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Pion-nucleus multiple scattering using separable t matrices. I. Multiple scattering formalism*

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The theory of pion-nucleus multiple scattering is developed in the fixed nucleon approximation, using a separable form for the pion-nucleon t matrices. The solution of the set of equations represents multiple scattering to all orders. The extension of the theory to describe charge-exchange pion scattering is discussed.

[NUCLEAR REACTIONS Pion-nucleus multiple-scattering theory, separable pion-nucleon t matrices, fixed nucleon approximation.]

INTRODUCTION

In this article we describe a method of calculating intermediate energy ($T\pi = 50$–250 MeV) pion-nucleus scattering from light nuclei. The technique employs multiple scattering to all orders and incorporates the finite range of the pion-nucleon interaction. Its main approximation, shared with many other multiple scattering calculations, is the assumption that the nucleons do not rearrange themselves during the encounter of the pion with the nucleus ("fixed nucleon approximation"). This is separate from the question of the effect of nucleon motion of the π-N scattering amplitude, which can be taken into account, at least approximately.

In our calculation, we make no large A approximation as is made in most optical models. In addition, we make no "low density" assumptions of the kind implicit in the expansion of the optical potential in successive orders of nucleon-nucleon correlations. In contrast to the Glauber model,[2] which was designed for application in the GeV range, we make no small angle approximation and do not require that the pion-nucleon amplitude be forward peaked. Although it must be admitted that the optical model has had some success in describing elastic scattering from nuclei as light as helium,[3] and that the Glauber series, for reasons ill understood, has been used with success at energies as low as 120 MeV,[4] it would clearly be advantageous to free the calculation from these assumptions.

Our method has historical roots in the early 1950's. Following a self-consistency technique introduced by Foldy,[5] Brueckner[6] extended earlier impulse approximation work of pion-deuteron elastic scattering to include all orders of multiple scattering. His solution for the pion wave is exact within the framework of pure p-wave pion-nucleon interactions and fixed nonoverlapping scattering centers. He then averaged the full pion-deuteron amplitude over the deuteron density. Restricted to the deuteron, the calculation could be done analytically and in an elegant fashion. Larger nuclei pose formidable computational difficulties because of the large system of equations needed to describe the multiple scattering and because of the multidimensional integrals arising from the integral over the nuclear density.

In a far-reaching treatment of the optical potential,[7] Foldy and Walecka returned to the problem of the multiple scattering of projectiles from a collection of fixed scatterers. By assuming that the projectile and nucleon interact through separable two-body potentials in each partial wave, they were able to sum the multiple scattering series to all orders. The next step, integrating the amplitude over $3A$ nuclear coordinates, is extremely complex. Since Foldy and Walecka were attempting to derive an expression for the optical potential, they introduced an additional low density assumption. This approximation allowed them to expand the complete solution in a multiple scattering series while simultaneously expanding the A-particle nuclear density in terms of increased orders of nucleon-nucleon correlations. This program resulted in an expression for the first order optical potential plus a series of correction terms.

Our work parallels that of Brueckner and of Foldy and Walecka. The multiple scattering equations developed are essentially equivalent to theirs, although we choose to work directly with

the pion-nucleon t matrix instead of with a potential. Like Breuckner, we use self-consistency equations to describe the multiple scattering rather than solving the Schrödinger equation directly.

We differ from Foldy and Walecka in that we do the integration of the pion-nucleus scattering amplitude over the overlap of the initial and final state waves functions *without further physical assumptions* such as neglect of higher order correlations implicit in low density approximations. Thus we bypass the introduction of the optical potential. This is accomplished by using the full power of large computing machines to evaluate the many dimensional integrals by Monte Carlo techniques. This integration technique gives us a great flexibility in the use of nuclear wave functions. Not only can we use actual shell model wave functions to calculate elastic and inelastic scattering, but we can also treat deformed nuclei, spin-flip, and charge exchange.

We conclude the introduction with a few heuristic remarks concerning the fixed nucleon approximation (FNA) since it underlies most multiple scattering calculations. The FNA rests on the observation that even at rather low incident energies the pion moves much faster than the bound-state nucleons. We take FNA to mean that the nucleons move a negligible distance during the encounter with the incident pion. For example, a pion of kinetic energy 180 MeV has a velocity of about $0.9c$, roughly 4 times greater than the typical of a bound nucleon, while even a pion with only 50 MeV moves about 3 times faster than a bound nucleon. Although a velocity ratio of 3–4 may not seem *a priori* large enough to justify the neglect of nucleon rearrangement, it is a reasonable starting point and has proved remarkably fruitful in practice.

It is interesting that in the very high (GeV) and very low (50 MeV) regions the FNA is not strongly tested by experiment. This is because the FNA is tested only if the pion *rescatters* on the same nucleon (triple scattering or higher). In the high energy regime, where Glauber theory is most successful, the pion-nucleon amplitude is peaked in the forward direction. Therefore, although multiple scattering may be important, backscattering is very unlikely and hence the pion strikes a given nucleon only once. In the other extreme, for example at 50 MeV, the pion-nucleon amplitude is quite weak and higher-order multiple scattering contributes relatively small corrections to single scattering.

In Sec. I we derive the multiple scattering equations and indicate how they are solved for separable interactions. An alternate derivation is given in Sec. II. Section III briefly discusses the inclusion of spin and isospin degrees of freedom. The final section sketches the Monte Carlo methods used in calculating the matrix elements.

In a latter article we will discuss the effect of the nucleon motion on the effective π-nucleon scattering amplitude and how it is related to the assumptions made about the precise way in which the t matrix is assumed to go off shell. In practice an adequate treatment of this question is essential for the description of elastic scattering from light nuclei. We will also discuss the inclusion of Coulomb effects and apply the technique to elastic scattering of pions from ^{4}He.

