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Numerical investigation of minimal three-body equations. II. Resonant pair interactions

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We apply the "minimal" three-body equations to a simple three-boson model with resonant pair interaction. We show that the isobar or sequential decay amplitude has important variation over the three-body phase space even for relatively weak coupling and we give an approximate analytic expression for that amplitude that reproduces the variation for a wide range of parameters.

[NUCLEAR REACTIONS Numerical and analytic investigation of unitarity and analyticity constraints in a resonant three-boson model.]

I. INTRODUCTION

In a recent series of papers we have stressed the role the general principles of quantum mechanics—unitarity and analyticity—play in determining the major features of three-body final states.[1-4] We have shown that in the sequential decay or isobar language it is possible to embody these constraints in a "minimal" integral equation. In this paper we investigate numerical solutions of that equation in the case of pairwise resonant final state interactions. This is the case for which the sequential decay or isobar formalism is intended. We pick a particularly simple model—three bosons with s-wave resonant interactions—and show that it is possible to find an analytic expression that represents the major features of the full numerical solution for a wide range of parameters and is also simple enough to permit its properties to be "read off".

There are two major points we are trying to make here. One is that even moderate strength final state resonant interactions can produce important variation in magnitude and phase of the amplitude, usually taken as a constant in phenomenological applications of the sequential decay or isobar method with resonant final state interactions. The second is that it is possible to give an analytic expression for this amplitude that is both simple and numerically valid over a wide range of parameters.

In Sec. II we present the formalism for the minimal three-body equation with resonant pair interactions and derive one simple approximate solution. In Sec. III we give the numerical results, stressing both the validity of the analytic form and the important variation of the amplitude. Section IV contains a brief discussion.

II. FORMALISM

Consider spinless bosons of mass 1 ($\hbar=1$) with two-body interactions in s waves only. Suppose further that that interaction produces a pair resonance at energy E_0 with width Γ so that near the resonance the two-body t matrix in the center of mass at energy E can be written[5]

$$t(E)=\frac{8\pi^2\Gamma}{E-E_0+\frac{1}{2}i(E/E_0)^{1/2}\Gamma} \tag{1}$$

or, in the narrow width approximation,

$$t(E)=\frac{8\pi^2\Gamma}{E-E_0+\frac{1}{2}i\Gamma}\,. \tag{2}$$

We wish to study a three-body final state of these bosons, in the phase-space region dominated by the pair resonances. In particular we want to study how the resonant final state interaction information is distributed over the phase space and how quantum-mechanical coherence effects that distribution. We do this in terms of the isobar or quasi-two-body amplitude that describes the transition from the initial state to a state of resonant pair and spectator particle. The full three-body final state is obtained from this amplitude by appending the resonant pair propagator and decay vertex and summing over all possible pairs. The quantum-mechanical constraints of unitarity and analyticity on the quasi-two-body amplitudes are expressed in terms of a linear integral equation very similar to the separable potential equations. We are not interested in this work in how resonant interactions affect the total reaction rate or produce three-body resonances. (This problem has been investigated in detail in a previous paper, where it was shown that very large effects are indeed possible.[6]) Rather, we are interested only in the dominant dependence of the two-body amplitude on the spectator momentum in the final-state phase space. For this purpose we do not need to construct the detailed two-body dynamics that leads to the t matrix (1), but can use it directly in the three-body formalism without concern for off-shell effects. The t of (1) will introduce no left-hand cuts and we can use it directly in the zero-range form of the three-body formalism.[1,3] In that case the three-body equation is

$$(I|f|\vec{p})=(I|R|\vec{p})+\frac{1}{(2\pi)^4}\int d^3p'\,\frac{t(E-\frac{3}{4}p'^2)(I|f|\vec{p}')}{E-p^2-p'^2-\vec{p}\cdot\vec{p}'}\,, \tag{3}$$

