}. \end{aligned} \qquad (116)$$

When $D(k)=0$ the Wronskian in Eq. (116) vanishes at $r=0$ and $r=\infty$, but clearly does not vanish for all r.

Continuum bound state solutions

From the form of $D^{\pm}(k)$ for the potential of Eq. (98) it is clear that choices of the parameters λ_1, λ_2, α_1, and α_2 exist such that the real and imaginary parts of $D^{\pm}(k)$ vanish simultaneously. From Sec. III it follows that for these values of the parameters $D(k)$ must also vanish at the same value of k. In principle, a set of parameters which yields a continuum bound state can be obtained by setting the real and imaginary parts of $D^{\pm}(k)$, given by Eq. (100), equal to zero. A more convenient approach, however, is to use the method of cancellation of the Green's function for generating a continuum bound state.[26] When this method is applied it leads directly to the fact that the potential of Eq. (98) yields a continuum bound state if and only if the functions $I_1(k)$ and $I_2(k)$ defined in Eqs. (104) and (105) are each identically equal to zero. Under these conditions

$$\lambda_1 = -\frac{2\alpha_1(\alpha_1+\alpha_2)(\alpha_1^2+k_0^2)}{(\alpha_1-\alpha_2)}, \qquad (117a)$$

$$\lambda_2 = \frac{2\alpha_2(\alpha_1+\alpha_2)(\alpha_2^2+k_0^2)}{(\alpha_1-\alpha_2)}, \qquad (117b)$$

where k_0 is the wave number at which the continuum bound state occurs. If these values of λ_1 and λ_2 are substituted into Eq. (99) for $D(k)$ and Eq. (100) for $D^{\pm}(k)$, they lead to

$$D(k_0)=R(k_0)=I(k_0)=0, \qquad (118)$$

which is the condition for a continuum bound state.

This behavior of the Fredholm determinants at a continuum bound state can be illustrated by considering the following set of parameters, consistent with Eqs. (117), which yields a continuum bound state at 400 MeV:

$\lambda_1 = 105.876\ \mathrm{fm}^{-3}$,

$\lambda_2 = -499.752\ \mathrm{fm}^{-3}$,

$\alpha_1 = 2.0\ \mathrm{fm}^{-1}$,

$\alpha_2 = 4.0\ \mathrm{fm}^{-1}$.

The Fredholm determinants for these parameters are given in Fig. 3. As expected, and in contrast

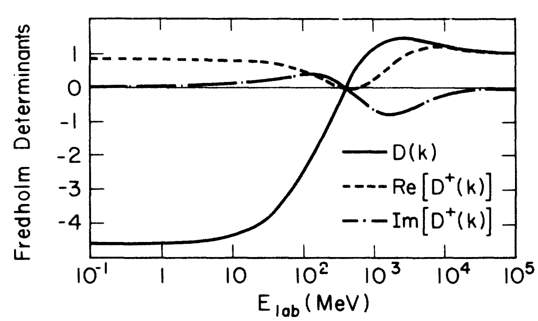

FIG. 3. Fredholm determinants for a two-term separable potential of the Mongan case IV form, with the parameters chosen so as to yield a continuum bound state at 400 MeV.

to the behavior of $D^{\pm}(k)$ at a spurious state, both the real and imaginary parts of $D^{\pm}(k)$ pass through zero at the continuum bound state, in addition to the zero of $D(k)$ at that energy.

The continuum bound state wave function can be calculated from either of the defining homogeneous integral equations, Eq. (30) or (51). It is

$$\psi_h^{\pm}(k,r) = N[e^{-\alpha_1 r} - e^{-\alpha_2 r}], \tag{119}$$

where N is a normalization factor. It follows immediately that

$$\psi_h^{\pm}(k,0)' = N(\alpha_2 - \alpha_1). \tag{120}$$

Thus the potential of Eq. (98) yields a continuum bound state of the type discussed in case (a) of Sec. IV $[\psi_h^{\pm}(k,0)' \neq 0]$ for $\alpha_1 \neq \alpha_2$.

For $\alpha_1 \neq \alpha_2$ the solution $\overline{\psi}_h^{\pm}(k,r)$ of the transposed homogeneous equation, Eq. (64), is easily obtained from Eq. (66). The result is

$$\begin{aligned}\overline{\psi}_h^{\pm}(k,r) &\propto \int_0^\infty V(r,s)\psi_h^{\pm}(k,s)ds \\ &= N[(\alpha_1^2+k^2)e^{-\alpha_1 r} - (\alpha_2^2+k^2)e^{-\alpha_2 r}].\end{aligned} \tag{121}$$

The orthogonality condition of Eq. (61) follows immediately upon substitution from Eq. (121) and performing the integration with $\sin kr$. As expected, in substituting from Eq. (121) into Eq. (63), the negative of the slope calculated in Eq. (120) is obtained.

The solution $\varphi_h(k,r)$, which along with $\psi_h^{\pm}(k,r)$ forms the linearly independent pair of solutions to Eq. (1), can be obtained by direct solution of the homogeneous integral equation, Eq. (34). The result is

$$\begin{aligned}\varphi_h(k,r) = M\bigg[&\left(\frac{\alpha_1}{k(\alpha_1^2+k^2)} - \frac{\alpha_2}{k(\alpha_2^2+k^2)}\right)\sin kr \\ &- \left(\frac{1}{\alpha_1^2+k^2} - \frac{1}{\alpha_2^2+k^2}\right)\cos kr \\ &+ \frac{1}{\alpha_1^2+k^2}e^{-\alpha_1 r} - \frac{1}{\alpha_2^2+k^2}e^{-\alpha_2 r}\bigg],\end{aligned} \tag{122}$$

where M is the arbitrary constant of normalization. That $\varphi_h(k,r)$ satisfies the boundary conditions of Eq. (74) and oscillates at infinity follows from Eq. (122).