I. MULTIPLE SCATTERING EQUATIONS

Let us consider the scattering of a pion wave from A fixed nucleons having position coordinates $\vec{x}_i$ $(i=1,2,\ldots,A)$. We define an operator $\hat{\psi}_i$ such that

$$\hat{\psi}_i e^{i\vec{k}\cdot\vec{x}} = \psi_i(\vec{k},\vec{x},\vec{x}_i) , \tag{1}$$

where[8] $\psi_i(\vec{k},\vec{x},\vec{x}_i)$ is the scattered part of the pion wave function assuming that the initial pion wave interacts with an isolated nucleon at $\vec{x}_i$. In the nucleus, however, the $(A-1)$ nearby nucleons will distort the wave incident on the nucleon at $\vec{x}_i$, changing $\psi_i(\vec{k},\vec{x},\vec{x}_i)$ into $\Psi_i(\vec{k},\vec{x};\vec{x}_1,\ldots,\vec{x}_A)$, where the dependence of the wave function on the coordinates of all of the nucleons has been made explicit. The total scattered pion wave is given by the sum of the distorted wave functions over the nucleons:

$$\Psi(\vec{k},\vec{x};\vec{x}_1,\ldots,\vec{x}_A)=\sum_{i=1}^{A}\Psi_i(\vec{k},\vec{x};\vec{x}_1,\ldots,\vec{x}_A) . \tag{2}$$

We now derive a self-consistency equation which determines the multiply scattered waves Ψ_i in terms of the singly scattered ones, ψ_i. The incident wave on the ith nucleon consists of two components—the initial plane wave $e^{i\vec{k}\cdot\vec{x}}$ and the previously scattered waves emerging from the other scattering centers,

$$\sum_{j\neq i}\Psi_j(\vec{k},\vec{x};\vec{x}_1,\ldots,\vec{x}_A) .$$

By Eq. (1) the wave emerging from the ith scattering center is found by applying $\hat{\psi}_i$ to the incident wave, hence

$$\Psi_i(\vec{k},\vec{x};\vec{x}_1,\ldots,\vec{x}_A) = \hat{\psi}_i\left[e^{i\vec{k}\cdot\vec{x}}+\sum_{j\neq i}\Psi_j(\vec{k},\vec{x};\vec{x}_1,\ldots,\vec{x}_A)\right], \tag{3}$$

The plane wave impulse approximation (PWIA) results from assuming only the plane wave part is present. The second term is a correction which includes multiple scattering to all orders. To evaluate the last term we first express $\Psi_j(\vec{k},\vec{x};\vec{x}_1,\ldots,\vec{x}_A)$ as a Fourier transform,

$$\Psi_j(\vec{k},\vec{x};\vec{x}_1,\ldots,\vec{x}_A)=\frac{1}{(2\pi)^3}\int d^3p\, e^{i\vec{p}\cdot\vec{x}}\Psi_j(\vec{k},\vec{p};\vec{x}_1,\ldots,\vec{x}_A)\ , \tag{4}$$

and then use Eq. (1) to evaluate $\hat{\psi}_i e^{i\vec{p}\cdot\vec{x}}$. This gives the integral equation

$$\Psi_i(\vec{k},\vec{x};\vec{x}_1,\ldots,\vec{x}_A)=\psi_i(\vec{k},\vec{x},\vec{x}_i)+\sum_{j\neq i}\int\frac{d^3p}{(2\pi)^3}\,\psi_i(\vec{p},\vec{x},\vec{x}_i)\Psi_j(\vec{k},\vec{p};\vec{x}_1,\ldots,\vec{x}_A)\ . \tag{5}$$

Knowing ψ_i allows us to solve the system of equations to obtain the full multiple scattered waves.

We shall rewrite Eq. (5) in terms of the scattering amplitudes. As is shown in Appendix A, the amplitude F_i is related to the Fourier transform of the wave function by

$$\Psi_i(\vec{k},\vec{q};\vec{x}_1,\ldots,\vec{x}_A)=G_0(q)F_i(\vec{k},\vec{q};\vec{x}_1,\ldots,\vec{x}_A)\ , \tag{6}$$

where G_0 is the pion propagator. When $|k|=|q|=k$, F_i is the on-shell amplitude. For arbitrary q, $F_i(\vec{k},\vec{q};\vec{x}_1,\ldots,x_A)$ is the half-on-shell amplitude. Taking the Fourier transform of Eq. (5) on the pion coordinate $\vec{x}$ and using Eq. (6) gives

$$F_i(\vec{k},\vec{q};\vec{x}_1,\ldots,\vec{x}_A)=F_i(\vec{k},\vec{q},\vec{x}_i)+\sum_{j\neq i}\int\frac{d^3p}{(2\pi)^3}\,F_i(\vec{p},\vec{q},\vec{x}_i)G_0(p)F_j(\vec{k},\vec{p};\vec{x}_1,\ldots,\vec{x}_A)\ , \tag{7}$$

where $F_i(\vec{k},\vec{q},\vec{x}_i)$ is the scattering amplitude for scattering from a single nucleon at x_i. Since $F_i(\vec{k},\vec{q},\vec{x}_i)$ is usually expressed in terms of a target at the origin, we must consider the displacement properties of the wave functions and amplitudes.