where I stands for the initial state and R is the inhomogeneous term. In order to keep our discussion as simple as possible, consider an initial state and inhomogeneous term that correspond to the weak three-body decay of a scalar state. $(I|R|\vec{p})$ then represents the amplitude for that decay without rescattering. We take this vertex to be structureless. Furthermore, f will be linear in the magnitude of R, the weak decay strength, hence we take it to be unity. Making this replacement in (3), taking the three-body S-wave projection, substituting (1) for t, and making the further simplification of using E_0 as the energy scale (this corresponds to putting $E_0=1$), we get

$$f(p)=1+\frac{\Gamma}{\pi p}\int\frac{p'dp'\ln[(pp'+p^2+p'^2-E)/(-pp'+p^2+p'^2-E)]f(p')}{\frac{3}{4}p'^2-E+1-i(E-\frac{3}{4}p'^2)^{1/2}\frac{1}{2}\Gamma}\,, \tag{4}$$

where the S-wave projection of $(I|f|p)=f(p)$.

We now want to solve (4). This is easily done by the now standard techniques of rotation of contours to avoid singularities and then replacing integrals by sums and using matrix inversion to solve the resulting algebraic equation. However, in this simple problem one might hope for a quasi-analytic solution which, if not valid for all values of the parameters, might at least have a wide range of validity. We have already shown both analytically as well as numerically that the p (but not the E) dependence of $f(p)$ is well represented even in the first iterate of the three-body equation.[7] Hence let us study that iterate, and to make matters simpler consider the narrow width approximation. We are then led to the integral (recalling that $R=1$)

$$\frac{\Gamma}{\pi p}\int_0^\infty\frac{p'dp'}{\frac{3}{4}p'^2-E+1-\frac{1}{2}i\Gamma}\ln\left(\frac{p^2+p'^2+pp'-E}{p^2+p'^2-pp'-E}\right). \tag{5}$$

This can be rewritten

$$\frac{2\Gamma}{3\pi p}\int_{-\infty}^\infty\frac{p'dp'}{p'^2-p_0^2}\ln\left[\frac{(p'-p_{--})(p'-p_{-+})}{(p'-p_{+-})(p'-p_{++})}\right], \tag{6}$$

where

$$p_0^2=\tfrac{4}{3}(E-1+\tfrac{1}{2}i\Gamma)$$

and $p_{\pm\pm}=\frac{1}{2}[\pm p\pm(4E-3p^2)^{1/2}]$, the roots of the argument of the ln in (5). Furthermore, in getting (6) we have used the symmetry of the integrand to extend the integral from $-\infty$ to $+\infty$. p_0 has a positive imaginary part, while $p_{\pm\pm}$ has a small positive or negative imaginary part depending on the sign of the radical because E is understood to have a small positive imaginary part. We can therefore write the ln as the sum of two terms, one with only upper half plane singularities and one with only lower half plane singularities. The integral is then done by closing the contour in the half plane without the logarithmic cuts and picking up only the pole at $p=\pm p_0$. One finds for (6)

$$\frac{4\Gamma i}{3p}\ln\left(\frac{p_0+p_{++}}{p_0+p_{-+}}\right)=\frac{4\Gamma i}{3p}\ln\left[\frac{p_0+(E-\frac{3}{4}p^2)^{1/2}+\frac{1}{2}p}{p_0+(E-\frac{3}{4}p^2)^{1/2}-\frac{1}{2}p}\right]. \tag{7}$$

This function has all the characteristics of the principal p dependence of f in the physical region. The boundaries of that region are $p=0$ and $p^2=\frac{4}{3}E$. Equation (7) is finite and analytic in p^2 at $p=0$, but has the $(E-\frac{3}{4}p^2)^{1/2}$ branch point required by unitarity. Furthermore, the argument of the ln is never near zero or ∞ in the physical region (even if the Γ in p_0 is put equal to zero), and hence there is no nearby logarithmic singularity. Across the $(E-\frac{3}{4}p^2)^{1/2}$ cut the logarithm becomes $\ln[(p_0+p_{+-})/(p_0+p_{--})]$. The argument of this ln can introduce logarithmic singularities for physically allowed momenta, but these are on the second sheet of the $(E-\frac{3}{4}p^2)^{1/2}$ cut. The relationship of these singularities to the (also second sheet) Peierls singularities has been discussed elsewhere.[8] It is precisely to avoid these unphysical singularities that analyticity must be added to the unitarity constraint.