From Eq. (120) it might be expected that if $\alpha_1 = \alpha_2$, the case (a) results would reduce to case (b). However, if $\alpha_1 = \alpha_2 = \alpha$, the potential of Eq. (98) reduces to a one-term potential with a Yamaguchi form factor and strength $\lambda = \lambda_1 + \lambda_2$. When Eqs. (117) are used, however, $\lambda = -4\alpha^2(3\alpha^2 + k_0^2)$, resulting in a situation in which neither $D^{\pm}(k)$ nor $D(k)$ is zero; hence only trivial homogeneous solutions can exist. That this is the case is clear from Eqs. (119) and (122), each of which becomes identically zero when $\alpha_1 = \alpha_2$. However, even though the potential of Eq. (98) does not admit case (b), there do exist potentials[31] which exhibit a case (b) type $[\psi_h^{\pm}(k,0)' = 0]$ continuum bound state.

VI. CONCLUSIONS

In the introduction, we emphasized and demonstrated with examples that certain analytic constraints are relaxed in going from a local to a nonlocal potential. The discussion of relaxation of constraints is based on the use of Fredholm determinants. Unlike the case for a short range local potential, for a nonlocal potential the Fredholm determinants $D(k)$, $\Delta(k)$, and $D^{\pm}(k)$ can vanish for nonzero values of k. Furthermore, we have shown that zeroes of these determinants are not independent. A zero of $D^{\pm}(k)$ at $k = k_0$ implies a zero of $D(k)$ at k_0, whereas a zero of $D(k)$ does not imply a zero of $D^{\pm}(k)$. Zeroes of the Fredholm determinants have been discussed previously in association with behavior not possible to a local potential. In the present paper, based on the dependence of these zeroes we have identified anomalous behavior of two types, spurious states $[D(k)=0,\ D^{\pm}(k)\neq 0]$ and continuum bound states $[D(k)=0,\ D^{\pm}(k)=0]$.

Limiting ourselves only by the conditions that the potential $V(r,r')$ be symmetric [in which case $\Delta(k) \equiv D(k)$] and satisfy Eq. (2), and considering only simple zeroes of the Fredholm determinants, we have compared the solutions for a nonlocal potential with those possible for a short range local potential. In the case of a short range local potential, the physical solution, regular solution, and Jost solutions are unambiguously defined for all $k \neq 0$ by integral equations which appropriately incorporate boundary conditions of choice. For a nonlocal potential we have shown that these solutions remain well defined as long as $D(k) \neq 0$. A more detailed analysis is required when $D(k) = 0$, in that certain of these solutions cease to exist and, in addition, solutions to the associated homogeneous equations must be considered.

For a spurious state we have pointed out that since $D^{\pm}(k) \neq 0$ the physical solution $\psi^{+}(k,r)$ and its conjugate $\psi^{-}(k,r)$ must exist; on the other hand, the regular solution $\varphi(k,r)$ and the Jost solutions $f^{\pm}(k,r)$ do not. However, it is possible to define for all $D(k)$ a modified regular solution $\Phi(k,r)$ and a real irregular solution $\theta(k,r)$. We demonstrate the linear independence of these two solutions. On the other hand, the integral equations for $\Phi(k,r)$ and $\theta(k,r)$ become homogeneous for $D(k)=0$. Moreover, at a spurious state it is not possible to impose arbitrary boundary conditions on the general solution $\alpha\Phi(k,r)$ and $\beta\theta(k,r)$.

For a continuum bound state for which the bound state wave function $\psi_h^{\pm}(k,r)$ exhibits nonzero slope, there is a solution to the inhomogeneous physical equation. This solution, however, is not unique, but is arbitrary with respect to the addition of any amount of $\psi_h^{\pm}(k,r)$. The regular solution $\varphi(k,r)$ exists, and is arbitrary with respect to the addition of any amount of $\varphi_h(k,r)$; the modified regular solution $\Phi(k,r)$ is identically zero. Neither the Jost solutions nor the real irregular solution $\theta(k,r)$ exists. We demonstrate that the solutions $\varphi_h(k,r)$ and $\psi_h^{\pm}(k,r)$ are linearly independent and that the general solution can be written in the form $\alpha\varphi_h(k,r)+\beta\psi_h^{\pm}(k,r)$.

For a continuum bound state for which $\psi_h^{\pm}(k,0)'=0$, all solutions of the inhomogeneous equation exist; again, however, $\Phi(k,r)=0$. On the other hand, all of the homogeneous solutions are equal to $\psi_h^{\pm}(k,r)$, and solutions of the inhomogeneous equations are arbitrary with respect to the addition of any amount of $\psi_h^{\pm}(k,r)$.

Thus in this paper we provide a framework for investigating some of the consequences of relaxed constraints associated with analyticity. We do not, however, present an explanation of the phenomena ascribed in the introduction as requiring a nonlocal potential. Such a study requires examination of the phase shift behavior in the vicinity of the zeroes of the Fredholm determinants. An investigation of this phase shift behavior is under way.[32] Also in preparation is an extension of the generalized Fredholm determinant previously defined for a one-term separable potential[13] to the case of a symmetric N-term separable potential.[33] Both areas of inquiry appear fruitful.

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