The wave function is shown in Appendix B to satisfy

$$\Psi_i(\vec{k},\vec{x};\vec{x}_1,\ldots,\vec{x}_A)=e^{-i\vec{k}\cdot\vec{D}}\Psi_i(\vec{k},\vec{x}+\vec{D};\vec{x}_1+\vec{D},\ldots,\vec{x}_A+\vec{D}) \tag{8}$$

under displacement $\vec{D}$. Its corresponding Fourier transform obeys

$$\Psi_i(\vec{k},\vec{q};\vec{x}_1,\ldots,\vec{x}_A)=e^{i(\vec{q}-\vec{k})\cdot\vec{D}}\Psi_i(\vec{k},\vec{q};\vec{x}_1+\vec{D},\ldots,\vec{x}_A+\vec{D})\ . \tag{9}$$

From Eq. (6) we see that $F_i(\vec{k},\vec{q};\vec{x}_1,\ldots,\vec{x}_A)$ transforms under displacements in the same way as $\Psi_i(\vec{k},\vec{q};\vec{x}_1,\ldots,\vec{x}_A)$. It is convenient to use the centered amplitudes defined by

$$F_i^c(\vec{k},\vec{q};\vec{x}_1,\ldots,\vec{x}_A)\equiv F_i(\vec{k},\vec{q};\vec{x}_1-\vec{x}_i,\ldots,\vec{x}_A-\vec{x}_i)=e^{i(\vec{q}-\vec{k})\cdot\vec{x}_i}F_i(\vec{k},\vec{q};\vec{x}_1,\ldots,\vec{x}_A)\ , \tag{10}$$

where the nucleus is positioned such that the ith nucleon is at the origin. It is clear that $F_i^c(\vec{k},\vec{q},\vec{x}_i)$, the scattering amplitude from a single nucleon i, located at the origin, is independent of $\vec{x}_i$, and thus will be renamed $f_i(\vec{k},\vec{q})$. In terms of the centered amplitudes, Eq. (7) becomes

$$F_i^c(\vec{k},\vec{q};\vec{x}_1,\ldots,\vec{x}_A)e^{i\vec{k}\cdot\vec{x}_i}=f_i(\vec{k},\vec{q})e^{i\vec{k}\cdot\vec{x}_i}+\sum_{j\neq i}\int\frac{d^3p}{(2\pi)^3}\,e^{i\vec{k}\cdot\vec{x}_j}f_i(\vec{p},\vec{q})e^{i\vec{p}\cdot\vec{r}_{ij}}G_0(p)F_j^c(\vec{k},\vec{p};\vec{x}_1,\ldots,\vec{x}_A)\ , \tag{11}$$

where $\vec{r}_{ij}=\vec{x}_i-\vec{x}_j$. In the energy regime below 300 MeV incident pion energy, we can restrict ourselves to s and p waves. In this case the integral equations (11) are most easily solved by partial wave analysis followed by matrix inversions. F_i^c now requires only as many partial waves in $\hat{q}$ as are contained in f_i, since the only $\hat{q}$ dependence in the right-hand side of Eq. (11) comes from f_i. By contrast, F_i would typically require many more partial waves to reproduce the exponentials which are explicitly extracted in Eq. (10). Thus the use of F_i^c enormously reduces the size of the matrix equations necessary to invert the set of integral equations. Even so, for s and p wave pion-nucleon interactions we require a set of $(1+3)\times A$ equations. For ^{12}C this is 48 simultaneous equations.

After the equations have been solved for the F_i^c, each amplitude is then shifted so that the center of mass of the nucleus, denoted by the vector $\vec{R}$, is now at the origin. This yields for the pion-nucleus half-shell amplitude

$$F(\vec{k},\vec{q};\vec{x}_1,\ldots,\vec{x}_A)=\sum_{j=1}^{A}e^{i(\vec{k}-\vec{q})\cdot(\vec{x}_j-\vec{R})}F_j^c(\vec{k},\vec{q};\vec{x}_1,\ldots,\vec{x}_A). \tag{12}$$

In the absence of multiple scattering, F_j^c becomes f_j and the expectation of Eq. (12) over the nuclear density expresses exactly the plane-wave impulse

approximation to elastic scattering.

As a preliminary to the partial wave analysis, let us define

$$G_i(\vec{k},\vec{q};\vec{x}_1,\ldots,\vec{x}_A)\equiv F_i^c(\vec{k},\vec{q};\vec{x}_1,\ldots,\vec{x}_A)e^{i\vec{k}\cdot\vec{x}_i}\ , \tag{13}$$

so that Eq. (11) now becomes

$$G_i(\vec{k},\vec{q})=f_i(\vec{k},\vec{q})e^{i\vec{k}\cdot\vec{x}_i}$$

$$=\sum_{j\neq 1}\int\frac{d^3p}{(2\pi)^3}f_i(\vec{p},\vec{q})G_0(p)e^{i\vec{p}\cdot\vec{r}_{ij}}G_j(\vec{k},\vec{p})\ , \tag{14}$$

where the $\vec{x}_1,\ldots,\vec{x}_A$ in the G_i and G_j have been suppressed. The partial wave expansions of the quantities in Eq. (14) are

$$G_0(k,p)e^{i\vec{p}\cdot\vec{r}_{ij}}\equiv 4\pi\sum_{\lambda\mu}A_\lambda(k,p,r_{ij})Y_\lambda^\mu(\hat{p})Y_\lambda^{\mu *}(\hat{r}_{ij})\ , \tag{15}$$

$$f_i(\vec{p},\vec{q})\equiv\frac{2\pi}{ik}\sum_{lm}f_i^l(p,q)Y_l^m(\hat{q})Y_l^{m*}(\hat{p})\ , \tag{16}$$

$$G_j(\vec{k},\vec{p})\equiv\frac{2\pi}{ik}\sum_{l'm'}g_j^{l'm'}(\vec{k},p)\left(\frac{2l'+1}{4\pi}\right)^{1/2}Y_{l'}^{m'}(\hat{p}) \tag{17}$$

where $g_j^{l'm'}$ is a function of all of the nucleon coordinates and k is the on-shell momentum. Substituting Eq. (15)–(17) into Eq. (14) and letting the incident momentum $\vec{k}$ lie along the $\hat{z}$ axis gives

$$g_i^{lm}(\vec{k},q)=f_i^l(k,q)e^{i\vec{k}\cdot\vec{x}_i}\delta_{m,0}$$

$$+\frac{1}{i\pi k}\sum_{j\neq i}\sum_{\lambda,\mu}\sum_{l'm'}\left(\frac{2l'+1}{2l+1}\right)^{1/2}G_{lm,l'm',\lambda\mu}^{lm}\int_0^\infty p^2dp\,A_\lambda(k,p,r_{ij})f_i^l(p,q)g_j^{l'm'}(\vec{k},p)Y_\lambda^{\mu *}(\hat{r}_{ij})\ , \tag{18}$$

where

$$G_{lm,l'm',\lambda\mu}^{lm}=\int d\Omega_p\,Y_l^{m*}(\hat{p})Y_\lambda^\mu(\hat{p})\,Y_{l'}^{m'}(\hat{p})\ . \tag{19}$$