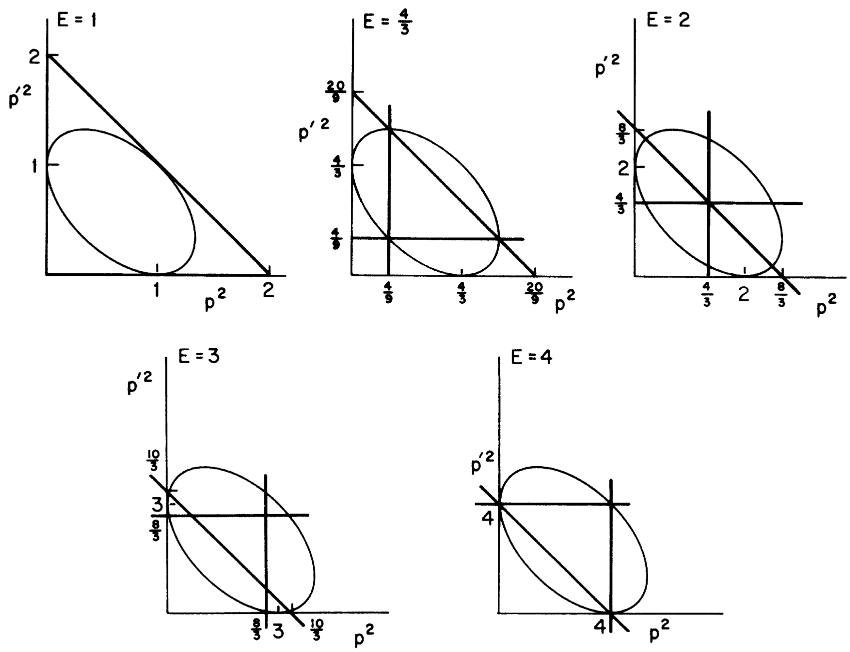

FIG. 1. The Dalitz ellipse, or phase space for total three-body energy $E=1$, $\frac{4}{3}$, 2, 3, and 4. The position of the pair resonance bands is indicated by the three full lines in each case.

With so much of the structure of f contained in (7), we might look for its use as an approximation to f. To improve the approximation we consider (7) not as the first term in the Neumann expansion of the equation, but as the first term in the expansion of the Fredholm numerator. To this order the Fredholm denominator D is given by

$$D=1-\mathrm{tr}K. \tag{8}$$

In terms of (4) again in the narrow width approximation, $\mathrm{tr}K$ is

$$\mathrm{tr}K=\frac{\Gamma}{\pi}\int_0^\infty \frac{dp'}{(\frac{3}{4}p'^2-E+1-\frac{1}{2}i\Gamma)}\ln\left(\frac{3p'^2-E}{p'^2-E}\right). \tag{9}$$

This integral can be done by precisely the same methods as were used to do (5)–(7), and we find

$$\mathrm{tr}K=\frac{4\Gamma i}{3p_0}\ln\left(\frac{3^{1/2}p_0+E^{1/2}}{p_0+E^{1/2}}\right), \tag{10}$$

with p_0 defined as in (6). We now have as our approximation for f

$$f(p)=1+\frac{4\Gamma i}{3pD}\ln\left(\frac{p_0+p_{++}}{p_0+p_{-+}}\right) \tag{11}$$

in terms of (6), (8), and (10).

In the next section we will compare (11) with the full solution of (4) for a range of numerical examples.

III. NUMERICAL RESULTS

In this section we compare numerical solution of the integral equation (4) for the isobar amplitude $f(p)$ with the analytic form (11). The parameters at our disposal are Γ the resonance width and E the total three-body energy. Since we keep the two-body resonance fixed at $E_0=1$, the width is now given in units of the resonance energy. We call it γ. p is the momentum of the spectator, while the isobar energy in its center of mass is $E-\frac{3}{4}p^2$.

TABLE I. Comparison of exact [from Eq. (4)] and approximate [Eq. (8)] values of the Fredholm denominator.