To proceed further we now assume that the amplitude f_i^l is separable. As a simple model we choose[7,9]

$$f_i^l(p,q)=\lambda_i^l(\omega)v^l(p)v^l(q)\ , \tag{20}$$

where for s and p waves we take

$$v^l(p)=\left(\frac{p}{k}\right)^l\frac{k^2+\alpha_l^2}{p^2+\alpha_l^2}\ , \tag{21}$$

with the α_l range parameters to be determined from experiment, and $\lambda_i^l(\omega)$ is evaluated in terms of pion-nucleon phase shifts. For example, if the incident meson is π^+, we have

$$\lambda^0{}_{\text{proton}}=\lambda(S^{31}),\quad \lambda^1{}_{\text{proton}}=\tfrac{2}{3}\lambda(P^{33})+\tfrac{1}{3}\lambda(P^{31}),$$

$$\lambda^0{}_{\text{neutron}}=\tfrac{2}{3}\lambda(S^{11})+\tfrac{1}{3}\lambda(S^{31}),\quad \lambda^1{}_{\text{neutron}}=\tfrac{2}{9}\lambda(P^{11})+\tfrac{1}{9}\lambda(P^{31})+\tfrac{2}{9}\lambda(P^{33})+\tfrac{4}{9}\lambda(P^{13})\ , \tag{22}$$

where $\lambda(L^{2I,2J})=(k/k_{\text{c.m.}})(e^{2i\delta_L^{2I,2J}}-1)$, the phase shift being evaluated in the pion-nucleon center-of-mass frame. The momentum ratio $(k/k_{\text{c.m.}})$ gives an approximate transformation from the center-of-mass (c.m.) to laboratory frame. The model amplitude which we have used in our calculations also includes a relativistic version of the angle transformation as will be discussed in a following article.

By inserting (20) into (18) we see that g_i factors; we therefore write

$$g_i^{lm}(\vec{k},q)\equiv g_i^{lm}(\vec{k})v^l(q)\ . \tag{23}$$

Substitution of this expression along with Eq. (20) into Eq. (18) yields

$$g_i^{lm}(\vec{k})=\lambda_i^l(\omega)\,\delta_{m,0}e^{i\vec{k}\cdot\vec{x}_i}+2\pi\lambda_i^l(\omega)\sum_{\lambda\mu}\sum_{l'm'}\sum_{j\neq i}Y_\lambda^{\mu *}(\hat{r}_{ij})G_{lm,l'm',\lambda\mu}^{lm}\left(\frac{2l'+1}{2l+1}\right)^{1/2}g_j^{l'm'}(\vec{k})Z_{ll'}^\lambda(k,r_{ij})\ , \tag{24}$$

where

$$Z^{\lambda}_{ll'} = -\frac{2i}{k(2\pi)^2}\int_0^\infty p^2\,dp\,v^l(p)v^{l'}(p)A_\lambda(k,p,r_{ij})\ . \tag{25}$$

To evaluate $Z^{\lambda}_{ll'}$, A_λ is needed. From Eq. (15), using the form for the free pion propagator

$$G_0(k,p) = \frac{4\pi}{p^2-k^2-i\eta}\ , \tag{26}$$

we obtain

$$\frac{e^{i\vec{p}\cdot\vec{r}_{ij}}}{p^2-k^2-i\eta} = \sum_{\lambda\mu} A_\lambda(k,p,r_{ij})Y^{\mu}_{\lambda}(\hat{p})Y^{\mu}_{\lambda}{}^{*}(\hat{r}_{ij})\ . \tag{27}$$

Expanding the exponential and equating coefficients we find

$$A_\lambda(k,p,r_{ij}) = \frac{4\pi i^\lambda j_\lambda(pr_{ij})}{p^2-k^2-i\eta}\ . \tag{28}$$

The form of G_0 away from its pole at $p=k$ is to some extent arbitrary since G_0 is multiplied by the off-shell amplitude. For example, Walker[10] has used

$$G_0 = \frac{4\pi}{2\omega(k)}\,\frac{1}{\omega(p)-\omega(k)-i\eta}\ .$$

This may be recast as

$$G_0 = \frac{4\pi}{2\omega(k)[\omega(p)-\omega(k)]/(p^2-k^2-i\eta)}\,\frac{1}{p^2-k^2-i\eta} \equiv \frac{4\pi N(p,k)}{p^2-k^2-i\eta}\ , \tag{29}$$

where it is easily seen that $N(k,k)=1$. The function $N(k,p)$ may be absorbed into $v^l(p)$, and so modifies the off-shell behavior only. Since $v^l(p)$, and hence the off-shell behavior, has been evaluated from experimental data[11] (π^+-d absorption), this procedure reduces the arbitrariness.