	E	ReD	ImD	Re$(1-\mathrm{tr}K)$	Im$(1-\mathrm{tr}K)$
$\gamma=0.3$	4	1.00	−0.06	1.00	−0.06
	3	1.00	−0.07	1.00	−0.07
	2	1.00	−0.10	1.00	−0.10
	$\frac{4}{3}$	0.98	−0.14	0.99	−0.14
	1	0.93	−0.18	0.94	−0.19
$\gamma=0.6$	4	0.99	−0.12	1.00	−0.13
	3	0.99	−0.14	0.99	−0.15
	2	0.97	−0.19	0.98	−0.20
	$\frac{4}{3}$	0.93	−0.25	0.95	−0.27
	1	0.85	−0.29	0.88	−0.32
$\gamma=1$	4	0.98	−0.20	0.99	−0.21
	3	0.96	−0.23	0.98	−0.25
	2	0.92	−0.29	0.96	−0.32
	$\frac{4}{3}$	0.84	−0.36	0.89	−0.41
	1	0.75	−0.38	0.80	−0.46

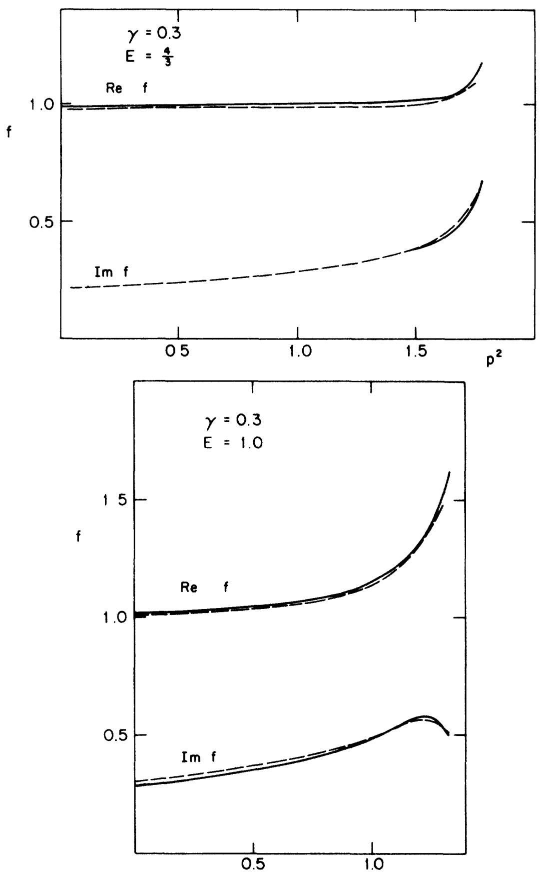

FIG. 2. Solution of Eq. (4) (solid line), the "exact" equation, compared with the approximate form, Eq. (11) (dashed line) for the real and imaginary parts of f for $\gamma = 0.3$ and $E = \frac{4}{3}$ and 1.

Hence at $E = 1$ the resonances will first appear, just at the edges of the kinematic region. As E increases they will sweep through the Dalitz ellipse. In Fig. 1 we show the Dalitz ellipse for $E = 1$, $\frac{4}{3}$, 2, 3, and 4. Some of these correspond to important kinematic limits, for example where all bands cross at the center ($E = 2$) or where two bands cross at the kinematic boundary ($E = 4$ or $E = \frac{4}{3}$). As we shall see, and as has been discussed analytically before, nothing spectacular happens on the physical sheet at these points.[9] For γ we take values 0.3, 0.6, and 1. In Table I we compare the exact [from numerical inversion of (4)] and approximate (8) values of the Fredholm denominator for our range of parameters. We see that at one extreme ($\gamma = 0.3$, $E = 4$) we have effectively very weak coupling and it is not surprising that the two agree, but even at the point of strongest coupling ($\gamma = 1$, $E = 1$) the two are in remarkably good agreement. Having seen this we should not be surprised to find (11) giving good results for $f(p)$. In Figs. 2, 3, and 4 we show $f(p)$ from (4) and from (11) for a typical range of γ and E. For $\gamma = 0.3$ we do not show $E = 2$, 3, or 4 because the coupling is so weak in these cases that f has no important variation. In all cases we see the $(E - \frac{3}{4}p^2)^{1/2}$ branch point. We see that certainly for weak coupling, but even for moderate coupling ($\gamma = 1$, $E = 1$) the analytic form (11) is doing a good job. But most importantly we see that even for relatively weak coupling ($\gamma = 0.3$, $E = 1$) and certainly for moderate coupling, $f(p)$ has important dependence on p and it is not all simply associated with the $(E - \frac{3}{4}p^2)^{1/2}$ singularity. Most phenomenological analysis takes f to be a constant, while our results here give an indication

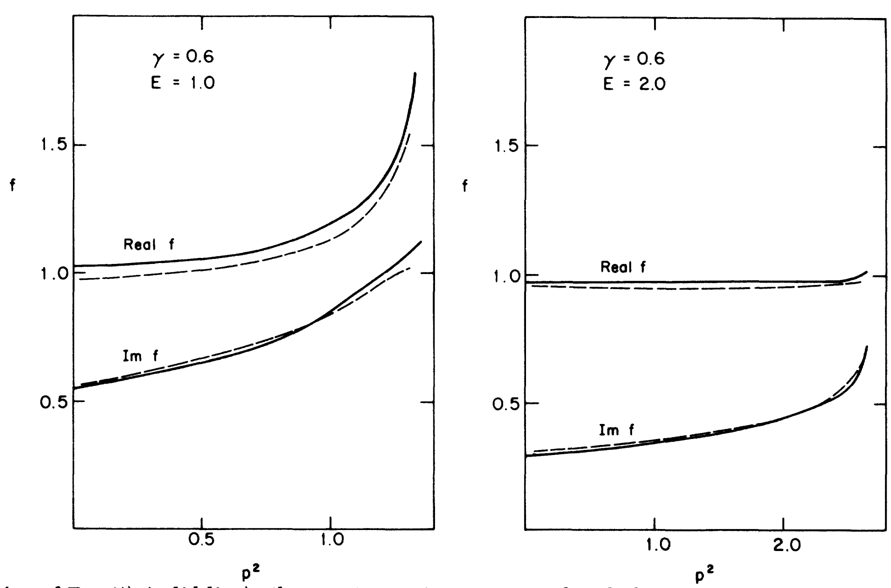

FIG. 3. Solution of Eq. (4) (solid line), the exact equation, compared with the approximate form, Eq. (11) (dashed line), for the real and imaginary parts of f for $\gamma = 0.6$ and $E = 1$ and 2.