Substituting (28) into (25) gives

$$Z^{\lambda}_{ll'}(k,r_{ij}) = -\frac{2i}{\pi k}\int_0^\infty \frac{p^2dp\,v^l(p)v^{l'}(p)j_\lambda(pr_{ij})i^\lambda}{p^2-k^2-i\eta}\ . \tag{30}$$

This integral is evaluated as in Ref. 7. By Eq. (19), only even $l+l'+\lambda$ contribute. This result combined with Eq. (21) implies that $v^l(p)v^{l'}(p)\times j_\lambda(pr_{ij})$ is an even function of p, hence we can extend the integration to $-\infty$. The spherical Bessel function is next written as $\frac{1}{2}[h^{(+)}_\lambda + h^{(-)}_\lambda]$, and the integral done by contour methods. The semicircular contour at infinity is taken in the upper half plane for the first term and in the lower half plane for the second. The poles at $p=\pm(k+i\eta)$ yield $i^\lambda h^{(+)}_\lambda(kr_{ij})$, independent of α_l. The poles of the v^l's contribute damped terms which exactly cancel the singularity in $h^{(+)}_\lambda$ at $r_{ij}=0$. Except for binding and recoil corrections these terms contribute all of the off-shell dependence. Their short range $[\exp(-\alpha_l r_{ij})]$ is in accord with Beg-Agassi-Gal-Hüfner theorem[12] which states that only on-shell information is required for multiple scattering from nonoverlapping fixed centers.

For the particular form of v^l used, we have

$$Z^{\lambda}_{ll'}(k,r_{ij}) = i^\lambda\left\{h^{(+)}_\lambda(kr_{ij}) - \frac{1}{(\alpha_{l'}{}^2-\alpha_l{}^2)k^{l+l'+1}}\left[(i\alpha_l)^{l+l'+1}(k^2+\alpha_{l'}{}^2)h^{(+)}_\lambda(i\alpha_l r_{ij}) - (i\alpha_{l'})^{l+l'+1}(k^2+\alpha_l{}^2)h^{(+)}_\lambda(i\alpha_{l'}r_{ij})\right]\right\}. \tag{31}$$

Using this form for the $Z^{\lambda}_{ll'}$, we can now solve the system of equations (24) for the $g^{lm}_i(\vec{k})$, from Eq. (23) obtain the $g^{lm}_i(\vec{k},q)$, and then from Eqs. (12), (13), and (17) the total pion-nucleus amplitude $F(\vec{k},\vec{q};\vec{x}_1,\ldots,\vec{x}_A)$. The final step is to average the amplitude over the nuclear wave functions, i.e.,

$$F_{if} = \langle\psi_f|F(\vec{k},\vec{q};\vec{x}_1,\ldots,\vec{x}_A)|\psi_i\rangle\ . \tag{32}$$

The details of this integration process will be included in Sec. IV.

II. AN ALTERNATE DERIVATION

The basic multiple-scattering equation which we have applied, Eq. (7), can also be obtained directly from formal scattering theory. We begin with Lippmann-Schwinger equations describing the pion wave scattered from all nucleons $|\Psi(\vec{k})\rangle$, and the pion wave scattered from only the ith nucleon $|\psi_i(\vec{k})\rangle$

$$|\Psi(\vec{k})\rangle = |\vec{k}\rangle + \sum_j G_0V_j|\Psi(\vec{k})\rangle \tag{33}$$

and

$$|\psi_i(\vec{k})\rangle = |\vec{k}\rangle + G_0V_i|\psi_i(\vec{k})\rangle\ , \tag{34}$$

where $|\vec{k}\rangle$ is the incident plane wave, V_i describes the pion-nucleon interaction, and G_0 is the free pion Green's function. Subtracting and rear-

ranging Eqs. (33) and (34),

$$|\Psi(\vec{k})\rangle - |\psi_i(\vec{k})\rangle = \sum_{j\neq i} G_0 V_j |\Psi(\vec{k})\rangle + G_0 V_i (|\Psi(\vec{k})\rangle - |\psi_i(\vec{k})\rangle) . \tag{35}$$

We can formally solve Eq. (35) for the quantity $|\Psi(\vec{k})\rangle - |\psi_i(\vec{k})\rangle$ to obtain

$$|\Psi(\vec{k})\rangle - |\psi_i(\vec{k})\rangle = \sum_{j\neq i} (1 - G_0 V_i)^{-1} G_0 V_j |\Psi(\vec{k})\rangle . \tag{36}$$

Multiplying by $\langle \vec{k}' | V_i$, inserting a complete set of momentum states, and making use of the property of G_0 being diagonal in the momentum,

$$\langle \vec{k}' | V_i | \Psi(\vec{k})\rangle - \langle \vec{k}' | V_i | \psi_i(\vec{k})\rangle = \sum_{j\neq i} \int \frac{d^3p}{(2\pi)^3} \langle \vec{k}' | V_i (1 - G_0 V_i)^{-1} | \vec{p}\rangle \times G_0(p) \langle \vec{p} | V_j | \Psi(\vec{k})\rangle . \tag{37}$$

From Eq. (34), with $\vec{k} \to \vec{p}$,

$$|\psi_i(\vec{p})\rangle = |\vec{p}\rangle + G_0 V_i |\psi_i(\vec{p})\rangle \tag{38}$$

or

$$|\psi_i(\vec{p})\rangle = (1 - G_0 V_i)^{-1} |\vec{p}\rangle . \tag{39}$$

This expression, along with the definitions

$$F_i(\vec{k}, \vec{k}') = \langle \vec{k}' | V_i | \Psi(\vec{k})\rangle , \tag{40}$$

$$f_i(k, k') = \langle \vec{k}' | V_i | \psi(\vec{k})\rangle , \tag{41}$$

when inserted in Eq. (37) yields

$$F_i(\vec{k}, \vec{k}') = f_i(\vec{k}, \vec{k}') + \sum_{j\neq i} \int \frac{d^3p}{(2\pi)^3} f_i(\vec{p}, \vec{k}') G_0(p) F_j(\vec{k}, \vec{p}) , \tag{42}$$

the same results as Eq. (7).

III. SPIN AND ISOSPIN DEGREES OF FREEDOM

The multiple scattering equations discussed in the previous section can be generalized to include isospin. This is done by interpreting the scattering equation Eq. (14), which can be rewritten as $G_i = f_i + f_i \sum_{j\neq i} G_j$ as a matrix in isospin space.