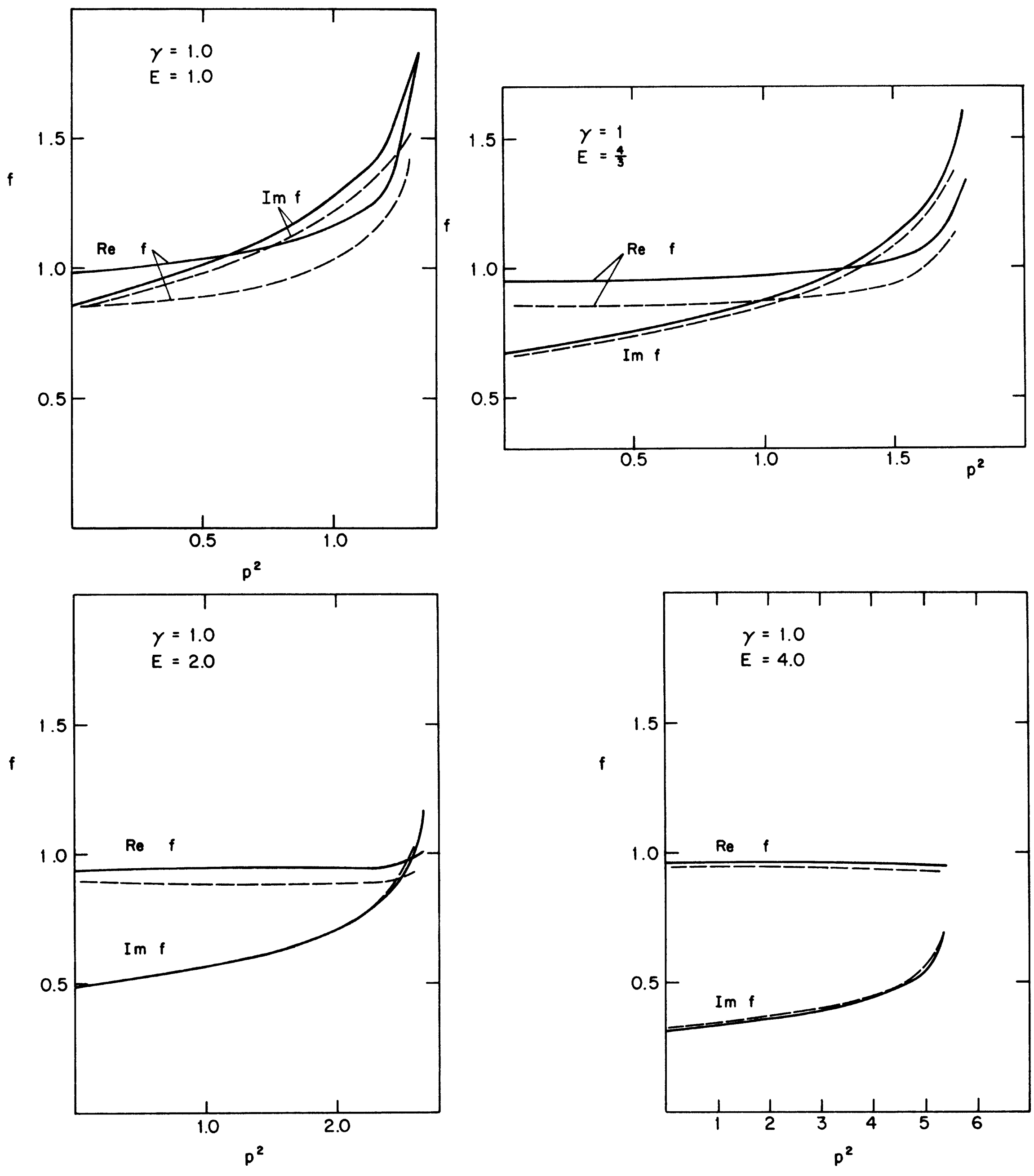

FIG. 4. Solution of Eq. (4) (solid line), the exact equation, compared with the approximate form, Eq. (11) (dashed line), for the real and imaginary parts of f for $\gamma=1$ and $E=1$, $\frac{4}{3}$, 2, and 4.

of the shaky validity of this assumption. Our success with (11) provides a form for repairing this assumption, at least in the simple three-boson case, but the fact that much of the dependence of f does not come from the unitarity singularity ($E - \frac{3}{4}p^2 = 0$) also indicates that the details of the full dynamics may play an important role in giving the dependence of f and hence a richer theory, and full solution of the equation corresponding to (4) may be required.

IV. DISCUSSION

We have studied in a simple spinless boson model with s-wave pair resonant interactions the "minimal" effect of unitarity and analyticity on three-body states. We have seen that even moderate strength final state interactions produce significant variation of the isobar amplitudes and that a simple analytic form gives a remarkably good account of that variation for a wide range of parameters. It may be that in actual physical problems the details of the dynamics and the complexity of the example will make such a simple form less useful. Even if it does, direct solution of the appropriate "minimal" equation corresponding to (4) is by no means difficult. We have examined a simple extension of our model to a three-body state with angular momentum 2 but still s-wave pair interaction. We find the three-body effects to be very weak in that case. We are examining a number of other cases and plan soon to attack the interesting and particularly simple case of the decay of C^{12*} $(J=1^+)\to 3\alpha$. We are also incorporating the major features of the results reported here in our relativistic analysis of the $\pi N\to\pi\pi N$ system[10] and of three-meson systems. In these cases relativistic kinematics make it much harder to do the integrals corresponding to (7) and we will presumably use direct solution of the equation corresponding to (4).

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