First let us consider an incident π^+ meson and assume that only one nucleon is allowed to charge exchange. The isospin state vectors are labeled by $|1\rangle$ (π^+ meson, neutron for the valence nucleon) and $|2\rangle$ (π^0 meson, proton as valence nucleon). Taking isospin matrix elements and inserting a complete set of isospin states between f_i and G_j, we have the coupled equations

$$\begin{aligned} \langle 1|G_i|1\rangle &= \langle 1|f_i|1\rangle + \sum_{j\neq i} \langle 1|f_i|1\rangle\langle 1|G_j|1\rangle + \sum_{j\neq i} \langle 1|f_i|2\rangle\langle 2|G_j|1\rangle , \\ \langle 2|G_i|1\rangle &= \langle 2|f_i|1\rangle + \sum_{j\neq i} \langle 2|f_i|1\rangle\langle 1|G_j|1\rangle + \sum_{j\neq i} \langle 2|f_i|2\rangle\langle 2|G_j|1\rangle . \end{aligned} \tag{43}$$

Of course, $\langle 2|f_i|1\rangle$ is zero unless i corresponds to the valence neutron. On the other hand, $\langle 2|G_j|1\rangle$ is nonzero for all j since the charge exchange could have occurred on a previous scatter.

Since charge exchange on a single nucleon doubles the number of equations, the required solutions become very time consuming. We have found it convenient to omit the coupling term $\langle 1|f_i|2\rangle\langle 2|G_j|1\rangle$ in the first equation. With this term omitted, the first equation may be independently solved and then substituted into the second equation which is then solved. We have calculated charge exchange cross sections both with and without this coupling term and find only minute changes (a few percent in the cases tested) in the results.

Assuming now that two nucleons can charge exchange, the isospsin state vectors are labeled by two indices. We label the state vectors by $|1\rangle \equiv |nn\rangle$ where both valence nucleons are neutrons, $|2\rangle \equiv |np\rangle$ where the first nucleon is a neutron and the second a proton, $|3\rangle \equiv |pn\rangle$ and $|4\rangle \equiv |pp\rangle$ analogously. The corresponding equations are

$$\begin{aligned} \langle 1|G_i|1\rangle &= \langle 1|f_i|1\rangle + \sum_{n=1,2,3} \sum_{j\neq i} \langle 1|f_i|n\rangle\langle n|G_j|1\rangle , \\ \langle 2|G_i|1\rangle &= \langle 2|f_i|1\rangle + \sum_{n=1,2,4} \sum_{j\neq i} \langle 2|f_i|n\rangle\langle n|G_j|1\rangle , \\ \langle 3|G_i|1\rangle &= \langle 3|f_i|1\rangle + \sum_{n=1,3,4} \sum_{j\neq i} \langle 3|f_i|n\rangle\langle n|G_j|1\rangle , \\ \langle 4|G_i|1\rangle &= \langle 4|f_i|1\rangle + \sum_{n=2,3,4} \sum_{j\neq i} \langle 4|f_i|n\rangle\langle n|G_j|1\rangle . \end{aligned} \tag{44}$$

The first equation describes elastic scattering; the second and third, single charge exchange; and the fourth, double charge exchange. Again we ignore back coupling so that $n=2$ and 3 are dropped in the first equation and $n=4$ in the second and third.

In similar fashion more valence neutrons can be accommodated, but the number of equations rapidly becomes unmanageable. For example, for only four valence neutrons, 16 equations must be used to keep track of the charge states.

Spin-flip can be formally included in a similar way. However, there are important complications since spin occurs dynamically in the equations.

This topic is discussed in detail in Ref. 13.

IV. INTEGRATION OVER THE NUCLEAR COORDINATES

In the plane-wave impulse approximation the nuclear scattering amplitude is

$$F(\vec{k},\vec{q};\vec{x}_1,\ldots,\vec{x}_A)=\sum_{i=1}^{A} f_i(\vec{k},\vec{q})e^{i(\vec{k}-\vec{q})\cdot(\vec{x}_i-\vec{R})}\,. \tag{45}$$

Hence each term in the sum involves the coordinates of a single nucleon. Supposing the nuclear density to factor into a product of terms each involving only a single nucleon, the integrals can easily be computed. Similar comments apply even if full shell model wave functions are used, although the angular momentum coupling can become complex. In Glauber theory, the nuclear scattering amplitude is expressible as a product of factors each of which involves a single nucleon's coordinates and thus the integrals can again be performed for each nucleon independently.

In contrast, if the multiple scattering series is summed exactly, the nucleon coordinates in $F(\vec{k},\vec{q};\vec{x}_1,\ldots,\vec{x}_A)$ become inextricably tangled, and we must compute a $3A$ dimensional integral. To get an idea of the magnitude of the problem, the integrations for ^{12}C are 36 (3×12) dimensional. At each integration point in this 36 dimensional space a 48 $[(1+3)\times12]$ dimensional set of coupled matrix equations must be solved to sum the multiple scattering series provided s and p waves describe the pion-nucleon interaction adequately. For reasonable convergence at the back angles (10–20%) several thousand integration points appear to be sufficient if the points are chosen properly. This requires about five minutes on a CDC-7600 computer for one value of the energy and for all angles.

We now discuss how to choose the points "properly" through a Monte Carlo technique. Let us suppose the wave function to be of the form $\Psi_{\rm int}(\bar{y}_1,\ldots,\bar{y}_A)Y_L^M(\hat{n})$, where the intrinsic coordinates are measured relative to the center of mass of the nucleus and relative to a body-fixed axis $\hat{n}$. The intrinsic wave function $\Psi_{\rm int}$ may be found from a (possibly deformed) shell model, for example. As an illustration, suppose that $|\Psi_{\rm int}|^2$ can be written as a product of single particle densities $\rho_1(\bar{y}_1)\cdots\rho_A(\bar{y}_A)$, and that $L=M=0$. The labels on the ρ's allow us to distinguish s from p shells, neutrons from protons, etc. In gambling parlance we "throw" a nucleus as follows. We first choose an axis in space relative to the beam axis about which to build the nuclear configuration. Since $|Y_0^0|^2$ is isotropic, this may be done by choosing two random numbers; $\cos\theta$ between -1 and $+1$ and ϕ between 0 and 2π. Nonaxially symmetric nuclei require a third angle to be selected about the axis. For each nucleon three random numbers are picked to establish $(r_i,\theta_i,\phi_i)_{\rm int}$ relative to $\hat{n}$ in such a way that the probability of choosing a point in a given volume is proportional to $\rho_i(\bar{y}_i)$ in that volume.[14] Although at this point $\sum_i \bar{y}_i \neq 0$, this condition will subsequently be satisfied through the translation in Eq. (12). Because of this the wave functions are automatically functions of the intrinsic coordinates and the customary problem of removing "center of mass correlations" common to most calculations is absent. From the $\bar{y}_i$ and $\hat{n}$ we next can calculate $\vec{x}_i$, the coordinates in the space fixed axes. Using these coordinates the multiple scattering equations are inverted to yield $F_i(\vec{k},\vec{k}^1;\vec{x}_1^1,\ldots,\vec{x}_A^1)$. The superscript 1 denotes the first Monte Carlo point. Now another nucleus is "thrown" by repeating the procedure. Continuing in this manner, we construct, after N such $3A+2$ dimensional tosses,

$$\langle F_i(\vec{k},\vec{k}')\rangle_N=\frac{1}{N}\sum_{n=1}^{N} F_i(\vec{k},\vec{k}';\vec{x}_1^n,\ldots,\vec{x}_A^n). \tag{46}$$

For sufficiently large N, $\langle F_i(\vec{k},\vec{k}')\rangle_N$ approaches the elastic scattering amplitude. Convergence is tested by varying N and by using different sequences of random numbers.

To apply the Monte Carlo method in cases for which the transition density $\Psi_f^*\Psi_i$ does not factor we first write

$$\Psi_f^*(\vec{x}_1,\ldots,\vec{x}_A)\Psi_i(\vec{x}_1,\ldots,\vec{x}_A) \equiv\rho_1(\vec{x}_1)\cdots\rho_A(\vec{x}_A)R(\vec{x}_1,\ldots,\vec{x}_A)\,, \tag{47}$$

where the normalized single particle densities $\rho_1(\vec{x}_1)\cdots\rho_A(\vec{x}_A)$ are chosen to match the overlap $\Psi_f^*\Psi_i$ as closely as possible. The defined function $R(\vec{x}_1,\ldots,\vec{x}_A)$ is then absorbed into the amplitude $F(\vec{k},\vec{k}';\vec{x}_1,\ldots,\vec{x}_A)$. The Monte Carlo procedure is applied to this modified amplitude and leads to

$$F(\vec{k},\vec{k}')\simeq\frac{1}{N}\sum_{n=1}^{N} F(\vec{k},\vec{k}';\vec{x}_1^n,\ldots,\vec{x}_A^n)R(\vec{x}_1^n,\ldots,\vec{x}_A^n)\,. \tag{48}$$

The method will be most successful (i.e., will converge most quickly) provided that the function $R(\vec{x}_1,\ldots,\vec{x}_A)$ does not oscillate to produce large cancellations in the integrand.

As an example of the use of nonfactorizable densities, we next assume a density of the form

$$\rho(\vec{r}_1,\ldots,\vec{r}_A)=\eta\rho_1(\vec{r}_1)\cdots\rho_A(\vec{r}_A)\prod_{i\neq j}J_{ij}(\vec{r}_i-\vec{r}_j)\,, \tag{49}$$

where η is a normalization factor. Such a density

puts correlations between all pairs of nucleons. A common form for $J(r)$ is $1-\exp[-(\beta r)^2]$, the Jastrow correlation function. Different factors could be used for n-n, n-p, and p-p pairs, if desired. The first step is to determine η and then adjust the rms size of the distribution. This is done by "throwing" a large number of nuclei to calculate

$$1=\int \rho(\vec{r}_i,\ldots,\vec{r}_A)d\vec{r}_1\cdots d\vec{r}_A$$

and

$$\langle r^2\rangle=\frac{1}{A}\int \rho(\vec{r}_i,\ldots,\vec{r}_A)\Big(\sum_{i=1}^{A} r_i^2\Big)d\vec{r}_1\cdots d\vec{r}_A\ . \tag{50}$$

The first of these equations determines η, the second the rms radius. The ρ's are then varied, for example by varying the oscillator parameter if the shell model description is used, to obtain the experimental value of $\langle r^2\rangle$. By using more parameters, higher moments could be fixed in a similar way. The elastic scattering amplitude is now

$$F(\vec{k},\vec{k}')=\frac{n}{N}\sum_{n=1}^{N} F(\vec{k},\vec{k}';\vec{x}_1^{\,n},\ldots,\vec{x}_A^{\,n})\prod_{i\neq j}\times J_{ij}(\vec{x}_i^{\,n}-\vec{x}_j^{\,n})\ . \tag{51}$$

In this fashion correlations are accounted for exactly and to all orders, within the computational limitations imposed by available computer time.

As a final example we consider the pion excitation of a nucleus from a $L=0$ ground state $[\Psi(\bar{y}_1,\ldots,\bar{y}_A)Y_0^0(\hat{a})]$ to a low-lying[15] (L,M) rotational level $[\Psi(\bar{y}_1,\ldots,\bar{y}_A)Y_L^M(\hat{a})]$; $\hat{a}$ is the body-fixed symmetry axis. We assume for simplicity the internal wave functions to be the same for both states. Using Eq. (48) the amplitude for the process is

$$F_L^M(\vec{k},\vec{k}')\simeq\frac{1}{N}\sum_{n=1}^{N} F(\vec{k},\vec{k}';\vec{x}_1^{\,n},\ldots,\vec{x}_A^{\,n})Y_L^M(\hat{a}^n)/Y_0^0(\hat{a}^n)\ . \tag{52}$$

The convergence is good provided L is 0 or 2, but deteriorates for larger values of L where the cancellations in the sum become more serious. This technique provides automatically the population of the different M states and hence also the spin density matrix of the residual nucleus.

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APPENDIX A

To derive Eq. (6) and to identify F as the usual scattering amplitude, we rewrite the Lippmann-Schwinger equations as

$$\Psi_T=\phi_0+G_0V\Psi_T\equiv\phi_0+G_0F, \tag{A1}$$

where Ψ_T is the full (incident plus scattered) state vector and $F\equiv V\Psi_T$ is the scattering operator.[16] The scattered part of the wave function is then $\Psi=G_0F$. Taking matrix elements of this expression and suppressing all nucleon coordinates we get

$$\langle\vec{x}|\Psi|\vec{k}\rangle=\int\langle\vec{x}|G_0|\vec{x}'\rangle\langle\vec{x}'|F|\vec{k}\rangle d\vec{x}' \tag{A2}$$

and

$$\langle\vec{q}|\Psi|\vec{k}\rangle=\int\langle\vec{q}|G_0|\vec{q}'\rangle\langle\vec{q}'|F|\vec{k}\rangle\frac{d\vec{q}'}{(2\pi)^3}\ . \tag{A3}$$

Using $\langle\vec{x}|G_0|\vec{x}'\rangle=G_0(\vec{x}-\vec{x}')$ and its Fourier transform, $\langle\vec{q}|G_0|\vec{q}'\rangle=G_0(\vec{q})(2\pi)^3\delta(\vec{q}-\vec{q}')$, (A1) and (A2) become, with a slight notational shift,

$$\Psi(\vec{k},\vec{x})=\int G_0(\vec{x}-\vec{x}')F(\vec{k},\vec{x}')d\vec{x}' \tag{A4}$$

and

$$\Psi(\vec{k},\vec{q})=G_0(\vec{q})F(\vec{k},\vec{q})\ . \tag{A5}$$

The second of these is Eq. (6). To identify F as the scattering amplitude we use

$$G_0(\vec{x}-\vec{x}')\underset{|\vec{x}|\to\infty}{\sim}\frac{e^{ikx}}{x}e^{-i\vec{k}'\cdot\vec{x}'}\ , \tag{A6}$$

where $\vec{k}'\equiv k\hat{x}$, in (A4) to get

$$\Psi(\vec{k},\vec{x})\sim\frac{e^{ikx}}{x}\int e^{-i\vec{k}'\cdot\vec{x}'}F(\vec{k},\vec{x}')d\vec{x}'$$

$$\sim\frac{e^{ikx}}{x}F(\vec{k},\vec{k}')\ , \tag{A7}$$

the usual expression for the scattered wave being the product of an outgoing spherical wave and a scattering amplitude.

APPENDIX B

The translation property (8) is easily proven for the case of fixed scattering centers described by (possibly energy dependent and nonlocal) potentials. Let the potential $V_i(\vec{x},\vec{x}')$ describe the ith scattering center when located at the origin. The pion wave satisfies

$$\psi_{\vec{k}}(\vec{x};\vec{x}_1,\ldots,\vec{x}_A)=e^{i\vec{k}\cdot\vec{x}}+\sum_{i=1}^{A}\iint G_0(\vec{x}-\vec{x}')V_i(\vec{x}'-\vec{x}_i,\vec{x}''-\vec{x}_i)\psi_k(\vec{x}'';\vec{x},\ldots,\vec{x}_A)d\vec{x}'d\vec{x}''\ , \tag{B1}$$

where $\vec{x}_1,\ldots,\vec{x}_A$, assumed fixed but arbitrary, are the positions of the scattering centers. We now shift

the positions of the scattering centers by a displacement $\vec{D}$, at the same time evaluating the wave function at the point $\vec{x}+\vec{D}$:

$$\psi_k(\vec{x}+\vec{D};\vec{x}_1+\vec{D},\ldots,\vec{x}_A+\vec{D})$$
$$= e^{i\vec{k}\cdot\vec{x}}e^{i\vec{k}\cdot\vec{D}} + \sum_{i=1}^{A}\int\int G_0(\vec{x}+\vec{D}-\vec{x}')V_i(\vec{x}'-\vec{D}-\vec{x}_i,\vec{x}''-\vec{D}-\vec{x}_i)\psi_k^-(\vec{x}'';\vec{x}_1+\vec{D},\ldots,\vec{x}_A+\vec{D})d\vec{x}'d\vec{x}''. \quad \text{(B2)}$$

Shifting the dummy variable by $\vec{D}$, the equation becomes, after multiplying by $e^{-i\vec{k}\cdot\vec{D}}$,

$$[e^{-i\vec{k}\cdot\vec{D}}\psi_k^-(\vec{x}+\vec{D};\vec{x}_1+\vec{D},\ldots,\vec{x}_A+\vec{D})]$$
$$= e^{i\vec{k}\cdot\vec{x}} + \sum_{i=1}^{A}\int\int G_0(\vec{x}-\vec{x}')V_i(\vec{x}'-\vec{x}_i,\vec{x}''-\vec{x}_i)[e^{-i\vec{k}\cdot\vec{D}}\psi_k^-(\vec{x}''+\vec{D};\vec{x}_1+\vec{D},\ldots,\vec{x}_A+\vec{D})]d\vec{x}'d\vec{x}''. \quad \text{(B3)}$$

Treating the bracketed quantity as the unknown function, Eqs. (B1) and (B3) are identical. By the uniqueness of solutions of integral equations, assuming well behaved potentials, we have

$$\psi(\vec{x};\vec{x}_1,\ldots,\vec{x}_A) = e^{-i\vec{k}\cdot\vec{D}}\psi_k(\vec{x}+\vec{D};\vec{x}_1+\vec{D},\ldots,\vec{x}_A+\vec{D}), \quad \text{(B4)}$$

which is Eq. (8).

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